Helicity moduli of three-dimensional dilute XY models

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The helicity moduli of various dilute, classical XY models on three-dimensional lattices are studied with a view to understanding some aspects of the superfluidity of ^{4}He in Vycor glass. A spin-wave calculation is used to obtain the low-temperature helicity modulus of a regularly-diluted XY model. A similar calculation is performed for the randomly bond-diluted and site-diluted XY models in the limit of low dilution. A Monte Carlo simulation is used to obtain the helicity modulus of the randomly bond-diluted XY model over a wide range of temperature and dilution. It is found that the randomly diluted models do agree and the regularly diluted model does not agree with certain experimentally found features of the variation in superfluid fraction with coverage of ^{4}He in Vycor glass.

There have been many studies of the effect of dilution on bulk thermodynamic properties of a variety of spin models.[1] However, very little attention has been paid to the effect of dilution on the helicity modulus[2] Υ, which can be viewed either as the interfacial free energy of an isotropic, n-component ($n>1$) spin system or a rigidity modulus describing its response to a weak, long-wavelength, helical magnetic field. In three dimensions, Fishman and Ziman[3] have calculated Υ analytically for a *regularly* (as opposed to *randomly)* diluted spherical (i.e., $n=\infty$) model. In two dimensions, Solla and Riedel[4] have used a Migdal-Kadanoff[5] real-space renormalization group to calculate Υ for a random, bond-dilute, classical (i.e., spin $S=\infty$) XY model ($n=2$). Most recently, Ebner and Stroud[6] have obtained Υ for a model of a granular superconductor. To the best of our knowledge, Υ has not been obtained for any other dilute spin model.

In this paper we report on three calculations—a low-temperature, low-dilution, spin-wave analysis, a Migdal-Kadanoff real-space renormalization group, and a Monte Carlo simulation—of Υ for the three-dimensional, random, bond-dilute, classical XY model.[7] The Hamiltonian for this model is

$$H=-\sum_{\langle i,j\rangle} J_{ij}\vec{s}_i\cdot\vec{s}_j\ , \tag{1a}$$

where the sum is over distinct, nearest-neighbor pairs of sites, $\langle i,j\rangle$, on a simple-cubic lattice, the spins $\vec{s}_i$ are two-component vectors of unit length, and the bonds J_{ij} are independent random variables with a distribution given by

$$P(J_{ij})=(1-q)\delta(J_{ij}-1)+q\delta(J_{ij})\ , \tag{1b}$$

where q is the concentration of missing bonds. We shall assume periodic boundary conditions in all directions.

There are a variety of models that can be mapped, under certain conditions, onto quantum (spin $S=\frac{1}{2}$, 1, $\frac{3}{2}$, etc.) or classical XY models.[8] These include quantum lattice fluids[8,9] and Heisenberg models with a strong anistropy along one direction.[8] There are other systems which resemble XY models by virtue of having an order parameter that has two components. The most common example is an interacting Bose fluid (e.g., ^{4}He).[8,9] The superfluid state and the superfluid density ρ_s of such a Bose fluid are the analogs of the ordered state and the helicity modulus Υ, respectively, of an XY model.[2] Our reason for studying model (1) is that it might shed some light on the superfluidity of ^{4}He in Vycor,[10] a porous glass consisting of a network of narrow, interconnected channels.[11] This network is not ordered in any obvious way, but, as far as we know, there is no information available on how random it is. In the absence of such information, it is not unreasonable to try and model the properties of ^{4}He in Vycor by a bond-dilute, quantum lattice gas.[12] The latter can be mapped onto the bond-dilute, spin $S=\frac{1}{2}$ XY model,[12] which we further approximate by its classical ($S=\infty$) version, Eq. (1). The point is to see whether Υ for the simplified model (1) captures in some gross way the qualitative[13,14] features of ρ_s for ^{4}He in Vycor.

Both Υ and the critical temperature T_c become depressed for any spin model ($n>1$) with *increasing* dilution [increasing q for model (1)], as do ρ_s and T_c with *decreasing* total density ρ of ^{4}He in Vycor. Clearly, a better criterion than the mere depression of Υ and T_c with increasing dilution is required to distinguish between different models. One such criterion[15] can be constructed by looking at plots of $R\equiv\Upsilon(T,q)/\Upsilon(T=0,q)$ versus $\tau\equiv T/T_c(q)$ for different values of q, the parameter which measures the degree of dilution. The analogous plot[16] of $\rho_s(T,\rho)/\rho_s(T=0,\rho)$ versus $T/T_c(\rho)$ for ^{4}He in Vycor is shown in Fig. 1(a) for different values of ρ. Note that, at least for low values[17] of ρ [for which $T_c(\rho)\lesssim 800$ mK], the curves in Fig. 1(a) show a distinct trend: for fixed $T/T_c(\rho)$, $\rho_s(T,\rho)/\rho_s(T=0,\rho)$ decreases as ρ decreases. Low values of ρ should correspond, roughly, to high values of the dilution q in a model similar to (1).

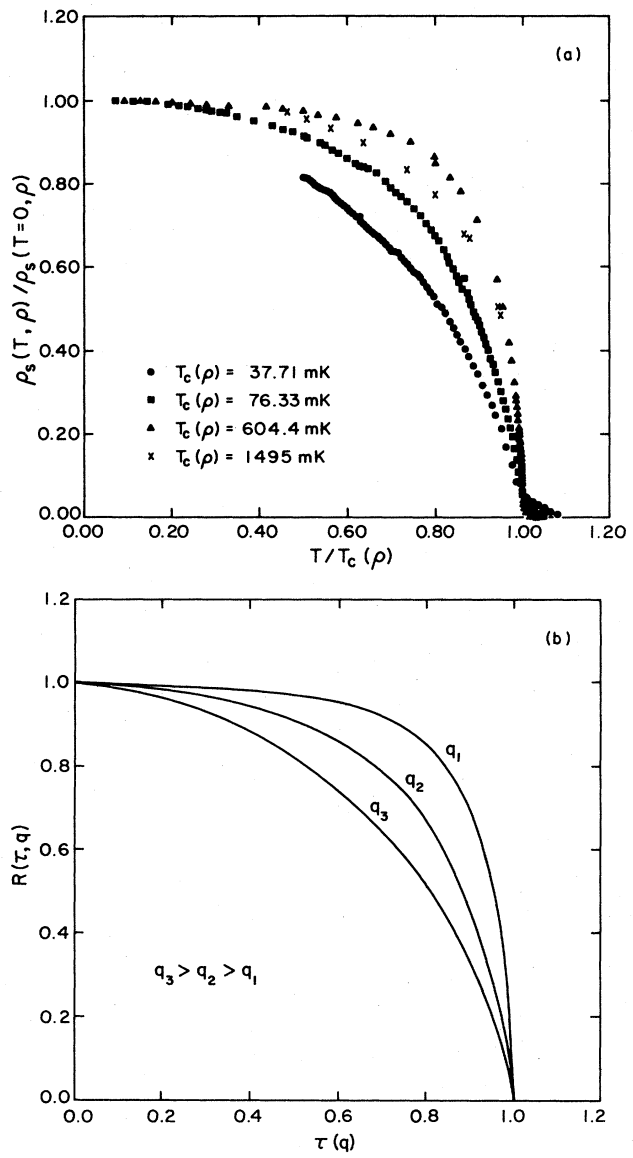

FIG. 1. (a) Plot of the scaled superfluid density $\rho_s(T,\rho)/\rho_s(T=0,\rho)$ vs the scaled temperature $T/T_c(\rho)$, where ρ is the total density of ^{4}He in the Vycor sample, and $T_c(\rho)$ is the critical temperature for the superfluid transition at this density. These experimental data are from Ref. 14 (also see Ref. 16). (b) Schematic plot of the expected variation with q of the scaled helicity modulus $R(\tau,q)\equiv\Upsilon(T,q)/\Upsilon(T=0,q)$ vs the scaled temperature $\tau\equiv T/T_c(q)$ for a dilute spin model that mimics the behavior of the superfluid density of ^{4}He in Vycor. The larger q is, the more dilute the model, as in model (1). [See discussion in the vicinity of inequality (2) and Ref. 17].

Thus, for a spin model that mimics the behavior of ^{4}He in Vycor, a plot of $R(\tau,q)$ versus τ should be similar to Fig. 1(b), i.e., for fixed τ, $R(\tau,q)$ should decrease as q increases (at least for sufficiently large[18] q),

$$R(\tau,q_2)\leq R(\tau,q_1),\quad q_1<q_2 \tag{2}$$

where the equality holds only for $\tau=0$ or 1. In the *regularly* diluted spherical model,[3] the analog of the inequality (2) is always violated.

In a simple effective-medium theory, in which model (1) is replaced by a pure one with bonds of strength $1-q$, $R(\tau,q)$ is independent of q. However, a simple, though crude, effective-medium argument for a weakly interacting Bose gas does give (at least at low temperatures) the trend we are seeking: $\rho_s(T,\rho)/\rho_s(T=0,\rho)$ decreases as ρ decreases for fixed $T/T_c(\rho)$. Such an argument begins with an interacting Bose gas (such as ^{4}He) and assumes that the *only* effect of the Vycor glass is the uniform reduction of the strength of the interaction with decreasing density ρ of ^{4}He in Vycor. For a weakly interacting Bose gas,[19]

$$\rho_s(T)/\rho\approx 1-2\pi^2T^4/45\rho c^5,\ \text{as } T\to 0\ , \tag{3}$$

where $c=[n_0(T)V(0)/m]^{1/2}$ is the speed of sound, $n_0(T)$ is the condensate density, $V(0)$ is the zero-momentum Fourier component of the interaction potential, and m is the mass of the bosons. [Note that $\rho_s(T=0,\rho)=\rho$, and $\hbar=k_B=1$.] To lowest order in T, we can replace c by $c(T=0)$ on the right-hand side of Eq. (3). Since $n_0(T=0)$ is proportional to[20] $\rho_s(T=0,\rho)=\rho$, we can rewrite Eq. (3) as

$$\rho_s(T)/\rho=1-W(T/T_c)^4\rho^{-5/6}\ , \tag{4}$$

where W is independent of T and ρ, and we assume that the critical temperature $T_c\sim\rho^{2/3}$. ($T_c\sim\rho^{2/3}$ for a noninteracting Bose gas,[21] and the corrections to this behavior are small for a weakly interacting Bose gas.) Equation (4) shows that, for fixed $T/T_c(\rho)$, $\rho_s(T)/\rho$ decreases as ρ decreases.

Rasolt and Stephen[22] have recently used a classical approximation to an interacting Bose gas to study the superfluid transition of ^{4}He in Vycor. Their study is confined to the vicinity of the critical point and does not consider the global trends [e.g., inequality (2)] that we concentrate on. We reserve a comparison of our approach with that of Rasolt and Stephen for the concluding remarks at the end of this paper.

Our principal results are as follows.

(i) A spin-wave analysis of the randomly bond-dilute model (1) indicates that the inequality (2) is satisfied for *sufficiently low* values of q and τ. [We consider configurations in which only one bond is missing, and obtain Υ *exactly* to orders q, T, and qT. To obtain a scaled plot such as Fig. 1(b), we need $(1/T_c)dT_c/dq\,|_{q=0}$. We have estimated this quantity from a variety of numerical results (see Table I).[23]]

(ii) A low-temperature, spin-wave analysis of the XY version (see below) of the regularly diluted spherical model[3] indicates that the appropriate analog of the inequality (2) is violated (at least for sufficiently large dilution and $\tau\ll 1$).

(iii) We have used a variety[24] of Migdal-Kadanoff[5] calculations to obtain $R(\tau,q)$ for the model (1). In all these calculations, the inequality (2) is violated for all τ. This contradicts the spin-wave result for $q_1,q_2,\tau\ll 1$ given in (i) above. Thus, Migdal-Kadanoff approximations cannot be relied upon[25] to predict the dependence of $R(\tau,q)$ on q.

(iv) Our Monte Carlo results for $R(\tau,q)$ versus τ for

TABLE I. Values of B for various randomly diluted, three-dimensional spin models [our Monte Carlo calculation yields $B \equiv -(1/T_c)dT_c/dq \mid_{q=0} \simeq 1.13$]. We can obtain an upper bound on B for the classical, bond-dilute XY model on a simple-cubic lattice from the data given here as follows. (i) B(bond dilute) $< B$(site dilute): the removal of one site disorders the system more than the removal of a bond because, in the former case, more bonds (equal to the coordination number of the lattice) are broken. We belive that the argument presented here can be made rigorous [Ref. 23(b)]. (ii) $B(S=\infty) < B(S=\frac{1}{2})$: We can plausibly expect that it is easier to disorder a quantum model than a classical one. This expectation is borne out by numerical calculations for the $n=3$ Heisenberg model, but we have not been able to make the argument rigorous. If we combine the inequalities (i) and (ii) with the data given in the table, we obtain the bound $B < 1.22$ for the $S=\infty$, bond-dilute XY model on the simple-cubic lattice. This bound is certainly not rigorous, but is based on arguments that are very plausible. Also, it is consistent with our Monte Carlo estimate of $B \simeq 1.13$.

Lattice	S	n	Dilution	B
sc	$\frac{1}{2}$	2	site	1.22±0.02[a]
fcc	$\frac{1}{2}$	3	bond	1.16±0.01[b]
fcc	$\frac{1}{2}$	3	site	1.36±0.03[a]
fcc	∞	3	site	1.15±0.01[c]

[a]Reference 23(c).
[b]Reference 23(b).
[c]Reference 23(a).

$q=0$, 0.4, and 0.55 are shown in Fig. 2. The conclusion that can be drawn from this figure is that $R(\tau,q)$ is not a very sensitive function of q [unlike $\rho_s(T,\rho)/\rho_s(T=0,\rho)$, which changes substantially with ρ, as shown in Fig. 1(a)].[26] Unfortunately, the error bars on our data are so large that it is impossible to tell whether the inequality (2) is satisfied or violated.

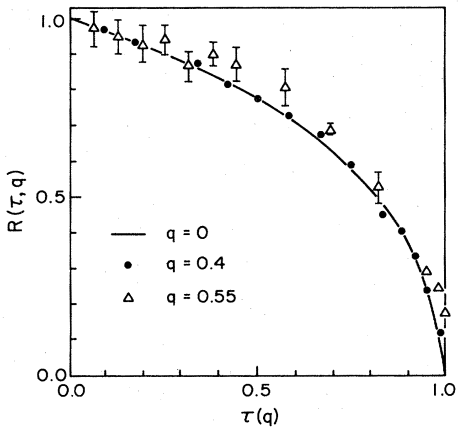

FIG. 2. Plots of the scaled helicity modulus $R(\tau,q) \equiv \Upsilon(T,q)/\Upsilon(T=0,q)$ vs the scaled temperature $\tau \equiv T/T_c(q)$ for the bond-dilute XY model for $q=0$ [$T_c(q)=2.208\pm0.005$ and $\Upsilon(T=0,q)=1$], $q=0.4$ [$T_c(q)=1.208\pm0.005$ and $\Upsilon(T=0,q)=0.405\pm0.004$], and $q=0.55$ [$T_c(q)=0.80\pm0.01$ and $\Upsilon(T=0,q)=0.178\pm0.004$]. Data are obtained from the Monte Carlo calculation described in the text. A solid line has been drawn through the $q=0$ data.

In the remainder of this paper we describe the spin-wave analysis and the Monte Carlo simulations referred to in (i), (ii), and (iv), respectively. We do not give any details of our Migdal-Kadanoff calculations[24] because they are completely unreliable for our purposes. We close with some concluding remarks on the relevance of our results to the superfluidity of ^{4}He in Vycor.

SPIN-WAVE ANALYSIS

Model (1)

We calculate $\Upsilon(T,q)$ by using a straightforward extension of a formula suggested by Rudnick and Jasnow[27] for pure spin systems, namely

$$\Upsilon(T,q) = \left[\frac{\partial^2 f(T,q;k_0)}{\partial k_0^2} \right]_{k_0=0} , \tag{5}$$

where $f(T,q;k_0)$ is the "quenched-averaged"[1] free energy density of the model (1) in its ordered state with a twist of wave vector k_0 in the order parameter (see below). We evaluate $f(T,q;k_0)$ and $\Upsilon(T,q)$ for low q and T via a classical version of the spin-wave approximation used by Banavar and Jasnow[28] for a pure [$q=0$ in Eq. 1(b)] quantum XY model. The strategy is simple: At low temperatures, and for a given configuration of present and missing bonds, we assume a "helical" state with $\langle s_i^x \rangle = M_0 \cos(\vec{k}_0 \cdot \vec{R}_i)$ and $\langle s_i^y \rangle = M_0 \sin(\vec{k}_0 \cdot \vec{R}_i)$, where M_0 is the magnitude of the magnetization per spin, x and y refer to the components of the spin, $\vec{R}_i$ is the position of the ith site, and the angular brackets denote a thermal average. The free energy density f is evaluated in the spin-wave approximation for each configuration, averaged over the distribution (1b), and differentiated as in Eq. (5) to obtain Υ.

It is convenient to introduce the variables

$$s_i^t = -s_i^x \sin(\vec{k}_0 \cdot \vec{R}_i) + s_i^y \cos(\vec{k}_0 \cdot \vec{R}_i)$$

and

$$s_i^n = s_i^x \cos(\vec{k}_0 \cdot \vec{R}_i) + s_i^y \sin(\vec{k}_0 \cdot \vec{R}_i) .$$

(Note that $\langle s_i^t \rangle = 0$ and $\langle s_i^n \rangle = M_0$.) We now make the spin-wave approximation,

$$s_i^n = 1 - \phi_i^2/2 + O(\phi_i^4), \quad s_i^t = \phi_i + O(\phi_i^3) ,$$

where $|\phi_i| \ll 1$ for low T. If we define the Fourier-transformed variables

$$\theta(\vec{k}) = (1/N)^{1/2} \sum_j \phi_j \exp(i\vec{k} \cdot \vec{R}_j) ,$$

where N is the total number of sites in the lattice, the spin-wave approximation to the Hamiltonian (1) may be written as

$$H_{SW} = C + \tfrac{1}{2} \sum_{\vec{k},\vec{k}'} \theta(\vec{k}) G^{-1}(\vec{k},\vec{k}') \theta(\vec{k}') - \sum_{\vec{k}} \theta(\vec{k}) V(\vec{k}) , \tag{6}$$

where

$$C \equiv -\sum_{\langle i,j \rangle} J_{ij} \cos(\vec{k}_0 \cdot \vec{R}_{ij}) , \tag{7}$$

$$G^{-1}(\vec{k},\vec{k}') \equiv (1/N) \sum_{\langle i,j \rangle} J_{ij} \cos(\vec{k}_0 \cdot \vec{R}_{ij}) u_{ij}(\vec{k}) u_{ij}^*(\vec{k}') , \tag{8}$$

$$V(\vec{k}) \equiv (1/N)^{1/2} \sum_{\langle i,j \rangle} J_{ij} \sin(\vec{k}_0 \cdot \vec{R}_{ij}) u_{ij}(\vec{k}) , \tag{9}$$

$$u_{ij} = e^{-i\vec{k}\cdot\vec{R}_i}(1 - e^{i\vec{k}\cdot\vec{R}_{ij}}) , \tag{10}$$

and $\vec{R}_{ij} \equiv \vec{R}_i - \vec{R}_j$. The total free energy of the model (6) follows by doing Gaussian integrals and is

$$F = C + (k_B T/2) \operatorname{Tr} \ln(G^{-1}/k_B T) - \tfrac{1}{2}(V^\dagger G V) + F_0 , \tag{11}$$

where F_0 contains only terms independent of k_0, k_B is the Boltzmann constant, and

$$V^\dagger G V \equiv \sum_{\vec{k},\vec{k}'} V^*(\vec{k}) G(\vec{k},\vec{k}') V(\vec{k}') .$$

(We shall use this condensed operator notation below.) By substituting Eq. (11) in Eq. (5), we obtain the unaveraged helicity modulus

$$\Upsilon_u = \lim_{N\to\infty} \left[(1/N) \sum_{\langle i,j \rangle}{}' J_{ij} - (k_B T/2N) \operatorname{Tr}(G G_x^{-1}) - (1/N)(V_x^\dagger G V_x) \right] , \tag{12}$$

where G is evaluated at $k_0 = 0$, the prime on the sum indicates that only bonds in the x direction enter the sum (this is because we have chosen, without any loss of generality, $\vec{k}_0 = k_0 \hat{x}$),

$$G_x^{-1}(\vec{k},\vec{k}') \equiv (1/N) \sum_{\langle i,j \rangle}{}' J_{ij} u_{ij}(\vec{k}) u_{ij}^*(\vec{k}') , \tag{13}$$

and

$$V_x(\vec{k}) \equiv (1/N)^{1/2} \sum_{\langle i,j \rangle}{}' J_{ij} u_{ij}(\vec{k}) . \tag{14}$$

We now average the helicity modulus Υ_u over the distribution (1b). Our result is accurate to $O(q)$ and is obtained by considering configurations in which all bonds are present ("zero impurity") and those in which only one bond is absent ("one impurity"). Note that the first term in Eq. (12) can be averaged directly and yields $1-q$. The second and third terms require more care and are best handled separately for the zero- and one-impurity cases.

Zero-impurity case

The Green function given by Eqs. (8) and (10) is (at $k_0 = 0$)

$$G_0^{-1}(\vec{k},\vec{k}') = f_{\vec{k}} \delta_{\vec{k},\vec{k}'} = (6 - 2\cos\vec{k}\cdot\hat{x} - 2\cos\vec{k}\cdot\hat{y} - 2\cos\vec{k}\cdot\hat{z}) \delta_{\vec{k},\vec{k}'} , \tag{15}$$

where the subscript 0 indicates that no bonds are missing, and we have set the lattice spacing equal to 1. From Eq. (13) we obtain

$$G_{x0}^{-1} \equiv f_{\vec{k}\cdot\hat{x}} \delta_{\vec{k},\vec{k}'} = (2 - 2\cos\vec{k}\cdot\hat{x}) \delta_{\vec{k},\vec{k}'} . \tag{16}$$

Lastly, $V_x = 0$ because no bonds are missing. Hence the zero-impurity contribution to the helicity modulus from the second and third terms of Eq. (12) is

$$\Upsilon_0 = (-k_B T/2N) \sum_{\vec{k}} f_{\vec{k}\cdot\hat{x}}/f_{\vec{k}} . \tag{17}$$

By symmetry, the sum over $\vec{k}$ is $N/3$ (N/d in d dimensions). Thus,

$$\Upsilon_0 = -k_B T/6 , \tag{18}$$

a result that must be multiplied by $(1-q)^{3N}$, the weight of the zero-impurity configuration.

One-impurity case

The Green function is obtained by combining Eqs. (8), (10), and (15),

$$G^{-1} = G_0^{-1} - (u_{12} u_{12}^\dagger/N) , \tag{19}$$

where we have assumed that the bond between sites 1 and 2 is missing. By using the identity

$$(u_{12}^\dagger G_0 u_{12})/N = (1/N) \sum_{\vec{k}} (2 - 2\cos\vec{k}\cdot\vec{a})/f_{\vec{k}} = \tfrac{1}{3}$$

[$\vec{a} \equiv \vec{R}_1 - \vec{R}_2$ and the sum is the same as the one in Eq. (17)], Eq. (19) can be rewritten as

$$(G u_{12})_{\vec{k}} = \tfrac{3}{2}(G_0 u_{12})_{\vec{k}} = 3u_{12}(\vec{k})/2f_{\vec{k}} , \tag{20a}$$

or

$$G = G_0 + (3/2N)(G_0 u_{12} u_{12}^\dagger G_0) . \tag{20b}$$

[In d dimensions, the factor $\frac{3}{2}$ is replaced by $d/(d-1)$.] G_x^{-1} follows from Eqs. (13) and (16),

$$G_x^{-1} = G_{x0}^{-1} - (u_{12} u_{12}^\dagger/N) \delta_{\vec{a},\hat{x}} , \tag{21}$$

where the second term contributes only if the missing bond is along the x direction (recall that $\vec{k}_0 = k_0 \hat{x}$). To average the second term in Eq. (12) we need $\operatorname{Tr}(G G_x^{-1})$. By combining Eqs. (19)–(21) we obtain

$$\operatorname{Tr}(G G_x^{-1}) = \sum_{\vec{k}} f_{\vec{k}\cdot\hat{x}}/f_{\vec{k}} + (3/2N) \sum_{\vec{k}} f_{\vec{k}\cdot\hat{x}} f_{\vec{k}\cdot\vec{a}}/f_{\vec{k}}^2 - (3/2N) \sum_{\vec{k}} |u_{12}(\vec{k})|^2 \delta_{\vec{a},\hat{x}}/f_{\vec{k}} . \tag{22a}$$

Since $\sum_{\vec{a}} f_{\vec{k}\cdot\vec{a}} = f_{\vec{k}}$, the contributions from the last two terms in the above equation cancel identically when we sum over the possible orientations of the missing bond. We are left with a sum of the type that appeared in Eq. (17), and so

$$\sum_{\vec{a}} \operatorname{Tr}(G G_x^{-1}) = 3N/3 = N . \tag{22b}$$

The last term in Eq. (12) is obtained by combining Eqs. (14), (19), and (20), and summing over the possible orientations of the missing bond, we obtain

$$\sum_{\vec{a}} V_x^{\dagger} G V_x = (1/N) \sum_{\vec{a}} (u_{12}^{\dagger} G u_{12}) \delta_{\vec{a},\hat{x}}$$
$$= (3/2N) \sum_{\vec{k},\vec{a}} (|u_{12}(\vec{k})|^2 / f_{\vec{k}}) \delta_{\vec{a},\hat{x}} = \tfrac{1}{2} . \quad (23)$$

Thus the one-impurity contribution to the last two terms in Eq. (12) is

$$\Upsilon_1 = -k_B T/2 - 1/2N . \quad (24)$$

Since we have already summed over the orientation of the missing bond at a fixed site, this term will have to be multiplied by the weight factor $N(1-q)^{(3N-1)}q$.

We now combine Eqs. (18) and (24) with their appropriate weights and $1-q$, the average of the first term in Eq. (12), to obtain[29]

$$\Upsilon(T,q) = 1 - 3q/2 - k_B T/6 + O(q^2, T^2) . \quad (25)$$

[Note that the coefficient of the qT term in this expansion is 0. Also, the corrections of $O(T^2)$ would require going beyond the spin-wave approximation.] Equation (25) can be rewritten as [see inequality (2)]

$$R(\tau,q) = 1 - A(q)\tau + O(\tau^2) , \quad (26)$$

where[29]

$$A(q) \equiv [k_B T_c(q=0)/6] \left[1 + \left[\frac{3}{2} + \frac{1}{T_c} \frac{dT_c}{dq} \bigg|_{q=0} \right] q + O(q^2) \right] . \quad (27)$$

The inequality (2) will hold at sufficiently low q if $A(q>0) > A(q=0)$, i.e., if

$$B \equiv -\frac{1}{T_c} \frac{dT_c}{dq} \bigg|_{q=0} < \frac{3}{2} . \quad (28)$$

To the best of our knowledge, B has not been calculated for the classical XY model on a three-dimensional simple-cubic lattice. Our Monte Carlo calculation indicates that $B \simeq 1.13$. We have also estimated B from its values for other models (see Table I) and find $B < 1.22$. Thus the inequality (2) should be satisfied by the model (1) for low q and T. (We have obtained similar results for the site-dilute, classical XY model.[30])

Regularly diluted XY model

A spin model on a lattice can be diluted regularly by "decorating" each bond between spins with l additional spins as shown in Fig. 3 for a square lattice.[3] The higher l is, the more dilute the system is.

We consider a classical XY model on such a three-dimensional, decorated simple-cubic lattice. The l spins decorating each bond form a one-dimensional chain. They can be integrated out directly without introducing non-nearest-neighbor couplings (as in any one-dimensional "decimation" transformation). However, the form of the interaction between nearest-neighbor spins is no longer as simple as it is in Eq. (1a). In spite of this complication, a spin-wave analysis can be used to obtain Υ for all l at low temperatures by using methods similar to the ones used to obtain Eq. (25) (of course, no configuration average is required). The analog of Eq. (25) is[29]

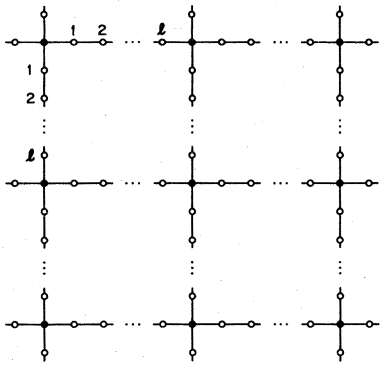

FIG. 3. Regularly diluted, two-dimensional square lattice. The spins at the vertices (solid circles) are connected via the l "decorating" spins (open circles). Only nearest-neighbor spins interact with one another as indicated by the lines or "bonds."

$$\Upsilon(l,T) = 1/(1+l)^2 - (1+3l)k_B T/6(1+l)^3 + O(T^2) , \quad (29)$$

which can be rearranged to give

$$R(\tau,l) \equiv \Upsilon(l,T)/\Upsilon(l,T=0) = 1 - D(l)\tau + O(\tau^2) , \quad (30)$$

where[29]

$$D(l) \equiv (1+3l)k_B T_c(l)/6(1+l) . \quad (31)$$

The analog of the inequality (2), namely $R(\tau,l_2) < R(\tau,l_1)$ for $l_1 < l_2$, is satisfied if $D(l) > D(0)$, which, by virtue of Eq. (31), requires

$$T_c(l)/T_c(l=0) > (1+l)/(1+3l) . \quad (32)$$

(In d dimensions, 3 is replaced by d.) The inequality (32) has to be violated for sufficiently large l, for $T_c(l) \to 0$ monotonically as $l \to \infty$. Thus, the regularly diluted XY model violates the analog of inequality (2).

MONTE CARLO RESULTS

Our Monte Carlo simulation of the model (1) is similar to the one of Ebner and Stroud[6] for a model of a granular superconductor. The only new feature of our calculation is that we use a phenomenological renormalization method[31] to evaluate $T_c(q)$. The helicity modulus is obtained by using the formula[32] [which can be derived from Eqs. (1) and (5)]

$$\Upsilon(T,q) = (1/N) \left\langle \sum_{\langle i,j \rangle}{}' J_{ij} \cos(\alpha_i - \alpha_j) \right\rangle - (1/N)(1/k_B T) \left\langle \left[\sum_{\langle i,j \rangle}{}' J_{ij} \sin(\alpha_i - \alpha_j) \right]^2 \right\rangle , \quad (33)$$

where $\alpha_i - \alpha_j$ is the angle between the spins at sites i and j, and the prime on the sum indicates that only bonds in

the x direction enter the sum. The thermal average is performed in the usual way,[33] and the average over the distribution (1b) is perfomed by evaluating Υ for several (typically 10) realizations of a lattice. We have done simulations on lattices of size $M\times M\times M$, with $M=5$, 8, and 10 for $q=0$, $M=8$, 10, and 12 for $q=0.4$, and $M=6$, 8, and 12 for $q=0.55$. In most cases we have used 5000 Monte Carlo steps per spin, although in a few difficult cases we have used 10 000.

To obtain $T_c(q)$, which is required for the scaled plot of Fig. 2, we use the phenomenological renormalization method described by Binder,[31(a)] and by Barber and Selke.[31(b)] We plot $1/M(|\psi|_M^2)^{-\nu/2\beta}$ versus T for different values of M, where

$$|\psi|_M^2=\left\langle\left|(1/N)\sum_j \exp(i\alpha_j)\right|^2\right\rangle ,$$

$N=M^3$, and the exponents $\beta=0.3455\pm0.002$ and $\nu=0.669\pm0.002$ are obtained from Ref. 34. (Note that $|\psi|_{M=\infty}^2\sim|T-T_c|^{2\beta}$ as $T\to T_c^-$.) The curves for two different values of M, for instance M_1 and M_2, intersect at a temperature $T(M_1,M_2)$. The temperatures $T(M_1,M_2)$ rapidly approach[35] the critical temperature for the infinite system as $M_1,M_2\to\infty$. By this method we obtain $T_c(q=0)=2.208\pm0.005$, which is in very good agreement with other calculations.[36] We also find $T_c(q=0.4)=1.208\pm0.005$ and $T_c(q=0.55)=0.80\pm0.01$.

In Fig. 2 we show plots of $R(\tau,q)$ versus τ for $q=0$, 0.4, and 0.55. For each value of q, our data are obtained from the largest lattice that we have simulated. The zero-temperature values of the helicity modulus are $\Upsilon(T=0,\ q=1)=1$, $\Upsilon(T=0,\ q=0.4)=0.405\pm0.004$, and $\Upsilon(T=0,\ q=0.55)=0.178\pm0.004$. The error bars are shown only for the case $q=0.55$, which yields our noisiest curve.[37] (A large fraction of this error comes from the average over the different configurations of the bonds.) Given our large error bars, it is impossible to decide whether the inequality (2) is satisfied or not. The only possible conclusion that we can draw is that the effect of dilution on model (1) is not as dramatic as for ^{4}He in Vycor [Fig. 1(a)];[26] however, even this conclusion is moot because we really do not have an explicit quantitative relation[18] between ρ in Fig. 1(a) and q in Fig. 2.

CONCLUDING REMARKS

We have investigated the effect of dilution on the helicity moduli of three-dimensional, simple-cubic, classical XY models. By considering configurations in which one bond is missing, we find, for the randomly diluted model (1) at sufficiently low dilution and temperature, that certain features of the helicity modulus [Fig. 1 and inequality (2)] are qualitatively similar to the behavior of the superfluid density of ^{4}He in Vycor. However, the helicity modulus of the regularly diluted model (Fig. 3) shows just the opposite behavior [see discussion in the vicinity of inequality (32)], at least for large dilution and at low temperatures, as does the helicity modulus of the regularly diluted *spherical* model of Fishman and Ziman.[3] It is tempting to conclude, therefore, that, in order to understand the superfluidity of ^{4}He in Vycor, it is crucial that the dilution be random. This should not come as a surprise because the structure of Vycor is far from regular, but some caution should be exercised in accepting such a conclusion for two reasons. (i) Our calculations are just not good enough to check whether model (1) satisfies the inequality (2) for large dilution. Unfortunately, the analog of inequality (2) for the experiments on ^{4}He in Vycor is satisfied unambiguously only at low densities of ^{4}He (i.e., large dilution).[17] Note that the lowest T_c in Fig. 1(a) is 2 orders of magnitude smaller than T_c for bulk ^{4}He. To obtain a comparable suppression of T_c in model (1), we need $q\gtrsim q_c\simeq0.753$, the percolation threshold for the simple-cubic lattice; however, Monte Carlo simulations in this regime are impractical. (ii) It is quite possible that the features of Fig. 1 that we have chosen to highlight are completely nonuniversal and depend in a complicated way on the geometry of Vycor. If this is the case, it is futile to use inequality (2) to decide whether randomly diluted spin models provide a better description of the superfluidity of ^{4}He in Vycor than regularly diluted ones or vice versa. Only more experiments on the superfluidity of ^{4}He in different porous media can decide how universal the trend embodied in inequality (2) is.

We have already noted that a model of a weakly interacting Bose gas, in which the strength of the interaction decreases with decreasing density of ^{4}He in Vycor, satisfies the appropriate analog of inequality (2) [see discussion in the vicinity of Eqs. (3) and (4)]. Such an approach is based on the assumption that Vycor allows us to observe a superfluid transition at low densities, but does not change the basic form of the Hamiltonian. Since Vycor provides, in all likelihood, a quenched random potential that affects the basic interactions, we feel that models which treat Vycor in such a crude "effective-medium" approximation must miss some features of this system. Rasolt and Stephen[22] have proposed a classical, n-component spin system as a model for a dilute, interacting Bose gas. The particle density, corresponding to the density of ^{4}He in Vycor, is controlled via the spherical-model constraint. Their treatment focuses on the crossover from ideal- to interacting-Bose-gas critical behavior and does not address the global trends in $\rho_s(T,\rho)$ discussed here. Their results are for $n=1$ (^{4}He corresponds to $n=2$), which leads to "Fisher renormalization" of the critical exponents[38] that characterize the superfluid transition in the interacting gas. It is not quite clear for what values of ρ and T the "effective-medium" approximation of Rasolt and Stephen[22] is valid.

For weak dilution, an argument due to Harris[39] shows that the critical exponents of a pure system are unchanged by a quenched random potential, if the specific-heat exponent $\alpha<0$ for the pure system. ($\alpha=-0.008\pm0.003$ for the XY model in three dimensions.[34]) As far as we know, there is no conclusive result for the effect of strong dilution on the critical exponents of an XY model in three dimensions. The Migdal-Kadanoff calculations we have done do not lead to new critical behavior even for strong dilution; however, it is not clear how much one should trust a Migdal-Kadanoff calculation in such matters, for it does not even yield a "random fixed point" for the three-dimensional Ising model (Kinzel and Domany[24]).

($\alpha \simeq 0.12 > 0$ for the three-dimensional Ising model, and thus on the basis of the Harris criterion,[39] new critical behavior, governed by a random fixed point, is expected.) At the moment it is not feasible to use Monte Carlo methods to obtain critical exponents for the dilute model (1). Even Monte Carlo renormalization-group calculations on the *pure XY* model in three dimensions have not succeeded in obtaining reliable values for critical exponents.[36(b)] The experiments are not yet at the stage where the critical exponents for the superfluid transition of ^{4}He in Vycor can be measured precisely.[14(d)]

Thus we feel that there are various open questions which need to be investigated, both theoretically and experimentally, before we can obtain a full understanding of the superfluidity of ^{4}He in Vycor, in particular, and in porous media in general.

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[1]For an overview, see S. Kirkpatrick, in *Ill Condensed Matter* (Les Houches, Session XXXI, 1978), edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979), p. 321; T. C. Lubensky, *ibid.*, p. 405.

[2]M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A 8, 1111 (1973).

[3]S. Fishman and T. A. L. Ziman, Phys. Rev. B 26, 1258 (1982).

[4]S. A. Solla and E. K. Riedel (unpublished); S. A. Solla, Ph. D. thesis, University of Washington, 1982.

[5]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys.—JETP 42, 743 (1976)]; L. P. Kadanoff, Ann. Phys. (N.Y.) 100, 359 (1976).

[6]C. Ebner and D. Stroud, Phys. Rev. B 28, 5053 (1983); 25, 5711 (1982).

[7]P. Shukla, J. Phys. C 11, L621 (1978) has used a Migdal-Kadanoff renormalization group to obtain percolation exponents by studying the zero-temperature limit of a classical, bond-dilute *XY* model in three dimensions. He also reports that such an approximate calculation does not yield very good results at finite temperature.

[8]D. D. Betts, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1974), Vol. 3, p. 578.

[9]M. E. Fisher, Rep. Prog. Phys. 30, 615 (1967); J. Stephenson, in *Physical Chemistry,* edited by D. Henderson (Academic, New York, 1971), Vol. VIII B.

[10]Vycor is a registered trademark of the Corning Glass Company.

[11]The typical size of a pore is 40–80 Å. For photographs of the structure of Vycor, see T. Takamori, J. Am. Ceram. Soc. 61, 434 (1978); D. D. Awschalom, Ph.D. thesis, Cornell University, 1983.

[12]We are thinking of a quantum lattice gas (see Ref. 8, pp. 574–577 for details) in which both the kinetic energy and potential energy terms can be random. Such a lattice gas maps onto a dilute, $S=\frac{1}{2}$ *XY* model in exactly the same way (Ref. 8) as the pure quantum lattice gas maps onto the pure, $S=\frac{1}{2}$ *XY* model. We study the bond-dilute model for simplicity (see Ref. 30).

[13]What counts as qualitative is, of course, a matter of opinion. One gross feature of ρ_s for bulk ^{4}He in three dimensions that Υ for a classical *XY* model does not mimic is the departure from the zero-temperature value; ρ_s falls off from its $T=0$ value as the fourth power of T, whereas Υ falls off linearly with T. A quantum *XY* model gives a T^4 behavior for Υ at low T (see Ref. 28).

[14]The superfluid density of ^{4}He in Vycor has been measured in many experiments. See, e.g., (a) E. N. Smith, D. J. Bishop, J. E. Berthold, and J. D. Reppy, J. Phys. (Paris) 39, C6-342 (1978); (b) D. J. Bishop, J. E. Berthold, J. M. Parpia, and J. D. Reppy, Phys. Rev. B 24, 5047 (1981); (c) B. C. Crooker, B. Hebral, E. N. Smith, Y. Takano, and J. D. Reppy, Phys. Rev. Lett. 51, 666 (1983); (d) B. C. Crooker, Ph.D. thesis, Cornell University, 1983.

[15]A similar criterion was used by Fishman and Ziman in Ref. 3.

[16]We are very grateful to Benjamin Crooker for making this composite plot. The data are from Ref. 14.

[17]From Fig. 1(a) it seems that the trend we are emphasizing reverses at high temperatures. However, there are reasons for believing that this might not be because of dilution alone. Rotons, for example, start contributing to the depletion of ρ_s around 1 K and they are certainly not accounted for in any simple spin model. Also, at high densities [such that $T_c(\rho) \gtrsim 800$ mK], when the coverage is more than a monolayer, the interpretation of the data is not as clean as it is at lower coverages.

[18]Strictly speaking, this is all we can expect from Fig. 1(a), for the trend we are emphasizing [inequality (2)] holds only for sufficiently small ρ for ^{4}He in Vycor. Also, it is not clear how to obtain an *explicit* relationship between q and ρ. An *implicit* relationship can be obtained by determining which values of q and ρ lead to the same T_c's for the model (1) and for ^{4}He in Vycor, respectively.

[19]A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), p. 495, Eq. (55.40), and p. 493, Eq. (55.36).

[20]This follows from G. Baym, in *Mathematical Methods in Solid State and Superfluid Theory* (Scottish Universities Summer School in Physics, 1967), edited by R. C. Clark and G. H. Derrick (Oliver and Boyd, Edinburgh, 1969), p. 153, Eq. (184).

[21]See, e.g., *Quantum Theory of Many-Particle Systems,* Ref. 19, p. 42.

[22]M. Rasolt and M. J. Stephen, preprint (1983).

[23]The data in the table are taken from (a) G. S. Rushbrooke, R. A. Muse, R. L. Stephenson, and K. Pirnie, J. Phys. C 5, 3371

(1972); (b) E. Brown, J. W. Essam, and C. M. Place, *ibid.* 8, 321 (1975); (c) J. S. Reeve and D. D. Betts, *ibid.* 8, 2642 (1975).

[24]The various calculations that we have done differ in the hierarchical lattices [see, e.g., T. W. Burkhardt, in *Real-Space Renormalization,* edited by T. W. Burkhardt and J. M. J. van Leeuwen (Springer, New York, 1982), pp. 33—56, especially p. 43, and references cited therein] we work on and the way in which we parametrize the renormalized probability distribution for the bonds [see, e.g., W. Kinzel and E. Domany, Phys. Rev. B 23, 3421 (1981), and references cited therein].

[25]Our spin-wave result for $\Upsilon(T,q)$ is *exact* to $O(T,q)$. To obtain $R(\tau,q)$ we need $B\equiv-(1/T_c)dT_c/dq\mid_{q=0}$ for which we only have an estimate (Table I). However, we believe our estimate is sound, and, coupled with the spin-wave calculation for $\Upsilon(T,q)$, yields a more trustworthy result than our Migdal-Kadanoff calculations. Also, for a pure spin system, a Migdal-Kadanoff calculation is *exact* on a hierarchical lattice (Ref. 24). The geometry of such a lattice is markedly different from that of a simple-cubic lattice. Thus, there really is no reason to believe that nonuniversal properties of the lattices should be even qualitatively similar.

[26]This will certainly not be true in the vicinity of $q_c\simeq0.753$, the percolation threshold for the simple-cubic lattice.

[27]J. Rudnick and D. Jasnow, Phys. Rev. B 16, 2032 (1977).

[28]J. R. Banavar and D. Jasnow, J. Phys. A 11, 1361 (1978).

[29]The d-dimensional analogs of Eqs. (25), (27), (29), and (31) are, respectively

$$\Upsilon(T,q)=1-qd/(d-1)-k_BT/2d+O(q^2,T^2)\ ,$$

$$A(q)\equiv[k_BT_c(q=0)/2d]\left[1+\left[d/(d-1)+\frac{1}{T_c}\frac{dT_c}{dq}\bigg|_{q=0}\right]q+O(q^2)\right]\ ,$$

$$\Upsilon(l,T)=1/(1+l)^2-(1+ld)k_BT/2d(1+l)^3+O(T^2)\ ,$$

and

$$D(l)\equiv(1+ld)k_BT_c(l)/2d(1+l)\ .$$

[30]For the site-dilute analog of model (1), in which q is the probability of a missing site, we find, in the spin-wave approximation [cf. Eqs. (25) and (27)]

$$\Upsilon(T,q)=(1-2.647q)[1-(k_BT/6)(1+1.647q)]+O(q^2,T^2)\ ,$$

$$A(q)\equiv[k_BT_c(q=0)/6]\left[1+\left[1.647+\frac{1}{T_c}\frac{dT_c}{dq}\bigg|_{q=0}\right]q+O(q^2)\right]\ .$$

(The numbers 2.647 and 1.647 are accurate to four significant figures and are obtained by the numerical evaluation of some integrals.) Thus, if $B<1.647$ for the site-dilute, $S=\infty$ model, which should be the case because $B=1.22\pm0.005$ for the site-dilute, $S=\frac{1}{2}$ XY model, and $B(S=\infty)<B(S=\frac{1}{2})$ (Table I), then the inequality (2) should be satisfied (for low q and T) even for the site-dilute version of model (1). The calculation of $\Upsilon(T,q)$ for the site-dilute problem is considerably more tedious than the one for the bond-dilute problem. Since the energy is independent of the missing spin at site 1, for instance, we must not integrate over the angle ϕ_1 corresponding to it in evaluating the partition function. We ensure this by inserting a δ function $\delta(\phi_1)$ and integrating over all angular degrees of freedom. Using the Fourier representation of the δ function, we are left with an integrand that is the exponential of a quadratic form in the ϕ_i's similar to Eq. (6). The evaluation of G requires the inversion of a 6×6 matrix. (In a simple-cubic lattice, six bonds are broken when a site is removed.) This matrix is singular; however, the calculation can be controlled by assuming that the strength of the six bonds emanating from a site is multiplied by a factor ϵ which is finally taken to zero at the end of the calculation.

[31](a) K. Binder, Phys. Rev. Lett. 47, 693 (1981); Z. Phys. B 43, 119 (1981); (b) M. N. Barber and W. Selke, J. Phys. A 15, L617 (1982), and references cited therein.

[32]See S. Teitel and C. Jayaprakash [Phys. Rev. B 27, 598 (1983)] for a similar calculation of Υ for the fully frustrated XY model in two dimensions.

[33]K. Binder, in *Monte Carlo Methods in Statistical Physics,* edited by K. Binder (Springer, New York, 1979), p. 1.

[34]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B 21, 3976 (1980).

[35]V. Privman and M. E. Fisher, J. Phys. A 16, L295 (1983), and references cited therein.

[36](a) M. Ferer, M. A. Moore, and M. Wortis, Phys. Rev. B 8, 5205 (1973); (b) R. H. Swendsen, *ibid.* 27, 391 (1983).

[37]There is also a small error in our τ coordinate (±0.002 for $q=0$, ±0.004 for $q=0.4$, and ±0.01 for $q=0.55$) because of our calculation of $T_c(q)$. Since this error is much smaller than that in $R(\tau,q)$, we have not shown it in Fig. 2.

[38]M. E. Fisher, Phys. Rev. 176, 257 (1968). If $\alpha<0$ there is no renormalization; if $\alpha>0$ all exponents are renormalized thus: $\beta\rightarrow\beta/(1-\alpha)$, $\gamma\rightarrow\gamma/(1-\alpha)$, $\alpha\rightarrow-\alpha/(1-\alpha)$, and all other exponents are obtained by assuming that the usual scaling laws hold for the renormalized exponents. The renormalization of the exponents depends sensitively on the number of components n of the spins. In three dimensions, α is $\simeq0.12$ for the Ising model ($n=1$) and -0.008 ± 0.003 for the XY model ($n=2$). Thus, the Fisher renormalization is appreciable for the Ising model, but does not take place for the XY model because $\alpha<0$.

[39]A. B. Harris, J. Phys. C 7, 1671 (1974).