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Ground-state degeneracy and fractionally charged excitations in the anomalous quantum Hall effect

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The Hamiltonian describing two-dimensional electrons in a high magnetic field is diagonalized exactly for a small number of particles. In addition to the energy spectrum the mean occupation number $\rho(j) = \langle C_j^\dagger C_j \rangle$ of the jth Landau state in the lowest Landau level is also calculated. For $\nu = n/m$ with m an odd integer, $\rho(j)$ has a period m $[\rho(j) = \rho(j+m)]$, and there are m distinct ground states—in striking analogy with a one-dimensional charge-density-wave system. In terms of $\rho(j)$, profiles of the $\frac{1}{3}$ kinks are obtained in the ground state for ν close to $\frac{1}{3}$. Creation energy of the kink is obtained from the energy gap. The $\nu = \frac{1}{2}$ case is markedly different.

I. INTRODUCTION

The discovery of the fractional quantized Hall effect[1,2] has stimulated much theoretical work. The wave-function approach by Laughlin[3] has provided a calculational tool as well as a good framework to understand the physics when the filling fraction ν of the lowest Landau level is very close to $\nu = 1/p$, where p is an odd integer. However, the nature of the ground state and the stability of the proposed fractionally charged excitations remain unclear. Moreover, the extension of this method to multiples of $1/p$ has led to fairly complicated constructions.[4,5] Therefore it might be worthwhile to pursue a more general, symmetry-related approach to understand some of the important features of the problem.

In this paper, following Yoshioka, Halperin, and Lee[6] we adopt an effectively one-dimensional Hamiltonian (Sec. II) and diagonalize it exactly for a small number of electrons. We calculate the low-lying spectrum for a fixed ν and for different total momentum J. For $\nu = \frac{1}{3}$ we find the system is locked into one of the three distinct equivalent ground states; for $\nu = \frac{2}{5}$ there are five distinct ground states, etc. For $\nu = \frac{1}{2}$ there seem to be an infinite number of ground states (Sec. III). The triply degenerate ground states imply the existence of fractionally charged kink excitations. In terms of the mean occupation number of the Landau state $\rho(j) = \langle C_j^\dagger C_j \rangle$, profiles of the $\frac{1}{3}$ kinks are obtained in the ground-state solution for ν very close to $\frac{1}{3}$ (Sec. IV). The creation energy of the kinks is estimated from the energy versus J curve in Sec. III. Section V concludes with a few comments.

II. MODEL HAMILTONIAN

We consider N_e electrons moving inside a square box of size L in the x-y plane with a high magnetic field B pointing in the z direction. The number of states in the lowest Landau level is given by the total flux in units of an elementary flux $N_s = BL^2/(hc/e)$. The magnetic length $l = \sqrt{\hbar c/eB}$ is related to N_s by $2\pi l^2 N_s = L^2$. In the Landau gauge $\vec{A} = (0, Bx)$, a basis set of the lowest Landau level consists of Landau states labeled by an integer j $(1 \leq j \leq N_s)$:

$$\tilde{\phi}_j(\vec{r}) = \left(\frac{1}{\sqrt{\pi}\, lL}\right)^{1/2} \exp\left(i\frac{x_j y}{l^2}\right) \exp\left(-\frac{(x-x_j)^2}{2l^2}\right) , \quad (2.1)$$

where

$$x_j = aj = \left(\frac{L}{N_s}\right) j \quad . \quad (2.2)$$

The Landau state $\tilde{\phi}_j$ is a plane wave in the y direction with momentum x_j/l^2 and a Gaussian distribution in the x direction centered about $x = x_j$. In Eq. (2.2) a is the separation between the centers of two neighboring Landau orbitals.

Because of (2.2) $\tilde{\phi}_j$ is periodic in the y direction; to make things periodic in the x direction as well, the following combination of $\tilde{\phi}_j$ should be used:

$$\phi_j = \sum \tilde{\phi}_i \; , \quad i \equiv j \; (\mathrm{mod} N_s) \quad . \quad (2.3)$$

We assume Coulomb interactions between the electrons. Imposing periodic boundary conditions and subtracting off the interaction with a uniform background charge distribution leads to the following second quantized Hamiltonian[6] in which $C_j^\dagger$ creates the Landau state ϕ_j:

$$H = \sum_J \epsilon_M C_j^\dagger C_j + \sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4} A_{j_1 j_2 j_3 j_4} C_{j_1}^\dagger C_{j_2}^\dagger C_{j_3} C_{j_4} \; , \quad (2.4)$$

where

$$A_{j_1j_2j_3j_4}=\frac{1}{2}\int d\vec{r}_2\int d\vec{r}_1\phi^*_{j_1}(\vec{r}_1)\phi^*_{j_2}(\vec{r}_2)V(\vec{r}_1-\vec{r}_2)\phi_{j_3}(\vec{r}_2)\phi_{j_4}(\vec{r}_1)$$

$$=\frac{e^2}{2l^2N_s}\sum_q{}' \frac{1}{|q|}\delta'_{j_1+j_2,j_3+j_4}\exp\left[-\frac{l^2q^2}{2}-iq_x(j_1-j_3)a\right] , \tag{2.5}$$

$$q_x=\frac{2\pi j_x}{L},\quad j_x=0,\ \pm1,\ \pm2,\ \ldots,$$

$$q_y=\frac{2\pi j_y}{L},\quad j_y\equiv j_d\ (\text{mod}N_s),\quad j_d=j_1-j_4\ , \tag{2.6}$$

$$\epsilon_M=-\frac{3.9e^2}{2L}\ . \tag{2.7}$$

δ' means momentum conservation $j_1+j_2\equiv j_3+j_4$ (modN_s). Zero is excluded in the summation of (2.5) over momenta. No matter how local $V(\vec{r})$ is the exponential factor in the scattering matrix element A is of the order 1 as long as $j_d\approx L/2\pi l$ because of the strong overlap between the Landau orbitals. ϵ_M in Eq. (2.7) is the Madelung energy per particle of a square lattice with lattice constant L.

III. GROUND-STATE DEGENERACY

As it stands H [Eq. (2.4)] is a one-dimensional Hamiltonian describing N_e electrons in an N_s site ring. For small N_e, H can be diagonalized exactly. Since the total momentum

$$J=\sum_{i=1}^{N_e}j_i$$

is a good quantum number, the result depends on both $\nu=N_e/N_s$ and J. For a given J the ground state Φ_J can be expressed as a linear combination of all possible Slater states $|j_1,j_2,\ldots\rangle=C^\dagger_{j_1}C^\dagger_{j_2}\cdots|0\rangle$ with the same total J:

$$\Phi_J=\sum_{j_1,j_2,\ldots}b_{j_1j_2}\ldots\delta'_{j_1+j_2+\cdots,J}|j_1,j_2,\ \ldots\rangle\ . \tag{3.1}$$

The energy $E(\nu,J)$ of the ground state Φ_J has the symmetry property

$$E(\nu,J)=E(\nu,J+N_e)=E(\nu,J+2N_e)=\cdots\ , \tag{3.2}$$

since a uniform shift of all the particles by one step $j_i\rightarrow j_i+1$ (and therefore $J\rightarrow J+N_e$) does not change the energy. It follows immediately from (3.2) that $E(\nu,J)$ is independent of J if N_e and N_s are relative primes, as mN_e can be made equal to any integer (modulo N_s) for a suitably chosen integer m. This happens when ν is almost irrational so that N_e and N_s are large relative primes. The analog of this in the charge-density-wave system is the independence of the total energy on the phase angle θ of the charge-density-wave condensate. This continuous translational symmetry in the charge-density wave, however, is "broken" down to a discrete symmetry when $\nu=n/m$ is rational. As a result of symmetry reduction the system is able to achieve a lower energy for some optimal choices of θ leading to m distinct ground states. The kinks connecting different ground states become elementary charged excitations instead of the electrons and holes.[7]

The analogy suggests that something phenomenologically similar may happen in the anomalous quantum Hall effect. Take, for example, $N_s=3N_e$. If $E(\nu,J)$ as a function of J has a minimum at $J=J_*$, it must have another one at J_*+N_e and J_*+2N_e. Then we would have the possibility of three stable degenerate ground states (Fig. 1). In principle, for $\nu=n/m$ there could be more than m equivalent ground states which are guaranteed by the symmetry argument. In the calculations that we have carried out the m-fold degeneracy is exact when m is odd. For $m=2$ the degeneracy is certainly more than twofold.

In Fig. 2 we display the low-lying spectrum for $\nu=N_e/N_s=\frac{1}{3}$, $N_e=4,5,6$, and for various J. Only the lowest-energy state is shown for each J in the six-particle case. The energy is measured in units of e^2/l. In Figs. 2 and 3

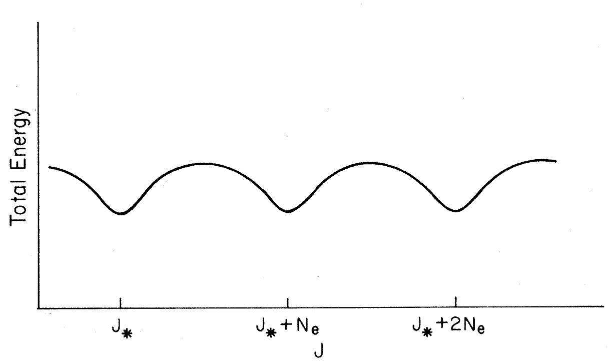

FIG. 1. Possible energy dependence on the momentum J at a fixed $\nu=\frac{1}{3}$.

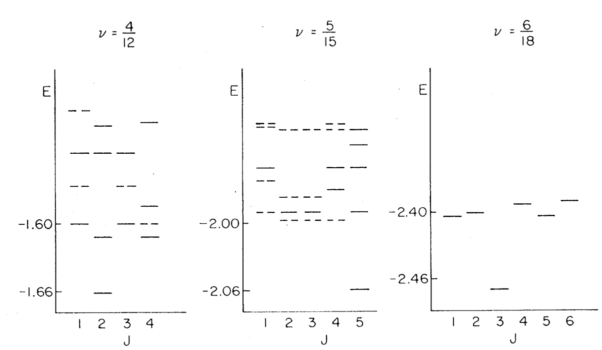

FIG. 2. Low-lying energy spectra of 4,5,6 electrons in a 12-, 15-, and 18-site ring, respectively.

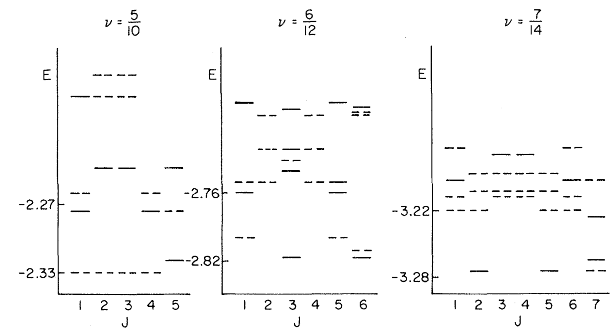

FIG. 3. Low-lying energy spectra of 5,6,7 electrons in a 10-, 12-, and 14-site ring, respectively.

shorter dashes are used to indicate degenerate states with the same J. The number of dashes gives the degeneracy. As is obvious from the figure the ground-state degeneracy is exactly threefold and there is an energy gap of about $0.06e^2/l$ separating the true ground states from the excited states. For $\nu = \frac{1}{2}$, $N_e = 5, 6, 7$, the energy spectra are shown in Fig. 3. The ground state is more than twofold degenerate and the degeneracy probably increases with the number of particles.

One quantity to partially characterize the degenerate ground states locally is the charge density of the Landau ring, i.e., the mean occupation number of each Landau state $\rho(j) = \langle C_j^\dagger C_j \rangle$. This quantity is plotted in Fig. 4 for $\nu = \frac{1}{3}$ and $\nu = \frac{2}{5}$. For $\nu = \frac{1}{3}$ and $N_e = 6$ the magnitude of the oscillation of $\rho(j)$ vs j is already very small so that $\rho(j)$ is practically a constant.

Another useful concept in this connection is the parent state.[8] A parent state is a certain unperturbed Slater state from which one can generate an eigenstate of the model Hamiltonian by adiabatically switching on the electron-electron interactions. Since the total momentum is conserved the J of the Slater state determines the J of the

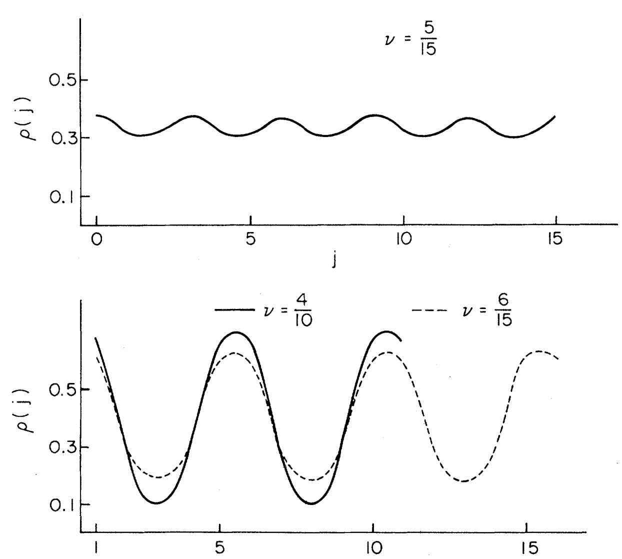

FIG. 4. (Top): Mean occupation number of the Landau states for $N_e = 5$, $N_s = 15$. (Bottom): Solid curve for $N_e = 4$, $N_s = 10$, and broken curve for $N_e = 6$, $N_s = 15$.

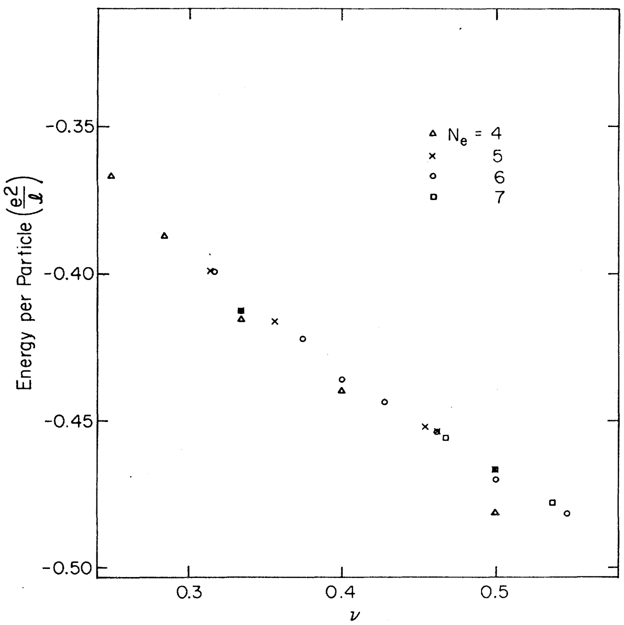

FIG. 5. Ground-state energy per electron as a function of ν.

eigenstate. It turns out that the ground states of $\nu = \frac{1}{3}$ can all be generated from some particularly nice parent states, i.e., Slater states which consist of orderly arrays of Landau orbitals. For example, the state $|1, 4, 7, 10, 13\rangle$ has a $J = 35 \equiv 5$ (mod15) corresponding to an optimal choice of J. Similarly, the state $|1, 5, 6, 10, 11, 15\rangle$ is a parent state of the eigenstate whose charge density $\rho(j)$ is shown as the dotted curve is Fig. 4. As anticipated by Tao and Thouless[9] a regular arrangement of the electrons in the space of Landau orbitals enhances the correlation energy.

In Fig. 5 we have collected data on the ground-state energy per particle for different ν and different N_e. It appears that the energy versus ν curve is nonanalytic near $\nu = \frac{1}{3}$. Just as in the case of ordinary one-dimensional charge-density-wave systems we believe that the qualitatively different behavior of the system at $\nu = \frac{1}{3}$ (the ground-state degeneracy) is responsible for this nonanalyticity. For $\nu = \frac{1}{2}$ the oscillation of the energy as a function of N_e seems to have stopped after $N_e = 7$. The $N_e = 8$ result is not displayed because it is indistinguishable from the $N_e = 7$ and $N_e = 5$ results. From the figure it seems there is no dip at all at $\nu = \frac{1}{2}$.

IV. KINK EXCITATIONS

The ground-state degeneracy established in Sec. III prompts the construction of a kink as a solution connecting one ground state on one side to another ground state on the other side. Since we impose periodic conditions, this requires that we consider a $3N_e \pm 1$ site ring.

Take the $\nu = N_3/N_s = \frac{5}{16}$ case. The parent state $\Phi = |1, 4, 7, 10, 13\rangle$ has $J = 35 \equiv 3$ (mod16). The parent state is an orderly array of Landau orbitals except for a "defect" between the last orbital and the first orbital. The exact ground-state charge-density distribution $\rho(j)$ for $J = 3$ (the bottom curve in Fig. 6) indeed shows a kink center about $j = 15$. This ground-state configuration which con-

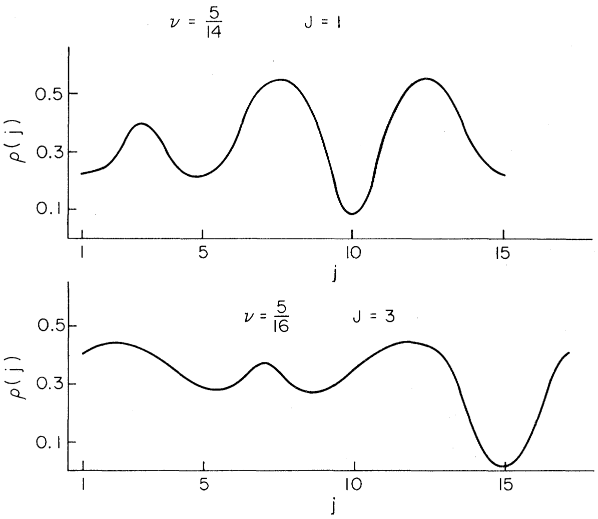

FIG. 6. Profiles of the $\frac{1}{3}$ kinks.

sists of a $\frac{1}{3}$ background plus a charge $-e/3$ kink seems to be very stable as it is separated from the first excited state by $0.06e^2/l$. The top curve in Fig. 6 is the profile of an oppositely charged kink.

The insight we have gained allows us to give a nice interpretation of the energy gap separating the ground state from the excited states in Fig. 2. Take the $N_e = 5$ case. The $J = 5$ and $J = 10$ ground states can be generated from the parent state $\Phi_1 = |1,4,7,10,13\rangle$ and $\Phi_2 = |2,5,8,11,14\rangle$, respectively. Φ_2 is obtained from Φ_1 by shifting all particles by one step. For $5 < J < 10$ we have to shift some particles. To keep the parent state as orderly as possible we should shift those particles in a row. For example, $|1,5,8,11,13\rangle$ would be a good choice. This is equivalent to creating a kink at $j = 3$ and an antikink at $j = 12$. Consequently, the difference in energy between the lowest-energy states of $J = 5, 10, 15$ and those of other values of J represents the creation energy of a kind-antikink pair. From Fig. 2 this energy is about $0.07e^2/l$.

Although we have presented detailed calculations for the $\nu = \frac{1}{3}$ case only, the conclusion can be generalized to higher commensurabilities. For $\nu = n/m$ there are m degenerate ground states and the kink excitations have charge $\pm(e/m)$.

As an illustration we consider the $\frac{2}{5}$ case. Starting from the parent state

$$\Phi = |\ldots,-9,-8,-4,-3,1,2,6,7,\ldots\rangle$$

by shifting the orbitals at $j = 2, 7, 12, 17, \ldots$ by one step we have created a kink at the center with charge $-\frac{1}{5}e$. The resulting parent state

$$|\ldots,-9,-8,-4,-3,1,3,6,8,11,\ldots\rangle$$

is equivalent to

$$|\ldots,-9,-8,-4,-3,1,4,5,9,10,\ldots\rangle$$

which clearly shows one ground state on the left-hand side and another ground state on the right-hand side.

V. CONCLUSION

In this paper we have tried to extract some essential features of the ground-state and elementary excitations of a two-dimensional electron system in a high magnetic field from some numerical calculations. We find that when the filling fraction ν is rational $\nu = n/m$ (m odd) the ground state is m-fold degenerate. The ground states can be partly characterized by the mean occupation number of the Landau states and by the parent states. The kinks connecting different ground states are fractionally charged elementary excitations. When ν differs slightly from $\nu = n/m$, the ground state consists of the $\nu = n/m$ ground states plus kinks which are probably pinned by the impurities, giving rise to the observed plateau in the Hall conductance.

To illustrate this for $\nu = \frac{1}{3}$ we use the argument that Laughlin used for the integral Hall effect.[10] In the presence of an electric field E in the x direction the energy of Φ_j is Ex_j. Under a virtual change of the magnetic potential $\delta A_y = (hc/e)(1/L)$, ϕ_j goes into ϕ_{j+1}. This is equivalent to shifting the entire condensate by one step. The change in the electric potential energy is therefore e^*EL, where $e^* = \nu e = \frac{1}{3}e$ is the average charge per Landau state. When ν deviates a bit from $\nu = \frac{1}{3}$, the system is composed of a uniform $\nu = \frac{1}{3}$ condensate plus some kinks which because of the excess charge they carry are pinned by the impurities. The change in the electrical potential energy is therefore due to the underlying $\frac{1}{3}$ condensate only, i.e., the Hall conductance is the same as that for $\nu = \frac{1}{3}$. We would like to emphasize that e^* is not the charge of a kink for $n \neq 1$ $(\nu = n/m)$.

In the absence of impurities because of the translational invariance in the y direction the kink is an extended homogeneous object in that direction in contrast to the vortexlike object that Laughlin proposed. It is, however, conceivable that the latter can be expressed as a linear combination of the kink solutions judging from the fact that both the kink and the vortex have the same charge and about the same creation energy.[3]

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