Quantum tunneling in the presence of an arbitrary linear dissipation mechanism

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This paper considers the tunneling out of a metastable state at $T=0$ of a system whose classical equation of motion is, in Fourier-transformed form, $K(\omega)q(\omega)=-[\partial V(q)/\partial q](\omega)$ where $V(q)$ is a conservative potential and $K(\omega)$ represents the effects of arbitrary linear dissipative and/or reactive elements. It is shown that, provided a few commonly satisfied conditions obtain, there is a simple prescription for writing down the imaginary-time effective action functional which determines the tunneling rate in the Wentzel-Kramers-Brillouin limit; namely, it contains the usual term in $V(q)$, plus a term of the form $(1/2\pi)\int_{-\infty}^{\infty}\frac{1}{2}\;K(-i|\omega|)|\tilde{q}(\omega)|^2 d\omega$, where $\tilde{q}(\omega)$ is the Fourier transform of the imaginary-time trajectory. Previously obtained results are special cases of this prescription. Applications are made to the case of "anomalous" dissipation (rate of dissipation proportional to the squared velocity of the momentum conjugate to the tunneling variable), to the "mixed" case (relaxation by collisions subject to a conservation law), and to more realistic models of a rf superconducting quantum-interference device.

I. INTRODUCTION

In the last three or four years there has been considerable interest in the phenomenon of quantum-mechanical tunneling of a macroscopic variable, particularly in systems involving the Josephson effect. In this context an important role is played by the question of the effect of dissipation on the tunneling process. In a previous paper,[1] Caldeira and the present author considered a system described by a coordinate q, with which is associated a potential energy $V(q)$ that has a metastable minimum, and obeying in the classically allowed regime a phenomenological equation of motion of the form

$$M\ddot{q}(t)+\eta\dot{q}(t)+\frac{\partial V(q(t))}{\partial q}=F_{\rm ext}(t)\;, \tag{1.1}$$

where the friction coefficient η is a constant. I shall refer to the case described by Eq. (1.1) as the case of simple normal Ohmic dissipation; note, in particular, that the rate of energy dissipation by the system into its environment is simply $\eta\dot{q}^2$, that is, it is proportional to the squared velocity of the tunneling variable. They showed that, provided any one degree of freedom of the environment is only weakly perturbed, the rate $P_{\rm QM}$ of quantum-mechanical tunneling out of the metastable minimum at zero temperature is given by an expression of the form

$$P_{\rm QM}=A\exp(-B/\hbar)\;, \tag{1.2}$$

where the effective WKB exponent B is the value, taken along the saddle-point trajectory ("bounce") of an effective action $S_{\rm eff}$ which is a function, apart from the mass M and potential $V(q)$, only[2] of the phenomenological dissipation coefficient η. Similarly, the prefactor A can be expressed in terms of the small fluctuations around the saddle-point trajectory and so is also a function only of the parameters appearing in Eq. (1.1). The authors of Ref. 1 showed that the exponent B in (1.2) is always greater than the value which would be obtained for a system which obeyed Eq. (1.1) [with the same $V(q)$] with $\eta=0$, so that the effect of the dissipation is always to suppress the tunneling rate.[4] In a subsequent paper[3] they elaborated the arguments of Ref. 1 in considerably more detail, and gave a quantitative discussion of the formula (1.2) for the physically interesting case of a quadratic-plus-cubic potential; they also, *inter alia*, indicated how to generalize their method to the case where the friction coefficient η depends on q or on frequency, but did not give an explicit prescription for going *directly* from the classical equations of motion to the tunneling formula for the frequency-dependent case.

The classical equation of motion (1.1) is sufficiently general to include (with the appropriate transcriptions) many cases which are of practical interest in the context of macroscopic quantum tunneling; in particular the case of a Josephson junction described by the so-called "resistively shunted junction" (RSJ) model.[10] However, it is clearly not the most general possible case. First, even when the general structure of the classical equation of motion resembles (1.1), the dissipation may be more complicated than can be characterized by a single constant η; for example, in the standard model of a tunnel oxide junction, described by the tunneling Hamiltonian, the effective conductance due to the normal quasiparticles [the analog of η in (1.1)] has a complicated dependence on both amplitude (q) and frequency. Second, the *conservative* terms in the classical equation of motion may be more complicated than in (1.1). For example, in a typical Nb point-contact superconducting quantum-interference device (SQUID) the junction (point contact) may itself have a small capacitance C, perhaps $\sim 10^{-15}$ F; however, it is backed by the geometrical capacitance C_G of the SQUID ring, which is usually much larger ($\sim 10^{-12}$ F). The appropriate circuit diagram[11] is then that shown in Fig. 1,

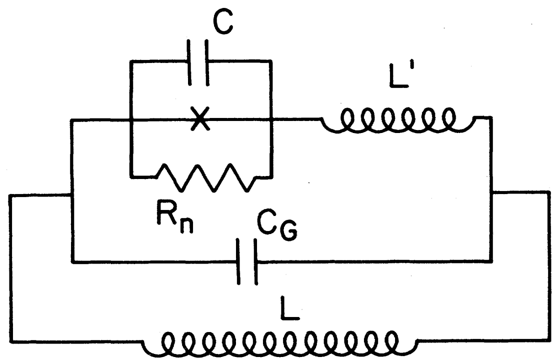

FIG. 1. More realistic circuit diagram for an rf SQUID incorporating a point contact.

and it is clear that the classical equation of motion will contain higher derivatives than in (1.1) (see Sec. III).

A third type of complication is exemplified by a formal model proposed by Widom and Clark,[9] which will be discussed in detail in Sec. III. This consists, in effect, of a parallel *LCR* circuit in which, rather than the effective inductance being strongly nonlinear, as is effectively the case in a SQUID, it is the capacitance which incorporates the nonlinear element, so that the quantity which tunnels is the charge on the capacitor plates. Although this model is very artificial (and in particular is not a realistic description of quantum tunneling in a ferroelectric—see Sec. IV), it does have the advantage of focusing attention on the fact that the rate of dissipation of energy by the system into its environment need not inevitably be proportional to the squared velocity of the tunneling variable q itself, but may, as here, be proportional to the rate of change of the momentum p conjugate to q.[12] As in I, Appendix C, I will refer to cases in which the rate of dissipation $\dot{W}$ is proportional to $\dot{q}^2$ as cases of "normal" dissipation, and those in which it is proportional to $\dot{p}^2$ as cases of "anomalous" dissipation (for a more precise definition see Sec. III). It is also possible, as we shall see in Sec. III, to have "mixed" cases in which the behavior effectively switches from normal to anomalous as the parameters are varied. A particularly interesting feature of the "pure anomalous" case is that (as demonstrated by Widom and Clark[9] for the special case considered by them) the sign of the effect of dissipation on tunneling is opposite to that in the normal case. Although it seems unlikely (for reasons to be discussed in Sec. IV) that the pure anomalous case will ever be of practical interest in the context of realistic macroscopic tunneling phenomena, it is of some theoretical interest to extend our quantitative methods to cover this case, and the case of mixed dissipation certainly may be of some real interest.

To date, very little quantitative work has been done on the effects of dissipation on tunneling in cases more general than that of the simple Ohmic dissipation described in Eq. (1.1). Indeed, the only such calculations known to the present author, apart from those reported in I, are those of Ambegaokar *et al.*,[13] which deal with the very specific case of an ideal tunnel oxide junction described by the traditional tunneling Hamiltonian, and that of Zwerger,[14] which takes into account the full frequency dependence of the complex dielectric constant for a linearly dissipative Josephson junction. Clearly a more general approach is desirable, if possible.

In this paper, I shall consider the effect of dissipation on tunneling (at $T=0$) for a class of problems which, although not the most general possible, is very much more general than that described by Eq. (1.1) and includes most of the situations mentioned above. To be specific, I consider a problem in which the nonlinear element which gives rise to the possibility of tunneling is shunted by an arbitrary generalized linear impedance mechanism, so that the Fourier-transformed equation of motion of the tunneling variable q reads

$$K(\omega)q(\omega)=-[\partial V(q)/\partial q](\omega)\ , \tag{1.3}$$

where $K(\omega)$ is an arbitrary function subject only to the usual constraints imposed by causality, etc. I then demonstrate, subject to a few very general conditions which are very likely to be satisfied in practice, that the tunneling rate out of the metastable minimum can be obtained by a strikingly simple prescription, which is given explicitly in Eqs. (2.22)—(2.25): In crude terms, every term in $(i\omega)$ in the generalized impedance $K(\omega)$ corresponds to a term in $|\omega|$ in the effective action whose saddle-point value is the WKB exponent (and the fluctuations of which determine the prefactor). This prescription enables one to read off the appropriate form of effective action directly from the classical equation of motion: The results derived earlier[1,3] for the case of simple Ohmic dissipation are just a special case of this, as are the results of Zwerger[14] for the specific case of a Josephson junction with weak linear dissipation. Thus the problem is reduced to the purely mathematical question of finding the saddle-point value of the effective action.

Once stated, the prescription is so spectacularly simple that the reader's initial reaction will almost certainly be that it must be a quite trivial consequence of considerations concerning the analytic properties of physical quantities when continued into the complex plane (cf. Ref. 14). If this is so, I have not found a method to prove it, and indeed I believe that the fact that the result certainly *fails*, even for the case of simple Ohmic dissipation, when the "strict linearity" condition specified in Sec. II is not satisfied (see I, Sec. IV) should induce extreme scepticism about the relevance of general analytic continuation arguments to this problem. The proof which I shall in fact give in this paper contains two main steps which are parallel to those necessary in the simple Ohmic case:[3] (1) I show that under the conditions stated the Lagrangian for the interaction of the system with its environment can always be cast, apart from possible mass and potential renormalization effects, in the "canonical" form of a coordinate-coordinate coupling to a bath of harmonic oscillators [cf. I, Eq. (3.12)], though with an "environment" spectral density which is not necessarily as simply related to the phenomenological dissipation coefficient as in I. (2) I then show that this form of interaction leads, in the tunneling problem, to an effective action of the specified form [Eq. (2.23)].[15] From there on the calculation proceeds exactly as in the simple Ohmic case.

In the next section I first give a precise definition of the problem, then demonstrate successively steps (1) and (2), and thereby establish the fundamental result of this paper [Eqs. (2.22)–(2.25)]. In Sec. III, I discuss some applications, and, in the Conclusion (Sec. IV), discuss the significance of the results obtained. I use throughout this paper the notation of I and refer to the latter paper for details of those parts of the argument which are similar to those used for simple Ohmic dissipation.

II. DERIVATION OF THE GENERAL FORMULA

A. Statement of the problem

I consider a system described by some variable q with which is associated a conservative potential energy $V(q)$ which has a metastable minimum (but see below). It will usually be convenient to choose the origin of both q and V to lie at this minimum; then $V(q)$ contains no terms linear in q. The phenomenological equation of motion of the variable $q(t)$ in the classically accessible regime is assumed to be of the form

$$\hat{K}q(t) = -\frac{\partial V(q(t))}{\partial q} , \tag{2.1}$$

where $\hat{K}$ is an arbitrary linear operator (in general of integro-differential type) subject only to the conditions imposed by causality. For example, in the case of simple Ohmic dissipation $\hat{K}$ is of the form $(Md^2/dt^2 + \eta\, d/dt)$. Note that in this case we can, if we wish, arbitrarily choose to include the term in $\partial V(q(t))/\partial q$ which arises from the quadratic part of $V(q)$ (or any part of it) in the operator $\hat{K}$; in more complicated cases, such as the one considered in Sec. III C, the assignment of the quadratic "potential" terms between the left- and right-hand sides of Eq. (2.1) may be uniquely determined.[16]

It is often convenient to visualize Eq. (2.1) in electrical engineering terms. We may represent it by a diagram of the form shown in Fig. 2, where the cross corresponds to the nonlinear element associated with $V(q)$ and the "black box" $\hat{K}$ represents an arbitrary linear impedance mechanism. For example, in a SQUID ring described by the simple RSJ model the natural representation would be to associate with the cross the Josephson junction and, in parallel with it, the bulk ring inductance, and with the black box the junction capacitance and (parallel) shunting resistance; however, as remarked above, it would be equally possible to transfer the bulk inductance term to the black box,[16] and in the more complicated case considered in Sec. III it is necessary to do so.

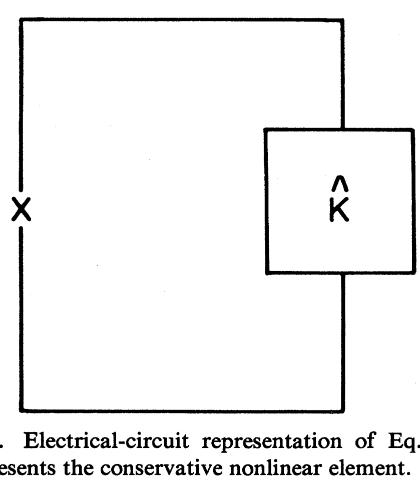

FIG. 2. Electrical-circuit representation of Eq. (2.1). The cross represents the conservative nonlinear element.

The form of Eq. (2.1) becomes simpler if we take its Fourier transform. In this paper we will define Fourier transforms by the prescription

$$f(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt ,$$
$$f(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega t} d\omega . \tag{2.2}$$

Then the Fourier transform of (2.1) has the form

$$K(\omega) q(\omega) = -[\partial V(q)/\partial q](\omega) , \tag{2.3}$$

where $K(\omega)$ is now in general a complex function of ω. In general, $K(\omega)$ will be related (though not necessarily identical) to some quantity which has the general nature of an impedance function. For example, in the simple RSJ case discussed above, in which the variable q is actually a magnetic flux, it is clear that we have simply

$$K(\omega) = i\omega Y(\omega) , \tag{2.4}$$

where $Y(\omega)$ is the admittance of the black box as conventionally defined. It is clear that the requirements of causality imply that both $Y(\omega)$ and its inverse, the impedance $Z(\omega)$, must be analytic functions of ω in the lower half of the complex plane, and that similar conditions must hold even when the variable q is not of an electrical nature. Consequently, we reach the important conclusion that, quite generally, *both* $K(\omega)$ *and* $K^{-1}(\omega)$ *are analytic functions of* ω *for* $Im\,\omega < 0$. As we shall see below, this condition is necessary to ensure that the formula for quantum tunneling to be derived is physically sensible.

To carry out the proof below we must impose three conditions on the interaction between the system and its environment. (1) Condition of small perturbation of the environment: Any *one* degree of freedom of the environment is sufficiently weakly perturbed that it is possible to neglect nonlinear effects.[17] The reasons for, and the plausibility (in macroscopic systems) of this condition is discussed in I, Appendix C. (2) Condition of "strict linearity:" This is a generalization of the postulate discussed in I for simple Ohmic dissipation, and can be stated in the form that the microscopic interaction Hamiltonian coupling the system to its environment contains only terms either (a) linear in the system variable q and its conjugate momentum p, or (b) quadratic in p and q, but not containing the environment variables. As in the case of simple Ohmic dissipation, we note that the fact that the (experimentally observable) phenomenological equation of motion for reasonable velocities is of the form (2.3) does not automatically guarantee this feature. However, in many cases of practical interest, and in particular in Josephson systems under most physically relevant conditions, we have sufficient *a priori* knowledge of the microscopic nature of the interaction to be reasonably sure that the condition holds. This question is discussed in some detail in I (Secs. II and III and Appendix C) and the discussion transposes straightforwardly to the more general case. (3) Condition of time-reversal invariance: This is probably not essential but is included to simplify the derivation given below. Note that it is not required that the conservative terms in the potential be explicitly time-reversal invariant.

B. Derivation of the "canonical" Lagrangian

Following the argument of I, Appendix C, and using condition (1) above, we can immediately infer that the most general form of Hamiltonian necessary to describe the coupled system and environment is

$$H(q,p;\{x_j,p_j\}) = \frac{p^2}{2M_0} + V_0(q) + \sum_j \left[\frac{p_j^2}{2m_j} + \tfrac{1}{2} m_j \omega_j^2 x_j^2 \right] - \sum_j [F_j(p,q)x_j + G_j(p,q)p_j] + \Phi(p,q) \,. \tag{2.5}$$

Here M_0 and $V_0(q)$ are respectively the "bare" mass and "bare" potential of the isolated system, the environment is represented as a bath of harmonic oscillators whose "coordinates" and "momenta" are chosen in an arbitrary way, F_j and G_j are real functions of p and q [see I, Eqs. (C8)], and $\Phi(p,q)$ is a real function which may depend on the parameters m_j, ω_j, F_j, and G_j, but is not a function of the environment dynamical variables p_j, x_j. We note that (2.1) contains no terms depending on the system variables p,q which are of order higher than linear in the environment variables; as discussed in I, Appendix C, the occurrence of such terms indicates that the perturbation of the environment is not "weak" and one must then use the adiabatic approximation. The net result of this maneuver, as explained in the above reference, is that we recover an expression of the type (2.5), but possibly with some renormalization of the effective potential (Ref. 18) $V_0(q)$. We will not distinguish this case explicitly in what follows, since the mass and potential are subject to further renormalization anyway.

We next introduce the condition (2) of Sec. II A: This immediately constrains F_j and G_j to be linear in p and q and $\Phi(p,q)$ to be bilinear. Moreover, applying condition (3) and choosing the environment "coordinate" x_j for the moment to have the *opposite* behavior to q under the operation of time reversal, we find that the most general form of the last three terms in (2.5), which we label H_{int}, is[19]

$$H_{\text{int}} = -\frac{p}{M_0} \sum_j F_j x_j - q \sum_j m_j^{-1} G_j p_j + \tfrac{1}{2} A p^2 + \tfrac{1}{2} B q^2 \,, \tag{2.6}$$

where we introduced factors of M_0 and m_j into the definitions of F_j and G_j, respectively, for subsequent convenience. We can immediately incorporate the last two terms into a renormalized mass and potential:

$$\widetilde{M}^{-1} \equiv M_0^{-1} + A\,, \quad \widetilde{V}(q) \equiv V_0(q) + \tfrac{1}{2} B q^2 \,. \tag{2.7}$$

It should be carefully noted that this may not necessarily correspond to a small correction: for example, in the case of pure anomalous dissipation, to be discussed in Sec. III A, a detailed examination (which I shall not give here) shows that the renormalized mass is actually zero (or, more precisely, of the order of the inverse of a large cutoff frequency).

Since we wish eventually to use a Lagrangian rather than a Hamiltonian technique, it is convenient at this stage not to carry out canonical transformations on the Hamiltonian (2.6) directly, but to write the corresponding Lagrangian, which is

$$\begin{aligned} \mathscr{L}(q,\dot{q};\{x_j,\dot{x}_j\}) = {} & \tfrac{1}{2}\widetilde{M}\dot{q}^2 - \widetilde{V}(q) + \sum_j \tfrac{1}{2} m_j \dot{x}_j^2 \\ & - \sum_j \tfrac{1}{2} m_j \omega_j^2 x_j^2 + \dot{q} \sum_j F_j x_j \\ & + q \sum_j G_j \dot{x}_j + q^2 \sum_j (G_j^2/2m_j) \\ & + \frac{1}{2M}\Big(\sum_j F_j x_j\Big)^2 \,. \end{aligned} \tag{2.8}$$

We immediately see that the term linear in $\dot{q}$ can be eliminated by adding to the Lagrangian the total time derivative $-(d/dt)\{q\sum_j F_j x_j\}$ (which, of course, does not affect the dynamics in any way), and incorporating the result in the sixth term in (2.8). Moreover, we can eliminate the last term by taking it together with the "unperturbed" environment Lagrangian (the third and fourth terms) and carrying out a linear transformation so as to diagonalize the resulting Lagrangian $\mathscr{L}'_{\text{env}}$:

$$\begin{aligned} \mathscr{L}'_{\text{env}} & \equiv \tfrac{1}{2} \sum_j m_j (\dot{x}_j^2 - \widetilde{\omega}_j^2 x_j^2) + \frac{1}{2\widetilde{M}} \Big[\sum_j F_j x_j \Big]^2 \\ & = \tfrac{1}{2} \sum_j \widetilde{m}_j (\dot{\xi}_j^2 - \widetilde{\omega}_j^2 \xi_j^2) \,, \end{aligned} \tag{2.9}$$

where the ξ_j are linear combinations of the x_j. Including the seventh term of (2.8) in (a further renormalized) $\widetilde{V}(q)$, we see that we now have a Lagrangian in which the only term coupling the system and (new) "environment" is of the form

$$\mathscr{L}_{\text{int}} = q \sum_j \widetilde{G}_j \dot{\xi}_j \,, \tag{2.10}$$

where the $\widetilde{G}_j$ are linear combinations of the original G_j. Finally, by carrying out the transformation explained in detail in I, Appendix A (which, of course, in the Hamiltonian formalism is just the canonical transformation which interchanges the environment "coordinates" and "momenta"), we reduce the whole Lagrangian to the "canonical" form

$$\begin{aligned} \mathscr{L}(q,\dot{q};\{\widetilde{x}_j,\dot{\widetilde{x}}_j\}) = {} & \tfrac{1}{2}\widetilde{M}\dot{q}^2 - \widetilde{V}(q) + \tfrac{1}{2}\sum_j \widetilde{m}_j (\dot{\widetilde{x}}_j^2 - \widetilde{\omega}_j^2 \widetilde{x}_j^2) \\ & - q \sum_j C_j \widetilde{x}_j - q^2 \sum_j C_j^2 / 2\widetilde{m}_j \widetilde{\omega}_j^2 \,, \end{aligned} \tag{2.11}$$

where the $\tilde{x}_j$ are new environment "coordinates" which have the same time-reversal parity as q, and where $C_j \equiv \tilde{\omega}_j \tilde{G}_j$. Note that since both steps of the transformation $x_j \to \xi_j \to \tilde{x}_j$ correspond to canonical transformations, the description (2.11) in terms of the $\tilde{x}_j$ is of equal validity to the original description (2.8), and we can use (2.11) in all the usual ways[20] (in functional integrals, etc.). However, one point should be carefully noted: Whereas in the original formulation (2.8) the parameters m_j and ω_j were characteristic of the "isolated" environment and the dissipative interaction between system and environment was contained only in the F_j and G_j, in the new description (2.11) the third term already incorporates some effects of the interaction (the $\tilde{m}_j$ and $\tilde{\omega}_j$ are in general themselves functions of the strength of the dissipation).

It remains to determine what constraints must be satisfied by the parameters in (2.11) in order that the classical equation of motion of the system coordinate $q(t)$ should have the specified form (2.1). As in the case of simple Ohmic dissipation, it is convenient to define the spectral density

$$J(\omega) \equiv \frac{\pi}{2} \sum_j (C_j^2/\tilde{m}_j \tilde{\omega}_j) \delta(\omega - \tilde{\omega}_j) , \qquad (2.12)$$

where ω is real, and also the quantity, defined for ω in the lower half of the complex plane,

$$\bar{K}(\omega) \equiv -\omega^2 \left[\frac{2}{\pi} \int_0^\infty \frac{J(\omega')d\omega'}{\omega'(\omega'^2 - \omega^2)} + \tilde{M} \right] , \qquad (2.13)$$

so that $\mathrm{Im}\bar{K}(\omega) = J(\omega)$ for $|\mathrm{Im}\omega| \to 0$. [Note that $\bar{K}(\omega)$ differs from the quantity defined in I, Eq. (C.35) by a sign, the domain of definition, and the addition of the mass term.] Now, just as in the simple Ohmic case, we calculate the classical equations of motion of q and the x_j from the Lagrangian (2.11), and eliminate the x_j. We find that the Fourier transform $q(\omega)$ of $q(t)$ obeys the simple equation (for $\omega = \mathrm{Re}\omega - i\epsilon$, $\epsilon \to +0$)

$$\bar{K}(\omega) q(\omega) = -[\partial \tilde{V}(q)/\partial q](\omega) \qquad (2.14)$$

so that $\bar{K}(\omega)$ is identical, in the limit $|\mathrm{Im}\omega| \to 0$, to the $K(\omega)$ occurring in the classical equation of motion (2.1), and $\tilde{V}(q)$ is identical to $V(q)$. This completes step (1) of the proof.

C. Tunneling rate

Since the Lagrangian (2.11) is of precisely the form used in I [see I, Eqs. (3.12) and (3.13)], we can now follow the arguments of Sec. IV of that reference word for word to obtain the tunneling rate. We find it is derived in the usual way (see below) from an effective "imaginary-time" action of the form

$$S_{\mathrm{eff}}(q(\tau)) = \int_{-\infty}^{\infty} \{ \tfrac{1}{2}\tilde{M}\dot{q}^2 + V(q) \} d\tau + \tfrac{1}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \alpha(\tau - \tau') [q(\tau) - q(\tau')]^2 , \qquad (2.15)$$

where $\alpha(\tau - \tau')$ is given by the expression

$$\alpha(\tau - \tau') \equiv \frac{1}{2\pi} \int_0^\infty J(\omega) e^{-\omega|\tau - \tau'|} d\omega . \qquad (2.16)$$

We now define the Fourier transform $\tilde{q}(\omega)$ of the imaginary-time trajectory $q(\tau)$ [so denoted to distinguish it from $q(\omega)$, the Fourier transform of the real-time classical trajectory $q(t)$] by the prescription[21]

$$\tilde{q}(\omega) \equiv \int_{-\infty}^{\infty} q(\tau) e^{-i\omega\tau} d\tau . \qquad (2.17)$$

Denoting the last term in Eq. (2.15) by ΔS_{eff}, we see that as a function of $\tilde{q}(\omega)$, it reads

$$\Delta S_{\mathrm{eff}} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} [\alpha(\omega) - \alpha(0)] |\tilde{q}(\omega)|^2 d\omega , \qquad (2.18)$$

where $\alpha(\omega)$ is the Fourier transform of $\alpha(\tau - \tau')$. Explicitly we have

$$\alpha(\omega) = (2\pi)^{-2} \int_{-\infty}^{\infty} d\tau \int_0^\infty d\omega' e^{-\omega'|\tau|} e^{-i\omega\tau} J(\omega') = (2\pi^2)^{-1} \int_0^\infty \frac{\omega' J(\omega')}{\omega'^2 + \omega^2} d\omega' . \qquad (2.19)$$

Inserting (2.19) into (2.18) and adding the mass term, which is clearly just $(2\pi)^{-1}$ times the integral of $\frac{1}{2}\tilde{M}\omega^2 |q(\omega)|^2$, we obtain

$$S_{\mathrm{eff}}(\tilde{q}(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\tilde{M}\omega^2 + \pi^{-1}\omega^2 \int_0^\infty d\omega' \frac{J(\omega')}{\omega'(\omega'^2 + \omega^2)} \right] |\tilde{q}(\omega)|^2 + S_V , \qquad (2.20)$$

where S_V is the potential contribution to the action, i.e.,

$$S_V(\tilde{q}(\omega)) \equiv \int_{-\infty}^{\infty} d\tau\, V(q(\tau)) \qquad (2.21)$$

expressed as a functional of $\tilde{q}(\omega)$. Comparing the term in the large parentheses with the definition (2.13) of $\bar{K}(\omega)$, which as we saw was identical on the real axis with $K(\omega)$, we see that apart from a factor of $\frac{1}{2}$ it is just $K(\omega)$ analytically continued to the negative[22] imaginary axis.[14] Thus we reach the fundamental conclusion of this paper: If the classical equation of motion is of the form (2.1), that is

$$K(\omega) q(\omega) = -[\partial V(q)/\partial q](\omega) , \qquad (2.22)$$

then the formula for the tunneling rate can be obtained in terms of an effective action S_{eff} which is given in terms of $\tilde{q}(\omega)$ by the expression,

$$S_{\mathrm{eff}}(\tilde{q}(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tfrac{1}{2} K(-i|\omega|) |\tilde{q}(\omega)|^2 d\omega + S_V(\tilde{q}(\omega)) , \qquad (2.23)$$

with $S_V(\tilde{q}(\omega))$ given by (2.21). To be precise, the expression for the tunneling rate Γ in the WKB approximation is

$$\Gamma = A\exp(-B/\hbar) , \tag{2.24}$$

where the exponent B is the saddle-point value of the action (2.23), and the prefactor A is given by the expression [cf. I, Eqs. (4.32)—(4.34), and erratum[3]],

$$A = (B^*/2\pi\hbar)^{1/2} \left| \frac{\det[\hat{\hat{K}} + V''(0)]}{\det'[\hat{\hat{K}} + V''\{q_{\rm cl}(\tau)\}]} \right|^{1/2} , \tag{2.25}$$

where $\hat{\hat{K}}$ is the operator whose Fourier representation $\hat{\hat{K}}_{\omega,\omega'}$ is given by $K(-i\,|\,\omega\,|\,)\delta(\omega-\omega')$, $q_{\rm cl}(\tau)$ is the saddle-point (bounce) trajectory, and B^* is twice the time integral of the kinetic energy in the bounce. For the meaning of det′, see I.

It is clear that the results of I, Sec. IV, are a special case of this general prescription: In this case the Fourier-transformed action [Eq. (4.27) of I] has the form (2.23) with the coefficient of $|\tilde{q}(\omega)|^2$ equal to $(1/2\pi)\frac{1}{2}(M\omega^2+\eta\,|\,\omega\,|\,)$ as it should [cf. Eq. (D.8d) of I and Ref. 16]. Note also that the fact, noted in Sec. II A, that both $K(\omega)$ and $K^{-1}(\omega)$ are analytic in the lower half-plane means that the effective action (2.23) can be neither zero nor (for sensible trajectories) infinite.

III. SOME APPLICATIONS

A. Case of "anomalous" dissipation

To orient our discussion of "anomalous" dissipation, let us first consider a simple LC circuit, where the question of tunneling does not arise. If we choose the flux ϕ through the inductance as our "coordinate" variable, then the Lagrangian is

$$\mathscr{L} = \tfrac{1}{2}C\dot{\phi}^2 - \phi^2/2L , \tag{3.1}$$

where L is the inductance and C the capacitance, and the momentum conjugate to ϕ is

$$p_\phi \equiv \frac{\partial \mathscr{L}(\phi,\dot{\phi})}{\partial \dot{\phi}} = C\dot{\phi} = -CV = -Q , \tag{3.2}$$

that is, the charge on the capacitance. Conversely, were we to choose Q as our "coordinate" the Lagrangian would be

$$\mathscr{L}(Q,\dot{Q}) = \tfrac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C} , \tag{3.3}$$

and the momentum conjugate to Q would be $L\dot{Q} = +\phi$. Evidently there is exact symmetry between these two possible descriptions. If the circuit is described quantum mechanically, then either description leads to the conclusion that the mean-square fluctuations of ϕ and Q in the ground state are given by

$$\langle \phi^2 \rangle = \hbar/2C\omega_0, \;\; \langle Q^2 \rangle = \tfrac{1}{2}\hbar\omega_0 C , \tag{3.4}$$

where $\omega_0 \equiv (LC)^{-1/2}$ is the resonant frequency of the circuit.

Now imagine that we connect a linear Ohmic resistance R in parallel with the inductance and capacitance. This destroys the symmetry between ϕ and Q, since the rate of energy dissipation is now a constant (R^{-1}) times $\dot{\phi}^2$, whereas if we express it in terms of $\dot{Q}^2$ the coefficient has a nontrivial frequency dependence. Moreover, if we apply to the system an external current $I_{\rm ext}$ (across L,C,R in parallel) then the equation of motion of ϕ has the form

$$C\ddot{\phi} + R^{-1}\dot{\phi} + L^{-1}\phi = I_{\rm ext} , \tag{3.5}$$

and the work done by the external "force" (current) is given by

$$\dot{W} = I_{\rm ext}\dot{\phi} . \tag{3.6}$$

If, on the other hand, we take Q as our coordinate variable, it is possible of course to write the equation of motion in a form similar to (3.5), namely

$$C\ddot{Q} + R^{-1}\dot{Q} + L^{-1}Q = F_{\rm ext}, \;\; F_{\rm ext} \equiv C\frac{dI_{\rm ext}}{dt} , \tag{3.7}$$

but it is then not possible to write the work done by the external current in the form $F_{\rm ext}\dot{Q}$. We will say that the dissipation mechanism is *normal* if ϕ is taken as the coordinate variable and *anomalous* if Q is taken as coordinate. In other words, in this simple case (involving a frequency-independent resistance) the dissipation is normal if the rate of dissipation $\dot{W}$ is proportional to the squared rate of change of the coordinate $\dot{q}^2$, and anomalous if it is proportional to the squared rate of change of (what in the absence of dissipation was) the momentum $\dot{p}^2$.

Provided the dissipative mechanism is "strictly linear" in the sense of I, Appendix C, it is very straightforward to calculate the ground-state density matrix of the damped circuit (see I, Appendix B) and to verify that the mean-square fluctuation of the flux, $\langle \phi^2 \rangle$ is *decreased* relative to its value (3.4) for the undamped circuit, whereas the charge fluctuation $\langle Q^2 \rangle$ is *increased* (of course in such a way that the uncertainty principle remains satisfied). For a series LCR circuit the situation is reversed: The dissipation is now proportional to $\dot{Q}^2R$, and so is normal if Q is chosen as coordinate and anomalous if ϕ is chosen. In this case the dissipation has the effect of decreasing $\langle Q^2 \rangle$ but increasing $\langle \phi^2 \rangle$. We see that in both cases, once we have made a particular choice of coordinate, we can say that normal dissipation decreases the mean-square value of the coordinate and increases that of the momentum, whereas anomalous dissipation has the reverse effect. Although we have proved this only for the simple case of frequency-independent Ohmic dissipation, it is very straightforward to follow through the argument of I, Appendix B, and prove the statement for a harmonic oscillator with arbitrary frequency-dependent dissipation, provided "normal" dissipation is defined in the more general way below.[23]

We now turn to the case of general motion in an anharmonic potential (which need not necessarily permit tunneling). We will always define the "coordinate" q in the obvious way, i.e., so that it is the potential energy $V(q)$ rather than the kinetic energy which is anharmonic. In particular, in the tunneling case it is the coordinate which has the metastable minimum. With this definition we have an absolute distinction between "normal" and "anomalous" dissipation: Crudely speaking, normal dissipation is proportional to the squared rate of change of the tunneling coordinate, whereas anomalous dissipation is proportional to the squared rate of change of the (original) conjugate momentum. Actually, this definition is technically ambiguous once we divert from the simple case of a constant dissipation coefficient, since it is always possible to write a frequency-dependent relation between position and (original) momentum. A more inclusive definition of "normal" dissipation in the general case is the following:[24] given a particular choice of coordinate q with associated potential (in general nonlinear) $V(q)$, then the dissipation is normal if it is possible to write *both*

$$M\ddot{q}(t)+\hat{\eta}\dot{q}(t)+\frac{\partial V(q(t))}{\partial q}=F_{\text{ext}}(t)\ , \tag{3.8}$$

where $\hat{\eta}$ has an arbitrary frequency dependence (i.e., in the time domain is an arbitrary integro-differential operator subject to the requirements of causality), *and*

$$\dot{W}(t)=F_{\text{ext}}(t)\dot{q}(t) \tag{3.9}$$

[cf. Eqs. (3.5)–(3.6)]. It would probably be possible to give an equally general definition of anomalous dissipation, but it is not worthwhile to do so in the present context, since "normal" and "anomalous" dissipation are themselves only special cases of a much more general situation. Rather than do so, we shall concentrate on the case of "simple Ohmic" anomalous dissipation so as to bring out the main qualitative features.

To study simple Ohmic anomalous dissipation and its effect on tunneling it is convenient to focus on a specific example. Let us therefore take the above parallel *LCR* circuit and make the *capacitance* strongly nonlinear: In fact, we shall assume that the potential energy $V(Q)$ associated with a charge Q on the capacitor plates is a function so nonlinear that it has a metastable minimum. The circuit so obtained is a slight variant of the one studied in Ref. (9); we note, however, that, contrary to the impression given there, it cannot realistically represent tunneling in a real ferroelectric material (see Sec. IV), but must be regarded purely as a formal model. It is clear that the classical equation of free motion of the circuit, written in terms of Q, has the form

$$L\frac{d^2}{dt^2}Q+\frac{L}{R}\frac{d}{dt}\left[\frac{\partial V}{\partial Q}\right]+\frac{\partial V}{\partial Q}=0\ . \tag{3.10}$$

Note that the kinetic energy is *not* $\frac{1}{2}L\dot{Q}^2$. Quite generally, the structure of the equation of motion of the system variable q in the case of pure Ohmic anomalous dissipation will have the form

$$M\frac{d^2}{dt^2}q+\tau\frac{d}{dt}\left[\frac{\partial V}{\partial q}\right]+\frac{\partial V}{\partial q}=0\ . \tag{3.11}$$

Although the "mass" M is the original mass of the system before the switching on of dissipation, the kinetic energy will not in general now be $\frac{1}{2}M\dot{q}^2$. We note that Eq. (3.11) can be formally cast in a form similar to (1.1), with a nonlinear "friction coefficient" $\eta(q)$, namely

$$M\ddot{q}+\eta(q)\dot{q}+\frac{\partial V(q)}{\partial q}=0\ , \tag{3.12}$$

but only at the cost of making the (pseudo) friction coefficient $\eta(q)\equiv\tau V''(q)$ *negative* over part of the tunneling region.

Now let us consider the effect of the dissipation on tunneling. It is immediately obvious from (3.11) that the correct form of $K(\omega)$ [defined in Eq. (2.3)] for this case is

$$K(\omega)=\frac{-M\omega^2}{1+i\omega\tau}\ . \tag{3.13}$$

Consequently, according to the prescription (2.23), the effective action which enters the expression for the tunneling rate is

$$S_{\text{eff}}(\tilde{q}(\omega))=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\frac{1}{2}M\omega^2}{1+|\omega|\tau}|\tilde{q}(\omega)|^2d\omega+S_V(\tilde{q}(\omega))\ . \tag{3.14}$$

Since the value of the effective action for any function $\tilde{q}(\omega)$ is less than it would be in the limit $\tau\to 0$ (no dissipation) for the same potential $V(q)$, it is immediately clear that the introduction of dissipation [without change in the phenomenological potential $V(q)$] in this case *increases* the tunneling rate, in agreement with the conclusion of Ref. 9. This is precisely what we should expect in view of the results derived above for the simple harmonic oscillator. It should of course be stressed that this conclusion is in no way in conflict with the results of Ref. 1 or of I, since in these references the formulation of the problem is such that it explicitly excludes the case of anomalous dissipation.

It is easy to make quantitative estimates of the effect on tunneling along the lines of I, Sec. V. Consider in particular the case of a cubic potential $V(q)=\frac{1}{2}M\omega_0^2q^2-\beta q^3$ as in I, and suppose $\omega_0\tau>1$. Using the obvious inequality,

$$\frac{M\omega^2}{1+|\omega|\tau}\geq\frac{M|\omega|}{\tau}-\frac{M}{\tau^2}\ , \tag{3.15}$$

we see that the effective action (3.14) is bounded below by the expression

$$S'(\tilde{q}(\omega))=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta|\omega|\ |\tilde{q}(\omega)|^2d\omega+S_{\bar{V}}(\tilde{q}(\omega))\ , \tag{3.16}$$

where

$$\bar{V}(q)\equiv\tfrac{1}{2}M(\omega_0^2-1/\tau^2)q^2-\beta q^3$$

and $\eta\equiv M/\tau$. But this is just the action for a system tunneling in the potential $\bar{V}(q)$ with *normal* simple Ohmic dissipation described by the friction coefficient η, without the kinetic energy term in $\frac{1}{2}M\omega^2$. In other words it is just the strong-damping limit of the problem discussed in I, Sec. V. Since we have an explicit solution for the saddle-point value of the action in this limit [I, Eq. (5.21)], we can immediately write a lower limit (which should be the exact value in the limit $\tau \to \infty$) for the WKB exponent B [Eq. (2.24)] for the case under consideration:

$$B \geq \frac{2\pi}{9} M (\widetilde{\Delta}q)^2/\tau, \quad \widetilde{\Delta}q \equiv \tfrac{1}{2} M(\omega_0^2 - 1/\tau^2)/\beta , \qquad (3.17)$$

where $(\widetilde{\Delta}q)^2$ is the "distance under the barrier" described by $\overline{V}(q)$. Other inequalities can be obtained in a similar way if desired.

B. Case of "mixed" dissipation

As my next illustration I discuss a kind of situation which may, in the tunneling context, be slightly nearer to reality. It is a situation which typically arises where there are relaxation processes in the system which are subject to the conservation of the momentum conjugate to the tunneling variable. As an example, let us consider the following (admittedly not very realistic) system: A cylinder of moment of inertia I_0 is suspended by a torsion thread with highly nonlinear properties, so that as a function of the rotation angle θ its potential energy $V(\theta)$ has a metastable minimum. It is then filled with a liquid such that the total moment of inertia of cylinder plus liquid is I. We assume (perhaps unrealistically) that the liquid is always in internal rotational equilibrium but that it relaxes to rotational equilibrium with the container, by collisions which conserve the total angular momentum, with some characteristic relaxation time τ. We then ask for the formula for the rate of tunneling out of the metastable minimum.

Intuitively, we would expect that if the characteristic inverse frequency $\overline{\omega}^{-1}$ associated with the tunneling process (e.g., the "bounce time" which is of the order of the inverse attempt frequency, cf. I, Sec. V) is small compared to the relaxation time τ, then the rate should be given by the standard WKB formula with "mass" (i.e., moment of inertia) equal to that of the cylinder alone, that is I_0; while if $\overline{\omega}^{-1}$ is long compared to τ, then we again obtain a WKB formula but with "mass" I, that is the total moment of inertia of cylinder plus liquid. Since I is larger than I_0, the tunneling rate would be expected to be slower in this limit. It also seems a reasonable guess that as we decrease τ, the rate would change monotonically from one extreme value to the other. Can we confirm these intuitive guesses?

The task of obtaining the classical dynamics of the system described is made considerably easier by the observation that apart from the existence of a metastable minimum of the potential and some complications associated with the so-called "Fermi-liquid" effects which have no analog here, the problem is isomorphic to that of the longitudinal magnetic resonance of the superfluid phases of liquid ^{3}He, which is discussed in some detail in Ref. 25. The correspondence is as follows: The (angular momentum of the) cylinder corresponds to the (spin of the) Cooper pairs in ^{3}He and the liquid filling it to the normal component, the analog of the potential energy $V(\theta)$ is the dipole energy $H_D(\theta)$ [where in the ^{3}He case the angle θ is the angle of rotation of the Cooper-pair spin coordinates $\vec{d}(\vec{n})$], and the (angular momentum-conserving) collisions which establish rotational equilibrium between liquid and cylinder are analogous to the (spin-conserving) collisions which equilibrate the Cooper pairs with the normal component. The moment of inertia is the analog of (γ^{-2} times) the susceptibility. Using this correspondence (or by direct analysis) we can immediately write the equation of motion of the torsion angle θ of the cylinder [see Eq. (6.36) of Ref. 25],

$$\dddot{\theta} + \frac{1}{\tau}\ddot{\theta} + I_0^{-1}\frac{\partial^2 V}{\partial \theta^2}\dot{\theta} + I^{-1}\frac{1}{\tau}\frac{\partial V}{\partial \theta} = 0 . \qquad (3.18)$$

We note that in both the "hydrodynamic" ($\omega\tau \ll 1$) and "collisionless" ($\omega\tau \gg 1$) limits the equation of motion (3.1) reduces to the simple form,

$$\ddot{\theta} + I_{\text{eff}}^{-1}\frac{\partial V}{\partial \theta} = 0 , \qquad (3.19)$$

but with different values of the effective moment of inertia ($I_{\text{eff}} = I$ in the first case, I_0 in the second). Moreover, by examining the first corrections to these limits we see that the dissipation looks "normal" when viewed from the collisionless end, but "anomalous" when viewed from the hydrodynamic end.

It is now very easy to apply our general prescription to calculate the tunneling rate. By rewriting the third term as $I_0^{-1}(d/dt)(\partial V/\partial\theta)$, we see that the function $K(\omega)$ [Eq. (2.3)] has the form

$$K(\omega) = -I\omega^2 \left[\frac{1 + i\omega\tau}{1 + i\omega\tau(I/I_0)} \right] . \qquad (3.20)$$

Hence the effective action for tunneling is, in an obvious notation,

$$S_{\text{eff}}(\widetilde{\theta}(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{1}{2} I\omega^2 \left[\frac{1 + |\omega|\tau}{1 + |\omega|\tau(I/I_0)} \right] \times |\widetilde{\theta}(\omega)|^2 d\omega + S_V(\widetilde{\theta}(\omega)) . \qquad (3.21)$$

We see that our intuitive expectations are confirmed: If all frequencies ω important in the tunneling process are large compared to τ^{-1} we obtain a simple WKB formula with effective moment of inertia I_0, whereas in the opposite limit we obtain a similar formula but with larger moment of inertia I. Moreover, since the quantity in large parentheses is a monotonically increasing function of τ, we see that the behavior of the rate between these two extremes is indeed a monotonic decrease.

As pointed out above, an equation of the form (3.18) is a general characteristic of a situation where the dissipation is due to processes which conserve the momentum conjugate to the tunneling variable. Thus, although the specific example considered above is unrealistic (and the ^{3}He analog cannot undergo tunneling, since in neither ^{3}He-A nor ^{3}He-B does the dipole energy have a metastable minimum), it does not seem out of the question that a realistic example of macroscopic quantum tunneling conforming to the above description may be found.

C. More realistic model of SQUID

As my final example I consider the circuit shown in Fig. 1, which although may not be a wholly realistic description of a SQUID ring is probably a considerable improvement on the simple RSJ model, at least for SQUID's incorporating Nb point contacts.[11] It is straightforward to verify that the classical equation of motion of the flux ϕ (or more precisely its deviation from the metastable equilibrium value) through the ring as a whole is of the form (2.3), with

$$K(\omega)=i\omega\left\{i\omega C+R_n^{-1}+\frac{i\omega C_G+1/i\omega L}{[1+(L'/L)]+(i\omega L')(i\omega C_G)}\right\} \tag{3.22}$$

and $V(q)$ given by the Josephson term in the potential energy only. Applying our general prescription, we see that the $\widetilde{K}(\omega)$ which enters the effective action (2.23) is of the form

$$\widetilde{K}(\omega)=\omega^2 C+|\omega|R_n^{-1}+\frac{\omega^2 C_G+1/L}{(1+L'/L)+\omega^2 L'C_G} . \tag{3.23}$$

For definiteness I shall assume in the following the order-of-magnitude conditions

$$C_G \gg C, \quad L\sim L' . \tag{3.24}$$

Then it is obvious that if all frequencies of importance in the tunneling process are small compared to the characteristic frequency $\overline{\omega}\sim(LC_G)^{-1/2}$ then the correct tunneling formula can be obtained from a simple RSJ model with effective capacitance and inductance given by

$$C_{\text{eff}}\cong C_G/(1+L'/L), \quad L_{\text{eff}}=L+L' , \tag{3.25}$$

while if the characteristic tunneling frequencies are large compared to $\overline{\omega}$ we expect the same to be again true but with

$$C_{\text{eff}}=C, \quad L_{\text{eff}}=L' . \tag{3.26}$$

This, of course, is exactly what we should have expected intuitively. In view of (3.24), the choice (3.26) will generally give a faster tunneling rate than (3.25), so that it is consistent to use this model *provided* that the attempt frequencies so generated are indeed large compared to $\overline{\omega}$. Whether or not this is so in any given experiment depends of course not only on the parameters in (3.22) but also on the critical current and external flux bias. In the general case it would presumably be necessary to insert expression (3.23) into (2.23) and find the saddle-point value of the effective action by numerical computation. Note that, in distinction to the real-time behavior of the circuit, the tunneling characteristics are not particularly sensitive to whether the characteristic frequency is near the resonance frequency $[(1+L'/L)(L'C_G)^{-1}]^{1/2}$ and we do not expect anything particularly spectacular to happen in this region.

It should be emphasized that although for given values of the parameters the use of a simple RSJ model with the parameters (3.26) may give a correct result for the tunneling transitions between different values of the total flux trapped in the SQUID ring, and predict quite high rates, it may be misleading to refer to the events so described as "macroscopic quantum tunneling." If we carry out a detailed analysis of what is going on by introducing explicitly as a second "system" variable the flux through the "small" circuit (the one containing C, R_n, and the junction), we find that the quantum tunneling is really carried out only by this variable, which is only dubiously macroscopic, and the total flux ϕ then adjusts to its new value by a purely classical process. A similar caveat, incidentally, should be applied to the case of tunneling with strong anomalous dissipation (Sec. III A): If the macroscopic variable tunnels rapidly *only* because of its coupling to the microscopic degrees of freedom, it is not entirely clear that the label "macroscopic" is correctly applied to the tunneling. Fortunately, as we shall see in the conclusion, this question is rather academic since the case of pure anomalous dissipation seems rather unlikely to be of much physical interest.

IV. CONCLUSION

In this paper I have developed a technique to calculate the effect on quantum tunneling of a quite arbitrary linear dissipative (or reactive) mechanism, irrespective of whether it is of normal, anomalous, or more general type and however complicated its frequency dependence. I now comment on some aspects of the results.

First, the technique given here automatically bypasses the rather tedious arguments about "frequency renormalization" effects which had to be gone through at length in I. While it is of course true, as stated in I, that a purely Ohmic (i.e., frequency-independent) resistance gives no frequency renormalization, this result rests essentially on the fact that the reactive part of the admittance [the $K(\omega)$ of I] of such an element is zero (to lowest order in ω/ω_c, where ω_c is some high cutoff frequency), and any frequency dependence of the resistance will in general give a renormalization, which will affect both the classical dynamics and the tunneling behavior. In the method of the present paper such effects are taken into account quite automatically, since we do not have to distinguish explicitly between the real and imaginary parts of (the present) $K(\omega)$.

Second, the results concerning the opposite effects of normal and anomalous dissipation can be understood very naturally within the framework of the quantum theory of measurement. As stressed by Zurek,[26] any interaction of a quantum system with its environment will tend to "collapse" the wave function of the system into an incoherent mixture, and the nature of the particular mixture into which the collapse takes place will depend on which system variable interacts with the environment: e.g., if the interaction is of the form $\hat{X}\hat{\Omega}$, where $\hat{X}$ is a variable of the system and $\hat{\Omega}$ one of the environment, then it is the variable X which is "observed" by the environment and the collapsed density matrix of the system is diagonal in the X representation (and hence, in general, not diagonal in a representation corresponding to the eigenfunctions of an operator which fails to commute with $\hat{X}$). Consider now, for example, the nonlinear *LCR* circuit of Sec. III A. The most natural description of the effect of the resistor consists in adding to the Hamiltonian a term of the form

$$H_{\text{int}} = -\phi \sum_\alpha C_\alpha x_\alpha + \tfrac{1}{2}\phi^2 \sum_\alpha C_\alpha^2 / m_\alpha \omega_\alpha^2 , \tag{4.1}$$

where the second term is the "counterterm" which cancels the unphysical potential renormalization which the first alone would produce (see I, Appendix A, for an exhaustive discussion of this point). Thus we expect the effect of the resistor to be to tend to collapse the system into an incoherent mixture of eigenfunctions of the flux ϕ. In the "normal" case (where it is the inductance which is nonlinear and the flux itself which is the tunneling variable) this impedes the tunneling, since the tunneling phenomenon depends essentially on the superposition of eigenstates corresponding to different values of the tunneling variable (cf., I, Sec. VI). On the other hand, the resistor tends to *increase* the degree of superposition of states corresponding to different values of the conjugate variable Q (the charge on the capacitor plates), and hence in the anomalous case, where it is this latter variable which does the tunneling, the dissipation actually assists the tunneling process.[27]

We may ask whether we are justified in using the terms "normal" and "anomalous" as we have done: In other words, is there any general reason why the environment should normally want to "observe" the tunneling variable itself rather than its conjugate momentum, or equivalently, why the dissipation should be proportional to the squared velocity of the former rather than the latter? Although it is difficult to give a completely general argument on this point, I believe that consideration of specific examples will make it plausible that this is so. Consider for example our *prima facie* counterexample, namely the circuit discussed in Sec. III A, with a highly nonlinear capacitance. Is this in fact a realistic description, as implied in Ref. 9, of tunneling in a ferroelectric system (assumed to be contained between the capacitor plates)? I believe it is not, for the following reason: In a real ferroelectric, the kinetic energy responsible for the tunneling comes, not from any external self-inductance, but from the actual motion of the ions of the sample themselves; it is easy to see that for any but the most extreme values of the sample dimensions, etc., the associated "kinetic inductance" $L_K \equiv 2E_{\text{kin}}/\dot{Q}^2$ is many orders of magnitude smaller than any external electromagnetic inductance. In fact, the L and R of the circuit are essentially irrelevant to the tunneling process, which would proceed just as well if the capacitor were completely open circuited.[28] Now the important point is that any dissipation associated with the actual motion of the ions will automatically be not in parallel but *in series* with the kinetic inductance, i.e., the rate of dissipation $\dot{W}$ will be proportional to $\dot{Q}^2$ and we are back to the normal case. More generally, it seems likely that the motion of the tunneling variable itself will always have some dissipation associated with it, even though there may possibly be other mechanisms as well: Thus, in any realistic physical system, the dissipation, though it may not be necessarily of pure normal type (cf. Sec. III B), is unlikely to be pure anomalous. In the absence of a plausible physical counterexample to this statement, therefore, we have to treat the case of pure anomalous dissipation as of primarily formal interest.

Finally, I comment briefly on the relation of the present work to that of Zwerger,[14] who considers specifically the case of a Josephson junction with weak linear dissipation and derives a result [his Eq. (31)] which is effectively equivalent, for this case, to Eq. (2.23). Apart from trivial differences of notation, etc., and some complications associated with the replacement of q in (2.11) by a sine function, the principal differences between Zwerger's approach and the present one are as follows: (1) Zwerger considers only the weak-coupling limit, while the results of the present paper are (I believe) valid for arbitrarily strong dissipation provided only that the condition (1) of Sec. II is satisfied. (2) In Ref. 14, the formula (31) is stated to refer to the *elastic* tunneling probability, and an inelastic probability is calculated separately: By contrast, I believe that formulas (2.23)–(2.25) refer to the *total* tunneling probability [see the remarks below Eq. (4.11) in I]. (3) The argument of Ref. 14, Sec. IV, relies on a formal analytic continuation procedure: As stated above, I believe that such a procedure is in general not justifiable [in fact, it follows from the results of I, Sec. IV, that when condition (2) above is not satisfied it gives quite the wrong answer]. It may well be valid when condition (2) or a similar condition is satisfied [as it is in Ref. 14, see Eq. (5)], but this condition needs to be noted explicitly: In general, a complete knowledge of the (experimentally measured) dielectric constant is insufficient to determine the tunneling behavior. In fact, it would appear from the results of Ref. 13 that the tunneling exponent for an ideal tunnel oxide junction as described by the standard microscopic tunneling Hamiltonian cannot be set in the form of Zwerger's Eq. (31) even in the limit of weak dissipation; this is not surprising since condition (2) is almost certainly not satisfied in this case. Nevertheless, as noted above, the condition *is* satisfied for a large number of experimentally relevant situations, and the formal analytic continuation procedure then gives the same results as the more laborious argument developed in Sec. II of the present paper.

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[3] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983) (hereafter referred to as I). An erratum concerning only the detailed form of the prefactor A in the tunneling expression is given in *ibid*. **153**, 445 (1984).

[4] A number of papers (Refs. 5—7) have appeared in the literature which dispute this conclusion. As far as the present author can see, they are all based either on an obvious logical *non sequitur* (Ref. 5), on a misunderstanding of the question being asked (Refs. 6 and 7, see Ref. 8), or on claims (Ref. 6) which appear to be unsubstantiated about alleged divergences in the calculations reported in Ref. 1. These arguments should not be confused with that given in Sec. IV of Ref. 9, which is, I believe, formally correct for the specific model studied; since the latter is a case not described by Eq. (1.1), this leads to no conflict with the conclusions of Refs. 1, 3, and 8. I would not agree with some of the conclusions apparently drawn in Secs. I and V of Ref. 9 concerning the general case, nor that the opinions that appear to be ascribed to Caldeira and myself in Sec. II of that reference are correctly so ascribed.

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[15] The results of step 2 are in some sense contained in the work of Zwerger (Ref. 14); cf., the remarks in the Conclusion.

[16] In such cases it is, of course, not necessarily true that $V(q)$ by itself has a metastable minimum: A necessary condition for metastability is that the quantity $\frac{1}{2}K(\omega=0)q^2+V(q)$ has such a minimum, where $K(\omega)$ is the Fourier transform defined below.

[17] Note that this does not necessarily imply that all the environment degrees of freedom are microscopic. For example, in the case discussed in Sec. III C it is implicit that one of the degrees of freedom of the "environment" is the flux through the "small" circuit, which is at least semimacroscopic. (Of course, we could have treated it equally well as a second "system" variable, thereby obtaining a two-dimensional tunneling problem, cf., Sec. III C.) The important point is that its behavior is purely harmonic.

[18] In some (unusual) cases the "bare" mass is also renormalized.

[19] Any terms of the form Cp or Dq can be removed by a trivial canonical transformation.

[20] Even the measure of the functional integral is unaffected, since the normalization of the x_j can be chosen so that the Jacobian of the transformation $x_j \to \tilde{x}_j$ is unity. Note however that in certain types of problems (not relevant in the present context) where precise initial conditions are important some care may be necessary in going between the two descriptions.

[21] This convention differs by a factor of $(2\pi)^{1/2}$ from that used in Appendix D of I.

[22] Recall that $\overline{K}(\omega)$ as defined by Eq. (2.13) has a branch cut along the real axis and tends to $K(\omega)$ only as this cut is approached from *below*. The function defined by (2.13) in the upper half-plane tends to $K^*(\omega)$ on the real axis: Hence we could equally well have written the bracket as $K^*(\omega)$ analytically continued to the positive imaginary axis. The identity of the two expressions is guaranteed by the Kramers-Kronig relations. It should be emphasized that the representation of $K(\omega)$ by approximate expressions which fail to satisfy these relations may lead to highly misleading and indeed inconsistent results.

[23] For the simple harmonic oscillator, "anomalous" dissipation can be defined quite generally as dissipation which is normal when the roles of coordinate and momentum are interchanged.

[24] Note that this condition is made explicit in the discussion in I, Appendix C, and is in fact essential to the validity of the generalization to frequency-dependent dissipation quoted there (though this may not be immediately obvious from the discussion).

[25] A. J. Leggett and S. Takagi, Ann. Phys. (N.Y.) **106**, 79 (1977).

[26] W. H. Zurek, Phys. Rev. D **24**, 1516 (1981).

[27] The above argument is at first sight vitiated by the fact that it is equally possible to write H_{int} in terms of the voltage across the circuit [see I, Eq. (A20)] and hence of the charge Q. However, there is then no counterterm and the "collapsing" effect of the interaction is outweighed by its effect in renormalizing the potential.

[28] In fact, it would proceed *better:* The external inductance actually hinders tunneling, just as does the capacitance C_G in the Josephson case [Fig. 1, cf. Eq. (3.23)]. Thus the effect of the resistor in assisting tunneling is simply that it shorts out an impeding mechanism. In the Josephson case a resistor in *series* with C_G would have the same effect.