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Theory of the spin-1 bosonic liquid metal: Equilibrium properties of liquid metallic deuterium

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The theory of a two-component quantum fluid comprised of spin-$\frac{1}{2}$ fermions and *nonzero* spin bosons is examined. This system is of interest because it embodies a possible quantum liquid metallic phase of highly compressed deuterium. Bose condensation is assumed present and the two cases of nuclear-spin-polarized and -unpolarized systems are considered. A significant feature in the unpolarized case is the presence, of a *nonmagnetic* mode with quadratic dispersion owing its existence to nonzero boson spin. The physical character of this mode is examined in detail within a Bogoliubov approach. The specific heat, bulk modulus, spin susceptibility, and thermal expansion are all determined. Striking contrasts in the specific heats and thermal-expansion coefficients of the liquid and corresponding normal solid metallic phase are predicted.

I. INTRODUCTION

Theoretical interest in the expected high-pressure states of metallic hydrogen and deuterium has most often centered on a possible solid crystalline phase.[1] Less frequently considered are dense liquid metallic phases, which undoubtedly exist at sufficiently high temperatures. An intriguing possibility, suggested in a number of detailed studies,[2] is that the liquid metallic phase may, at a density near that at the metal-insulator transition, continue to exist even at temperatures sufficiently low that the "ionic" degrees of freedom must be treated quantum mechanically. The physical basis for this low-temperature liquidity is attributable to the low ionic mass and concomitant high zero-point energy (as is also the case in liquid ^{3}He and ^{4}He).

A theory of a "quantum" liquid metallic phase of dense hydrogen (i.e., a two-component Fermion fluid of electrons and protons) has been given earlier.[3] This paper presents a corresponding study of the low-temperature phase of liquid metallic deuterium (LMD). Liquid metallic deuterium constitutes a two-component boson-fermion fluid with, most importantly (and in contrast to ^{3}He-^{4}He mixtures), the boson component having *nonzero* spin. The presence of these bosonic spin degrees of freedom leads to a new quasiparticle branch (a Goldstone mode) which, as will be seen, has significant implications for both the equilibrium and transport properties of the system. In further contrast to ^{3}He-^{4}He mixtures, LMD is distinguished by (i) long-range Coulomb forces and (ii) an enormous component-mass ratio.

No definitive static metallization either of H or D has been achieved, though there are some claims to the contrary.[4] Transient metallization involving dynamic techniques[5] has also been reported, but is also not definitive. Aside from possible detection of metallic conductivity no empirical knowledge of the preferred phase or its properties has been advanced. Several efforts aimed at metallization are, however, underway or envisaged.[4] Thus it is appropriate to attempt to anticipate properties of the liquid metallic phase in order to provide a basis for experimental discrimination between liquid and solid phases if metallization is eventually achieved. The system is also of intrinsic theoretical interest.

Section II discusses a model for LMD which is appropriate for a first study. Two cases are considered: deuterons both spin-polarized and unpolarized, respectively. Section III describes the quasiparticle spectrum and in particular clarifies the nature of the Goldstone branch. We concentrate here on equilibrium properties, and these are dealt with in Sec. IV. The nature of the quasiparticle interactions in this system and their transport properties are also of considerable interest. These are planned to be discussed in a separate paper.

II. MODEL AND MAGNETIC PREPARATION

We consider a mixture of N spin-$\frac{1}{2}$ fermions (electrons) of mass m_e and charge $-e$, and of N spin-1 bosons (deuterons) of mass m_d and charge $+e$. We assume the interactions to be Coulombic, and we assume that there are no spin-dependent interactions. Note that unlike ^{3}He-^{4}He mixtures[6] no macroscopic fermion-boson phase separation can be possible here, because of the long-range nature of the particle forces. Further, unlike the case of *isotopic* boson mixtures with isotope-independent interactions,[7] LMD, viewed as an effective three-component Bose fluid, will not undergo bosonic phase separation.

For purposes of this analysis we may assume the existence of a Bose-condensed but otherwise "normal" liquid metallic phase. Accordingly, we omit from consideration a number of open and interesting issues, e.g., the existence of electronic superconductivity,[8] electron ferromagnetism, liquid crystallinity, charge-density-wave formation, and other states of order. Furthermore, since the first successful experimental metallization will likely involve a very small, possibly microscopic sample of LMD which will be very much confined in the high-pressure maintaining apparatus, we will not address the issue of superfluid hydrodynamics.

We treat two cases: (A) "chemical" equilibrium among all boson components, (B) "chemical" nonequilibrium with a majority of bosons being in one spin sublevel (α) and a small fraction ($\lesssim 10^{-2}$) in a second level (β).[9] Case A is expected to be realized experimentally, almost by default. Case B requires preparation of the sample through the use of a large magnetic field. In practice, case B will eventually relax to case A because of the presence of dipolar forces in actual LMD. In both cases we consider the temperature to be much *lower* than the Bose-condensation temperature. Note that in case A all boson spin levels are in general condensed, whereas in case B there is the possibility that for sufficiently high temperatures, level α, for example, will be condensed while level β is not; in case B we restrict our attention to this situation.

The Bose-condensation (BC) temperature $\widetilde{T}$ for an ideal Bose gas is given by[10]

$$\widetilde{T}=\frac{3.31\hbar^2 n^{2/3}}{g^{2/3} m k_B} , \tag{1}$$

where g is the spin degeneracy, n is the density, m is the particle mass, and k_B is Boltzmann's constant. We will be interested in an r_s range[11] of 1.0–1.6 thought to be near the $T\approx 0$ metal-insulator transition boundary.[2] For the ideal case of A, $g=3$, $m=m_d=3.34\times 10^{-24}$ gm, and the BC temperature $\widetilde{T}_{(A)}\approx 21$ K (53 K) for $r_s=1.6$ (1.0). For the ideal case of B, it is relevant to consider two BC temperatures $\widetilde{T}_\alpha$ and $\widetilde{T}_\beta$ for the α and β levels, respectively. Then $\widetilde{T}_\alpha \approx 43$ K (110 K) for $r_s=1.6$ (1.0). For minority-spin concentration c of 1% and 0.1%, $\widetilde{T}_\beta\approx 2.0$ K (5.1 K) and 0.43 K (1.1 K), respectively, for $r_s=1.6$ (1.0).

These results give a very rough guide to the corresponding fully interacting BC temperatures (denoted here with asterisks). We expect the T^*'s to be somewhat lower, but probably of the same order of magnitude as the ideal estimates. A reasonable estimate for the deuteron effective mass ($m_d^*\sim 2m_d$) suggests by a simple estimate[12] that the actual T^*'s are a factor of $\lesssim 10$ lower than these results. In any event it seems likely that the temperature regimes $T \ll T^*_{(A)}$ or $T^*_\beta < T \ll T^*_\alpha$ will be easily attainable experimentally.

Preparation for case B. Assuming equilibrium conditions and using $\mu_d=0.857\mu_N=4.33\times 10^{-24}$ erg G^{-1}, a spin sublevel ratio n_β/n_α is produced with a magnetic field H given by $H\sim -3\times 10^7 T\ln(n_\beta/n_\alpha)$ G. Using the maximum presently available static fields ($\sim 10^5$ G) and existing millikelvin techniques, a 0.1% "solution" will result. The thermalizing nuclear spin depolarization which will then take place after the field is removed is primarily attributable to the nuclear-spin—conduction-electron hyperfine interaction. The effect of nuclear dipole-dipole interactions is somewhat smaller both because of the smallness of the ratio of nuclear to electronic moments ($\sim 10^{-3}$) and because of the smallness of the internuclear relative velocity as compared to the typical nuclear-electron relative velocity. Accordingly, a longitudinal magnetic relaxation time τ may be estimated from the Heitler-Teller formula,[13]

$$\frac{1}{\tau}\sim\frac{64\pi^2}{9}\frac{\gamma_e^2\gamma_d^2\hbar^3}{\epsilon_F}\,|\psi(0)|_F^4\,\frac{k_B T}{\epsilon_F} , \tag{2}$$

where γ_e (γ_d) is the electron (deuteron) gyromagnetic ratio, ϵ_F is the Fermi momentum of the electrons, and $|\psi(0)|_F^2$ is the Fermi surface average of the electronic wave functions at the position of the nucleus. For a rough estimate, it is sufficient to take $|\psi(0)|^2$ to be its value in the 1s hydrogenic state: $|\psi_{1s}(0)|^2=2.1\times 10^{24}$ cm^{-3}. This gives $\tau T\approx 3.7\times 10^3$ (2.3×10^4) sec for $r_s=1.0$ (1.6). Thus for $T\lesssim 1$ K the nuclear magnetic relaxation times should be more than ample to permit experimental studies of LMD under case B conditions. We do not consider the question of either the existence or the effect of nuclear magnetic domains.

III. ELEMENTARY EXCITATIONS

We adopt in this initial description of LMD a phenomenological approach; we first identify the quasiparticle excitations and their approximate dispersions. In the spirit of the Landau-Khalatnikov[14] approach used for ^{4}He, we determine the finite-temperature equilibrium and near-equilibrium properties under the assumption that these quasiparticles are distributed according to the equilibrium or near-equilibrium quantum ideal-gas distribution functions for the appropriate statistics.

A qualitative identification of the quasiparticles proceeds by first taking the point of view that LMD (case A) is an effective three-component Bose fluid with short-range species-independent effective interactions. Here of course the electronic degrees of freedom are ignored except insofar as they implicitly enter into the screening of the effective deuteron-deuteron interaction. From the appropriate effective deuteron Hamiltonian we expect to be able to identify the form of those quasiparticle branches possessing boson character. As will be seen, this approach actually leads to the identification of *two* types of bosonic branches. For small k, one is the expected longitudinal phonon while the other is a particle-type excitation whose presence is entirely a consequence of the nonzero boson spin.

A complementary point of view is to regard LMD as an effective electron gas moving in a "heavy" neutralizing and dynamically responding background. The background augments the screening due to the electron gas itself. This does not, however, change the qualitative fact that (renormalized) fermionic particle-hole branches will continue to be present in the original problem, again, assuming no electronic ordering such as superconductivity.

For the case of a mixture of spin-0 condensed bosons and fermions a picture corresponding to that just given for LMD is supported by microscopic theory[15] in which a joint random-phase approximation is made for both species. It is concluded that there are *two* quasiparticle types: phonons and fermion quasiparticle-hole pairs, each renormalized by all the interactions in the problem. This picture, even for the fully interacting case, can be established by a fairly general diagrammatic analysis. Consider the diagrammatic expansion of boson single-particle propagators.[11] Since the system is condensed there will exist two "anomalous" single-particle propagators involving, respectively, two creation and two annihilation operators. The three types of propagators will be coupled together via a Beliaev-Dyson equation.[10]

It is clear that the electron propagators will enter only in the form of polarization insertions (see Fig. 1). Thus, formally, the full diagrammatic expansion for the (coupled) boson propagators in the presence of the electrons is *topologically identical* to that for a system with bosons alone. The basic interaction unit involved is not, however, a static bare long-range interaction; rather it is a dynamically screened short-range interaction. To the extent that the electrons adiabatically follow the ions, it is reasonable to take this effective interaction as a renormalized static interaction, justifying the picture just given above.

A similar argument applies to the electron single-particle propagators. Here, the boson propagators enter only in polarization insertions within electron-electron bare interaction lines (see Fig. 1). Unlike the previous case, the replacement within a diagram of a basic bare Coulomb interaction unit with the dynamic, ionically screened effective interaction does not directly lead to the view that the electrons are interacting via a new static force.

Even though the effective electron-electron interaction is dynamic, experience with the electron-phonon interaction in metals[16] and the results of the calculation of Ref. 15 together give us some confidence that the above viewpoint is nevertheless often correct. In some cases, of course, it is distinctly incorrect, e.g., when the effective forces are attractive. Neglecting such possible effects, the above argument at the very least furnishes a framework within which we may predict the existence of normal fermionic quasiparticles associated with the electron propagator. The energy $\epsilon_e(k)$ of such a quasiparticle of momentum $\vec{k}$ is given by

$$\epsilon_e(k) \approx \frac{k_F}{m_e^*}(k-k_F)+\mu_e, \quad k \approx k_F \tag{3}$$

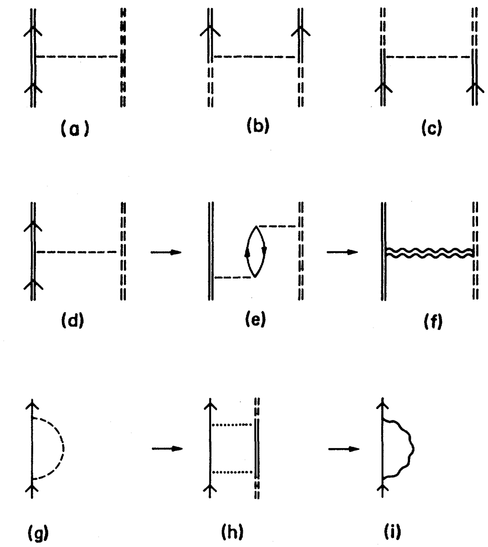

FIG. 1. (a), (b), and (c) show first-order contributions to "normal," anomalous-"out," and anomalous-"in" single-particle boson propagators, respectively. Double solid lines and double dashed lines denote bare boson propagators and "condensate factors," respectively (see Ref. 10). Single dashed lines denote bare Coulomb interaction between like charges. In (d), (e), and (f) the renormalization of a typical boson diagram, (a), due to the electrons is indicated: (e) shows a typical screening of the boson-boson interaction by an electron particle-hole pair [solid single lines denote bare electron (hole) propagators]. (f) symbolizes the effect of all types of electron polarization insertions within the given boson diagram topology, the double wavy line denoting the fully electronically screened boson-boson effective interaction. Such a procedure would apply to all boson diagrams. A similar argument applies to the bosonic renormalization of the electron propagator, illustrated for a typical electron diagram in (g)—(i). The dotted line denotes the bare electron-boson Coulomb interaction.

where k_F is the Fermi momentum, m_e^* is the effective fermion mass (renormalized by interactions with *all* species in the system), and μ_e is the fully renormalized electron chemical potential.

Bosonic modes. We now consider in some detail the bosonic quasiparticle branches. The effective bosonic Hamiltonian H is

$$H = E_0 + \sum_{\vec{k}} \sum_{\lambda=1}^{\nu} \frac{k^2}{2\tilde{m}_d} a^{\dagger}_{\lambda \vec{k}} a_{\lambda \vec{k}} + \frac{1}{2V} \sum_{\substack{\vec{k}_1 \vec{k}_2, \\ \vec{k}_3 \vec{k}_4}} \sum_{\lambda,\lambda'=1}^{\nu} U(|\vec{k}_1-\vec{k}_3|)\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4) a^{\dagger}_{\lambda \vec{k}_3} a^{\dagger}_{\lambda' \vec{k}_4} a_{\lambda' \vec{k}_1} a_{\lambda \vec{k}_2}\,, \tag{4}$$

where $\tilde{m}_d$ is an electronically renormalized deuteron effective mass, ν is the number of spin components ($\nu=3$ for LMD), U is the effective spin-independent short-range interaction, V the volume, and $a^{\dagger}_{\lambda \vec{k}}$ ($a_{\lambda \vec{k}}$) the creation (destruction) operator for a boson of spin λ and momentum $\vec{k}$. Here E_0 is a constant resulting from the elimination of the electron degrees of freedom.

The multicomponent Hamiltonian has been considered in a number of studies: Nepomnyaschii analyzed the formal structure of the Green's functions and, using an argument similar to that of Gavoret-Nozières,[17] established the form of the quasiparticle spectra for $k \rightarrow 0$.[18] Bassichis gave a very formal treatment of Eq. (4) for arbitrary ν within the Bogoliubov approximation.[19] Colson and Fetter also studied the somewhat analogous problem of ^{4}He-^{6}He Bose mixtures.[20] And Halperin considered the question of hydrodynamic modes.[21] All of these authors pointed out the existence of new particle-like modes.

In the present approach, as a qualitative guide to LMD, we analyze in some detail Eq. (4) for $\gamma=3$ within the Bogoliubov approximation. We utilize a less general, though more direct, procedure than in Ref. 19. We also more fully discuss and clarify the physical meaning of the new particle-like modes.

A. Case A

Assuming the most general form of Bose condensation, we write

$$N_{0\lambda}=N_0U_\lambda^2 , \tag{5a}$$

$$\sum_{\lambda=1}^{3} U_\lambda^2=1 , \tag{5b}$$

where $N_{0\lambda}$ is the condensate number in the λth spin sublevel, N_0 is total condensate number, and U_λ are the components of an arbitrary real unit vector. As in the usual spin-0 Bogoliubov procedure we treat the zero-momentum operators as c numbers and retain terms of order N_0 and N_0^2 only.[22] Using

$$N_0^2\approx N^2-2N_0\sum_{\vec{k}\,(\neq\vec{0})}\sum_{\lambda=1}^{3}a^\dagger_{\lambda\vec{k}}a_{\lambda\vec{k}} ,$$

we are then led to

$$H_{(A)}\approx E_0+\sum_{\vec{k}}\sum_{\lambda=1}^{3}\frac{k^2}{2\widetilde{m}_d}a^\dagger_{\lambda\vec{k}}a_{\lambda\vec{k}}+\frac{1}{2V}\left[N^2U(0)+2N\sum_{\vec{k}\,(\neq\vec{0})}\sum_{\lambda,\lambda'=1}^{3}U(k)a^\dagger_{\lambda'\vec{k}}a_{\lambda\vec{k}}U_\lambda U_{\lambda'}\right.$$
$$\left.+N\sum_{\vec{k}\,(\neq\vec{0})}\sum_{\lambda,\lambda'=1}^{3}U(k)(a_{\lambda\vec{k}}a_{\lambda',-\vec{k}}+a^\dagger_{\lambda\vec{k}}a^\dagger_{\lambda',-\vec{k}})U_\lambda U_{\lambda'}\right] . \tag{6}$$

In view of the form of Eq. (6) we introduce new boson operators $\widetilde{a}^\dagger,\widetilde{a}$ by

$$\widetilde{a}^\dagger_{\vec{k}}\equiv\sum_{\lambda=1}^{3}U_\lambda a^\dagger_{\lambda\vec{k}},\quad \widetilde{a}_{\vec{k}}\equiv\sum_{\lambda=1}^{3}U_\lambda a_{\lambda\vec{k}} . \tag{7}$$

It is trivially verified that the $\widetilde{a}^\dagger$, $\widetilde{a}$ satisfy the canonical Bose commutation relations $[\widetilde{a}_{\vec{k}},\widetilde{a}^\dagger_{\vec{k}'}]=\delta_{\vec{k}\vec{k}'}$, $[\widetilde{a}_{\vec{k}},\widetilde{a}_{\vec{k}'}]=[\widetilde{a}^\dagger_{\vec{k}},\widetilde{a}^\dagger_{\vec{k}'}]=0$. Using Eqs. (7) and adding and subtracting a kinetic energy operator in the new fields, we have from Eq. (6),

$$H_{(A)}\approx E_0+H_1+H_2 , \tag{8a}$$

$$H_1=\sum_{\vec{k}}\sum_{\lambda=1}^{3}\frac{k^2}{2\widetilde{m}_d}a^\dagger_{\lambda\vec{k}}a_{\lambda\vec{k}}-\sum_{\vec{k}}\frac{k^2}{2\widetilde{m}_d}\widetilde{a}^\dagger_{\vec{k}}\widetilde{a}_{\vec{k}} , \tag{8b}$$

$$H_2\equiv\sum_{\vec{k}}\frac{k^2}{2\widetilde{m}_d}\widetilde{a}^\dagger_{\vec{k}}\widetilde{a}_{\vec{k}}+\frac{1}{2V}\left[N^2U(0)+2N\sum_{\vec{k}\,(\neq\vec{0})}U(\vec{k})\widetilde{a}^\dagger_{\vec{k}}\widetilde{a}_{\vec{k}}+N\sum_{\vec{k}\,(\neq\vec{0})}U(k)(\widetilde{a}_{\vec{k}}\widetilde{a}_{-\vec{k}}+\widetilde{a}^\dagger_{\vec{k}}\widetilde{a}^\dagger_{-\vec{k}})\right] . \tag{8c}$$

We observe that H_2 has the same form as the full Hamiltonian (with Bogoliubov replacement) for the case of spin 0.[22] Evidently H_2 is then diagonalized in the canonical representation of α defined by

$$\widetilde{a}_{\vec{k}}=\frac{1}{(1-A_k^2)^{1/2}}(\alpha_{\vec{k}}+A_k\alpha^\dagger_{-\vec{k}}),\quad \widetilde{a}^\dagger_{\vec{k}}=\frac{1}{(1-A_k^2)^{1/2}}(\alpha^\dagger_{\vec{k}}+A_k\alpha_{-\vec{k}}) . \tag{9a}$$

The inverse relations are

$$\alpha_{\vec{k}}=\frac{1}{(1-A_k^2)^{1/2}}(\widetilde{a}_{\vec{k}}-A_k\widetilde{a}^\dagger_{-\vec{k}}),\quad \alpha^\dagger_{\vec{k}}=\frac{1}{(1-A_k^2)^{1/2}}(\widetilde{a}^\dagger_{\vec{k}}-A_k\widetilde{a}_{-\vec{k}}) , \tag{9b}$$

with

$$A_k=\frac{V}{U(k)N}\left\{-\frac{k^2}{2\widetilde{m}_d}-U(k)\frac{N}{V}+\left[\left[\frac{k^2}{2\widetilde{m}_d}+U(k)\frac{N}{V}\right]^2-\left[U(k)\frac{N}{V}\right]^2\right]^{1/2}\right\} . \tag{10}$$

Then,

$$H_2=\frac{N^2}{2V}U(0)+\sum_{\vec{k}}\epsilon_b(k)\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}} \tag{11}$$

where the bosonic excitation energy $\epsilon_b(k)$ in the Bogoliubov approximation is

$$\epsilon_b(k)=\left[\left[\frac{k^2}{2\widetilde{m}_d}+U(k)\frac{N}{V}\right]^2-U(k)\frac{N}{V}\right]^{1/2}. \tag{12}$$

As is well known, in the limit of $k\to 0$ these modes are phonon-like with linear dispersion; for $k\to\infty$, they are particle-like quadratic modes.

We turn to the diagonalization of H_1 which is rewritten as

$$H_1=\sum_{\vec{k}}\sum_{\lambda,\lambda'=1}^{3}\frac{k^2}{2\widetilde{m}_d}(\delta_{\lambda\lambda'}a^\dagger_{\lambda\vec{k}}a_{\lambda'\vec{k}}-U_\lambda U_{\lambda'}a^\dagger_{\lambda\vec{k}}a_{\lambda'\vec{k}}). \tag{13}$$

Next, operators β_{1k} and β_{2k} are introduced by the definitions

$$\beta_{i\vec{k}}\equiv\sum_{j=1}^{3}X_{ij}a_{j\vec{k}},\quad \beta^\dagger_{i\vec{k}}\equiv\sum_{j=1}^{3}X_{ij}a^\dagger_{j\vec{k}},\quad i=1,2 \tag{14}$$

where X_{ij} are real c numbers to be determined by requiring that H_1 may be written as

$$H_1=\sum_{\vec{k}}\sum_{i=1}^{2}\frac{k^2}{2\widetilde{m}_d}\beta^\dagger_{i\vec{k}}\beta_{i\vec{k}}, \tag{15}$$

i.e., as a sum of *two* degenerate quadratic branches. Note that each spin sum in Eq. (13) is over *three* components.

Substituting Eqs. (14) into Eq. (15), using Eq. (5), and comparing to Eq. (13), we are led to the relations

$$X_{11}^2+X_{21}^2=U_2^2+U_3^2, \tag{16a}$$

$$X_{12}^2+X_{22}^2=U_1^2+U_3^2, \tag{16b}$$

$$X_{13}^2+X_{23}^2=U_1^2+U_2^2, \tag{16c}$$

$$X_{11}X_{12}+X_{21}X_{22}=-U_1U_2, \tag{16d}$$

$$X_{11}X_{13}+X_{21}X_{23}=-U_1U_3, \tag{16e}$$

and

$$X_{12}X_{13}+X_{22}X_{23}=-U_2U_3. \tag{16f}$$

These coupled nonlinear equations are solved by introducing three *two-component* vectors $\vec{v}_i$ defined as

$$\vec{v}_i\equiv(X_{1i},X_{2i}),\quad i=1,2,3. \tag{17}$$

The relations (16) can then be written as

$$\vec{v}_1\cdot\vec{v}_1=U_2^2+U_3^2,\quad \vec{v}_2\cdot\vec{v}_2=U_1^2+U_3^2,\quad \vec{v}_3\cdot\vec{v}_3=U_1^2+U_2^2, \tag{18a}$$

$$\vec{v}_1\cdot\vec{v}_2=-U_1U_2,\quad \vec{v}_1\cdot\vec{v}_3=-U_1U_3,\quad \vec{v}_2\cdot\vec{v}_3=-U_2U_3. \tag{18b}$$

Denoting the angle between $\vec{v}_i$ and $\vec{v}_j$ by θ_{ij} we then find

$$\cos\theta_{12}=\frac{-U_1U_2}{|\vec{v}_1||\vec{v}_2|}=\frac{-U_1U_2}{(U_2^2+U_3^2)^{1/2}(U_1^2+U_3^2)^{1/2}}. \tag{19}$$

We may choose $\vec{v}_1$ to be in the $\hat{x}$ direction [in the (v_x,v_y) plane], for which

$$X_{11}=(U_2^2+U_3^2)^{1/2} \tag{20a}$$

and

$$X_{21}=0. \tag{20b}$$

Then

$$X_{12}=v_{2x}=v_2\cos\theta_{12}=\frac{-U_1U_2}{(U_2^2+U_3^2)^{1/2}}, \tag{20c}$$

and

$$X_{22}=v_{2y}=v_2\sin\theta_{12}=\frac{+U_3}{(U_2^2+U_3^2)^{1/2}}. \tag{20d}$$

Similarly,

$$X_{13}=\frac{-U_1U_3}{(U_2^2+U_3^2)^{1/2}} \tag{20e}$$

and

$$X_{23}=\frac{-U_2}{(U_2^2+U_3^2)^{1/2}}. \tag{20f}$$

(The determination of the relative sign between X_{22} and X_{23} follows from the requirement $\vec{v}_2\cdot\vec{v}_3=-U_2U_3$.) The β_{ik} are then written as

$$\beta_{1\vec{k}}=(U_2^2+U_3^2)^{1/2}a_{1\vec{k}}-\frac{U_1}{(U_2^2+U_3^2)^{1/2}}(U_2a_{2\vec{k}}+U_3a_{3\vec{k}}) \tag{21a}$$

and

$$\beta_{2\vec{k}}=\frac{1}{(U_2^2+U_3^2)^{1/2}}(U_3a_{2\vec{k}}-U_2a_{3\vec{k}}). \tag{21b}$$

As is readily verified, this choice of the new operators satisfies canonical commutation relations: $[\gamma_{i\vec{k}},\gamma_{j\vec{k}'}]=[\gamma^\dagger_{i\vec{k}},\gamma^\dagger_{j\vec{k}'}]=0$, $[\gamma_{i\vec{k}},\gamma^\dagger_{j\vec{k}'}]=\delta_{ij}\delta_{\vec{k}\vec{k}'}$, where $\gamma_{i\vec{k}}\in\{\alpha_{\vec{k}},\beta_{1\vec{k}},\beta_{2\vec{k}}\}$. Within the Bogoliubov approximation, these new quadratic modes are unrenormalized.

The physical nature of the quadratic modes is clarified by considering the two-component analog of Eqs. (8). There are then two branches $\bar{\alpha}$ and $\bar{\beta}$ (linear and quadratic, respectively, as $k\to 0$), i.e.,

$$\bar{\alpha}^\dagger_{\vec{k}}=\frac{1}{(1-A_k^2)^{1/2}}\sum_{\lambda=1}^{2}(U_\lambda a^\dagger_{\lambda\vec{k}}-A_kU_\lambda a^\dagger_{\lambda,-\vec{k}}) \tag{22a}$$

and

$$\bar{\beta}^\dagger_{\vec{k}}=U_2a^\dagger_{1\vec{k}}-U_1a^\dagger_{2\vec{k}}. \tag{22b}$$

We express the longitudinal partial density and transverse spin-density fluctuation operators in the Bogoliubov approximation, i.e.,

$$\rho^{\dagger}_{1\vec{k}} \equiv \sum_{\vec{q}} a^{\dagger}_{1,\vec{k}+\vec{q}} a_{1\vec{q}} \approx N_0^{1/2} U_1 (a^{\dagger}_{1\vec{k}} + a_{1,-\vec{k}}) , \quad (23a)$$

$$\rho^{\dagger}_{2\vec{k}} \equiv \sum_{\vec{q}} a^{\dagger}_{2,\vec{k}+\vec{q}} a_{2\vec{q}} \approx N_0^{1/2} U_2 (a^{\dagger}_{2\vec{k}} + a_{2,-\vec{k}}) , \quad (23b)$$

$${}^{(-)}\sigma^{\dagger}_{\vec{k}} \equiv \sum_{\vec{q}} a^{\dagger}_{1,\vec{k}+\vec{q}} a_{2\vec{q}} \approx N_0^{1/2} (U_2 a^{\dagger}_{1\vec{k}} + U_1 a_{2,-\vec{k}}) , \quad (23c)$$

and

$${}^{(+)}\sigma^{\dagger}_{\vec{k}} \equiv \sum_{\vec{q}} a^{\dagger}_{2,\vec{k}+\vec{q}} a_{1\vec{q}} \approx N_0^{1/2} (U_1 a^{\dagger}_{2\vec{k}} + U_2 a_{1,-\vec{k}}) . \quad (23d)$$

By multiplying Eq. (23b) by U_1/U_2, subtracting this from Eq. (23c), and using Eq. (22b), we have

$$N_0^{1/2} \bar{\beta}^{\dagger}_{\vec{k}} = {}^{(-)}\sigma^{\dagger}_{\vec{k}} - \frac{U_1}{U_2} \rho^{\dagger}_{2\vec{k}} . \quad (24)$$

Similarly, by multiplying Eq. (23a) by U_2/U_1 and subtracting Eq. (23d) from this,

$$N_0^{1/2} \bar{\beta}^{\dagger}_{\vec{k}} = \frac{U_2}{U_1} \rho^{\dagger}_{1\vec{k}} - {}^{(+)}\sigma^{\dagger}_{\vec{k}} . \quad (25)$$

Combining Eqs. (24) and (25), we have the symmetric form

$$\bar{\beta}^{\dagger}_{\vec{k}} + \frac{1}{2N_0^{1/2}} \left[\left[\frac{U_2}{U_1} \rho^{\dagger}_{1\vec{k}} - \frac{U_1}{U_2} \rho^{\dagger}_{2\vec{k}} \right] + ({}^{(-)}\sigma^{\dagger}_{\vec{k}} - {}^{(+)}\sigma^{\dagger}_{\vec{k}}) \right] . \quad (26)$$

Thus, for $U_1 = U_2$, $\bar{\beta}_{\vec{k}}$ creates a combination of relative longitudinal density fluctuations and relative transverse spin fluctuations.

As noted earlier,[21] these modes may be thought of as broken-symmetry (Goldstone) modes: When the system condenses into a particular direction in "$\vec{U}$ space," the ground state breaks a continuous symmetry of the Hamiltonian. This leads to a new excitation branch with dispersion and damping both tending to zero as $k \to 0$. These modes, since they are in addition quadratic, may be thought of as "magnons" in $\vec{U}$ space. However, it is important to realize that they arise without the presence of any explicit magnetic (spin-flip) interactions in the Hamiltonian.

From Eqs. (21) it is also evident that the β modes represent a coherent superposition over the spin components of bare particle excitations. In addition to the important spin coherence, these excitations are analogous to those of the ideal Bose gas. Thus, if we consider, for example, the *spin-independent* scattering of an electron quasiparticle against a β mode, the relevant physical picture is one of the former scattering off a bare *particle*, and is rather different from scattering off of a longitudinal density fluctuation.

Naturally, if one goes beyond the Bogoliubov approximation, we expect both the normal bosonic dispersion and what may be considered as an "impurity"-like dispersion to be further renormalized, i.e., beyond any renormalization contained in Eqs. (8) resulting from the electrons. The analysis of Ref. 18 suggests that the form of the dispersion relations for $k \approx 0$ would, however, be preserved. For larger k there is a possibility of a roton-like feature in the phonon-like branch though since the ion "cores" here are softer and the kinetic energy higher than in ^{4}He, such a feature is expected to be weaker. In any case we consider temperatures sufficiently low that only the $k \approx 0$ modes are relevant.

Finally it is interesting to consider the $T=0$ dynamic structure factor $S(k,\omega)$ for the multicomponent Bose fluid:

$$S(k,\omega) \equiv \sum_n | \langle \Psi_n | \rho^{\dagger}_k | \Psi_0 \rangle |^2 \delta(\omega - \omega_{n0}) , \quad (27)$$

where $|\Psi_0\rangle$ and $|\Psi_n\rangle$ are the many-body ground and nth excited states and ω_{n0} is the difference in energy of states $|\Psi_0\rangle$ and $|\Psi_n\rangle$. Within the Born approximation the scattering probability of a longitudinal density probe (e.g., particle) transferring momentum $\vec{k}$ and energy ω is proportional to $S(k,\omega)$.[23] If we continue to work in the Bogoliubov approximation, the total longitudinal density fluctuation operator takes the form

$$\rho^{\dagger}_{\vec{k}} \equiv \sum_{\vec{q},\lambda} a^{\dagger}_{\lambda,\vec{k}+\vec{q}} a_{\lambda\vec{q}} \approx N_0^{1/2} \sum_{\lambda} (U_\lambda a_{\lambda,-\vec{k}} + U_\lambda a^{\dagger}_{\lambda\vec{k}})$$
$$= N_0^{1/2} (\tilde{a}_{-\vec{k}} + \tilde{a}^{\dagger}_{\vec{k}}) . \quad (28)$$

Using Eqs. (9) in Eq. (28) we then have, from Eq. (27),

$$S(k,\omega) = N_0^{1/2} \left[\frac{1+A_k}{1-A_k} \right]^{1/2} \delta(\omega - \epsilon_b(k)) . \quad (29)$$

Thus in this approximation a longitudinal density probe will see only a normal bosonic excitation branch contribution. The multicomponent Bose liquid is in this respect analogous to ^{4}He. It may seem strange that the impurity-like modes do not contribute to $S(k,\omega)$: They are somewhat analogous to "particles" excited out of the condensate and are thus similar to the excitations of an ideal Bose gas. These *do* contribute to the structure factor, i.e.,

$$S_{\mathrm{ideal}}(k,\omega) = N\delta(\omega - k^2/2m) . \quad (30)$$

The point is that the impurity-like modes are a coherent superposition of particle modes in different spin states. As seen from Eq. (29), such a superposition is not excited by a longitudinal density probe; it would be excited, however, by a spin-discriminating probe.

Finally we remark that in case A in equilibrium (and at finite temperatures) we have three types of quasiparticles present with quasiparticle distribution functions given by

$$n_e(\vec{k}) = \frac{1}{e^{\beta[\epsilon_e(k) - \mu_e]} + 1} , \quad k \approx k_F \quad (31)$$

$$n_b(\vec{k}) = \frac{1}{e^{\beta\epsilon_b(k)} - 1} , \quad k \approx 0 \quad (32)$$

$$n_i^A(\vec{k}) = \frac{1}{e^{\beta\epsilon_i^A(k)} - 1} , \quad k \approx 0 \quad (33)$$

where $\beta=(k_B T)^{-1}$. In Eqs. (31)–(33) all quasiparticle energies are now fully renormalized.

B. Case B

For this case we have condensation only in the $\lambda=1$ spin sublevel, and the effective Hamiltonian analogous to Eq. (6) is applied only to that selected level. This Hamiltonian is again diagonalized in the Bogoliubov representation. In addition, the impurity spin gives rise to a quadratic impurity branch with a shift (the Landau-Pomeranchuk spectrum[24])

$$H_{(\mathrm{B})} \approx \sum_k \epsilon_b(k)\alpha_k'^{\dagger}\alpha_k' + \sum_k \epsilon_i^B(k) a_{2k}^{\dagger} a_{2k} + E_G^B \ . \tag{34}$$

Here the impurity quasiparticle energy is $\epsilon_i^B(k) = \epsilon_0 + k^2/2\tilde{m}_d'$, and $\epsilon_b(k)$ and $\tilde{m}_d'$ are the normal bosonic excitation energy and deuteron impurity effective mass in the limit in which the impurity concentrations tend to zero. Note that ϵ_0 is the impurity chemical potential, also in the low-impurity-concentration limit. Finally, E_G^B is the ground-state energy of the effective one-component Bose fluid of majority spins.

Note that the number of impurity excitations in case B is temperature independent whereas in case A the number of "impurity"-like excitations is temperature dependent. Further, the impurity excitations of case B lack the spin coherence of the "impurity"-like modes and moreover are not classified as broken-symmetry modes. Thus even though the impurity and "impurity"-like modes share quadratic dispersion, they are physically rather different.

For the low concentrations of impurities characteristic of case B we can clearly justify a direct single-particle one-to-one correspondence between impurity quasiparticle excitations and boson impurities. The usual combinatoric arguments for the entropy then lead to the impurity quasiparticle distribution function n_i^B which has the same form as that for the ideal Bose gas. For $T \gg T_\beta$ we thus may express the number N_i of impurities as

$$N_i = Nc = \sum_k e^{-\beta[\epsilon_i^B(k)-\mu_i]} \ . \tag{35}$$

Using the form for ϵ_i^B given above this leads to

$$\mu_i = \epsilon_0 + \frac{1}{\beta}\ln\left[\frac{Nc}{2V}\left[\frac{2\pi\hbar^2\beta}{\tilde{m}_d'}\right]^{3/2}\right] = \epsilon_0 + \mu^* \ . \tag{36}$$

Thus the impurity-mode "chemical" potential μ_i is shifted by ϵ_0 from the ideal classical chemical potential μ^* evaluated at the impurity effective mass $\tilde{m}_d'$. This in turn leads to the distribution function,

$$n_i^B(k) = Nc\left[\frac{\beta}{2\pi\tilde{m}_d'}\right]^{3/2} e^{-(k^2/2\tilde{m}_d')\beta} \ , \tag{37}$$

and is quite different from the number nonconserving form Eq. (33) appropriate to the "impurity"-like excitations of case A.

IV. EQUILIBRIUM PROPERTIES

The low-temperature specific heat, bulk modulus, spin susceptibility, and thermal-expansion coefficient for the model of LMD described above are contrasted with those both of a normal solid metallic phase of deuterium and of a normal liquid metallic phase of hydrogen.[3]

A. Specific heat

The low-temperature specific heat at constant volume includes contributions from each of the three quasiparticle branches. The phonon and quasielectron contributions are of the usual form, i.e.,

$$c_{V_{\mathrm{ph}}} \approx \frac{2\pi^2}{5} k_B \left[\frac{k_B T}{\hbar s}\right]^3 \tag{38}$$

and

$$c_{V_e} \approx 0.642 m_e^* \frac{k_B^2 T}{\hbar^2 r_s} \ , \tag{39}$$

where s is the fully renormalized phonon velocity. The new feature in LMD (case A) is the "impurity"-like contribution. In view of Eq. (33) this corresponds to the specific heat of a condensed ideal Bose gas with two identical components. Thus[10]

$$c_{V_i}^{(\mathrm{A})} \approx 0.639 \frac{k_B}{\hbar^3}(m_d^* k_B T)^{3/2} \ , \tag{40}$$

where m_d^* is fully renormalized deuteron mass. This result is independent of volume except insofar as m_d^* is itself volume dependent. In case B, however,

$$c_{V_i}^{(\mathrm{B})} \approx \tfrac{3}{2} c n k_B \ . \tag{41}$$

We estimate s from the Bohm-Staver formula: $s=(k_f^2/3m_e^* m_d^*)^{1/2}$. For $m_e^* \approx m_e, m_d^* \approx 2m_d$, we have $s \approx 3\times10^6/r_s$ cm/sec. (The Debye temperature Θ_D is given approximately by $\Theta_D \approx 1\times10^4/r_s^2$ K.) We then have (in erg/cm^3),

$$c_{V_{\mathrm{ph}}} \sim 5\times10^{-2}(T r_s)^3 \ , \tag{42}$$

$$c_{V_e} \sim 2\times10^3 \frac{T}{r_s} \ , \tag{43}$$

$$c_{V_i}^{(\mathrm{A})} \sim 2\times10^6 T^{3/2} \ , \tag{44}$$

and

$$c_{V_i}^{(\mathrm{B})} \sim 3\times10^8 \frac{c}{r_s^3} \ . \tag{45}$$

Because of the very large deuteron-electron mass ratio, the "impurity"-like contribution in case A dominates for all but extremely low temperatures. (The crossover temperature $T_{i\text{-}e}$ where $c_{V_i}^{(\mathrm{A})} = c_{V_e}$ is given by $T_{i\text{-}e} \sim 10^{-6}/r_s^2$ K.) A distinct and novel feature is the resulting $T^{3/2}$ behavior of the specific heat of unpolarized LMD over a large (and experimentally accessible) temperature range.

In case B, the impurity contribution dominates for concentrations $c \gg 10^{-5} r_s^2 T$. For the expected temperature range $T \lesssim 1$ K and for $r_s \sim 1.5$, the lowest concentration available with current magnetic fields (see Sec. II) fall well within this range. Nuclear polarized, partially condensed LMD would therefore exhibit a substantial temperature-independent specific heat.

The specific heat of *normal* solid metallic deuterium will possess only electron and phonon contributions, each with roughly the magnitude and form given in Eqs. (42) and (43). The crossover temperature where $c_{V_e} \approx c_{V_{\rm ph}}$ is approximately $T_{e\text{-ph}} \sim 2 \times 10^2 / r_s^2$ K. Hence under the expected temperature conditions, solid normal metallic deuterium will have a specific heat markedly smaller than that for unpolarized LMD: $c_{V_{\rm solid}}/c_{V_{\rm liquid}} \sim 10^{-3}/T^{1/2} r_s$ as $T \to 0$. Of course, ultimately, as T tends to zero, $c_{V_{\rm solid}} \sim c_{V_{\rm liquid}} \sim T$. For polarized LMD in case B, $c_{V_{\rm solid}}/c_{V_{\rm liquid}} \sim 10^{-5} T r_s^2 / c$, again a fairly small value for the likely attainable concentrations.

Finally, for normal liquid metallic hydrogen (LMH), a two-Fermion component proton-electron fluid, the specific heat is given by[3]

$$c_{V_{\rm LMH}} \sim c_{V_e} \frac{m_p^*}{m_e^*} \tag{46a}$$

$$\sim 7 \times 10^6 \frac{T}{r_s} \text{ erg/cm}^3 , \tag{46b}$$

m_p^* being the proton effective mass. As in LMD there is here a dominating "ionic" contribution. The temperature dependences are quite different, however, a direct consequence of the different quantum-mechanical origins (protonic Fermion quasiparticles in LMH versus deuteronic "particle"-like excitations out of a condensate in LMD). Of course, in contrast to the case of LMD, in LMH, $c_{V_{\rm solid}}$ and $c_{V_{\rm liquid}}$ do not ultimately become comparable as $T \to 0$, the ratio always remaining about m_e^*/m_p^*.

B. Bulk modulus

As in most metals we would expect that the electrons will make an important contribution to the isothermal bulk modulus $B \equiv -V(\partial P/\partial V)_T$, with P being the pressure. We estimate this electronic contribution on the basis of a free-electron-like model for which $B_e = \frac{2}{3} n \epsilon_F^*$, where ϵ_F^* is the Fermi energy evaluated with the effective mass. In reality there will be further "Fermi-liquid" renormalizations beyond those contained in this simple formula.[23] In any case B_e is temperature independent for $T \approx 0$, i.e.,

$$B_e = \frac{8.5 \times 10^{13}}{r_s^5 (m_e^*/m_e)} \text{ dyn/cm}^2 . \tag{47}$$

The "impurity"-mode contribution (case A) to the energy E_i is obtained using Eq. (33) and has the same form as for the ideal condensed Bose gas ($g=2$). The free energy $F_i = E_i - TS_i = -\frac{2}{3} E_i$ (where S_i is the "impurity"-mode entropy) and is given by[10]

$$F_i = \frac{0.17}{\hbar^3} m_d^{*3/2} (k_B T)^{5/2} V . \tag{48}$$

For the ideal case, the pressure $P = -(\partial F/\partial V)_T$ is volume-independent, implying that B_i vanishes. In the present case a new feature is that m_d^* is volume-dependent. Accordingly B_i is nonzero, and in fact,

$$B_i \approx 0.25 \frac{m_d^{*3/2}}{\hbar^3} (k_B T)^{5/2} \left[\frac{\partial(\ln m_d^*)}{\partial \ln V} + \frac{3}{2} \left[\frac{\partial(\ln m_d^*)}{\partial \ln V} \right]^2 + \frac{\partial^2(\ln m_d^*)}{\partial(\ln V)^2} \right] . \tag{49}$$

For LMD parameters and on the assumption that the logarithmic derivatives are no more than order unity, we find $B_i \sim 10^5 T^{5/2}$ dyn/cm^2, which is negligible compared to B_e. The impurity contribution to B in case B is also clearly negligible for small concentrations.

The phonon contribution to the free energy ($F_{\rm ph} = -\frac{1}{3} E_{\rm ph}$) is given as usual by

$$F_{\rm ph} = -\frac{\pi^2}{30} \frac{(k_B T)^4}{(s\hbar)^3} V . \tag{50}$$

The corresponding contribution to the bulk modulus is nonzero because of the volume dependence of s. In fact,

$$B_{\rm ph} = -\frac{\pi^2 (k_B T)^4}{10 (s\hbar)^3} \left[\frac{\partial(\ln s)}{\partial \ln V} - 3 \left[\frac{\partial(\ln s)}{\partial \ln V} \right]^2 + \frac{\partial^2 \ln s}{\partial(\ln V)^2} \right] . \tag{51}$$

For LMD, $B_{\rm ph} \sim 10^{-2} T^4$, and again it is negligible compared to B_e.

The zero-point kinetic energy of the ions, although important in many respects (e.g., in the determination of the prefered phase[2]) will not be comparable to the electronic kinetic energy. There remains the ion-ion repulsive part of the energy. This should be comparable to the corresponding contribution in the solid phase, or in LMH.

Very roughly, we therefore expect $B \sim B_e$. Our major conclusion is that there will be no major consequences for the bulk modulus that are attributable either to the boson character of the ions, or to the presence of the several spin degrees of freedom. Indeed solid and liquid metallic deuterium and liquid metallic hydrogen should all have roughly comparable bulk moduli.

C. Magnetic susceptibility

For LMD under the assumed conditions, $\chi \equiv \partial M/\partial H$, with M the total magnetization, will be dominated by the electronic contribution ($\mu_e/\mu_d \sim m_d/m_e$). Our model implies that there will be temperature-independent paramagnetic (Pauli) and diamagnetic (Landau) contributions of the same order, e.g., $\chi_{\mathrm{Pauli}} \sim 0.049 m_e^* \gamma_e^2/(1+{}^aF_0) r_s$. The computation of the Landau enhancement factor ${}^aF_0$ requires a detailed microscopic treatment; thus we cannot be certain now of how close LMD actually is to the magnetic instability associated with the condition $1+{}^aF_0=0$. For the small r_s values relevant here[23] we might expect on the basis of previous estimates for jellium that ${}^aF_0$ is not too close to -1. (Recent Monte Carlo studies for jellium suggest magnetic instabilities only at a much higher r_s value.[25]) Accordingly our preliminary expectation is that χ should be temperature independent and comparable among our models for LMD, solid metallic deuterium, and LMH, with no major consequence arising from the bosons.

D. Thermal expansion

The isobaric thermal-expansion coefficient $\alpha_p \equiv V^{-1}(\partial V/\partial T)_P = B^{-1}(\partial P/\partial T)_V$ also contains the familiar electron and phonon contributions[26]

$$\alpha_{p_e} \sim \frac{2}{3B} c_{V_e} \sim 2\times 10^{-11} r_s^4 T \tag{52}$$

and

$$\alpha_{p_{\mathrm{ph}}} \sim \frac{1}{B} \gamma c_{V_{\mathrm{ph}}} \sim 6\times 10^{-16} \gamma r_s^8 T^3 \ , \tag{53}$$

in units of K^{-1}, where γ is the dimensionless Grüneisen parameter ($\gamma \sim 1$).

Ignoring the volume dependence of m_d^* in computing the "impurity"-mode contribution to the pressure $P_i^{(\mathrm{A})}$ [using Eq. (48)] we find for case A,

$$\frac{\partial P_i^{(\mathrm{A})}}{\partial T} = 0.43 m_d^{*3/2} k_B^{5/2} T^{3/2}/\hbar^3 \ . \tag{54}$$

Because of the large deuteron-electron mass ratio, the "impurity"-like modes dominate the expansion coefficient, i.e. (in units of K^{-1}),

$$\alpha_{p_i}^{(\mathrm{A})} \sim 2\times 10^{-9} r_s^5 T^{3/2} \ . \tag{55}$$

This interesting behavior is in marked contrast to that of the solid metallic phase which as usual is expected to have a linear dependence on T comparable to that given in Eq. (52). This also contrasts with LMH where the protons give a dominating linear term $\alpha_{p_{\mathrm{proton}}} \sim (m_p^*/m_e^*)\alpha_{p_{\mathrm{electron}}}$, which may actually be negative[3] (again a consequence of Fermi-liquid effects in the proton fluid). Finally, in case B, $\partial P_i/\partial T = cnk_B$. It follows that

$$\alpha_{p_i} \sim 1.1\times 10^{-6} r_s^2 c \ , \tag{56}$$

in units of K^{-1}.

V. CONCLUSIONS

The intent of this study has been to elucidate the low-temperature properties of a dense metallic liquid state of deuterium, especially insofar as they contrast with those of the corresponding solid metallic phase, or with those of a quantum liquid metallic phase of hydrogen. A major aspect of LMD is the presence of more than one bosonic spin degree of freedom. For the Bose-condensed (although otherwise normal) phases considered, this leads to the presence of an additional quasiparticle excitation with quadratic dispersion. It is shown that this must follow even though there are no spin-dependent terms in the Hamiltonian. These excitations together with the phonon and electron quasiparticle-hole pair excitations are involved in predicting low-temperature thermal properties. When chemical equilibrium among the three boson spin levels is present these "impurity"-like excitations, whose number is temperature-dependent, are seen to resemble "bare" particles removed from the condensate. Spin-coherence effects are involved, however, in these excitations as well. Conditions under which LMD might be prepared in a way that one spin level is predominantly occupied and condensed (while the other minority levels are noncondensed) have also been discussed. Here of course the additional impurity modes directly correspond to "genuine," number-conserved, dressed impurity particles.

For both unpolarized and "partially condensed" cases we find the "impurity"-like excitations to play a significant role under the expected experimental conditions of temperature and density. For example, in unpolarized LMD the "impurity"-like modes lead to a dominating $T^{3/2}$ contribution for both specific heat and thermal expansion. Very interestingly, both quantities are in fact much larger and have different temperature dependences when compared to the corresponding quantities in the normal solid metallic phase. Within our model, however, the bulk modulus and spin susceptibility, both receiving significant contributions from the electrons, appear to exhibit no new behavior.

It is interesting to note that in LMH the heavy, "ionic" modes also contribute decisively to c_V and α_p, although, because of the Fermi statistics governing the protons they are of rather different form than we have found for LMD. We stress that were LMD to involve spin-*zero* bosons, the *fermionic* modes would immediately dominate the specific heat and thermal expansion. There would also then be no major differences between the model liquid-metallic and normal solid-metallic phases, at least insofar as static equilibrium properties are concerned.

More detailed microscopic studies may be helpful in solidifying this qualitative picture. In a future paper we will address the questions of quasiparticle interactions and transport properties.

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