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Crystallization of the incompressible quantum-fluid state of a two-dimensional electron gas in a strong magnetic field

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The energy of the incompressible quantum-fluid state proposed by Laughlin for electrons in the lowest Landau level of a two-dimensional electron gas [Phys. Rev. Lett. **50**, 1395 (1983)] is compared with that of the lattice state. It is found that the lattice state has lower energy for Landau-level filling factors $\nu < \frac{1}{9}$. An accurate parametrization is given for the dependence of the energy of Laughlin's state on the Landau-level filling factor.

Theoretical work on the fractional quantum Hall effect[1] has uncovered a connection between the quantum system of electrons in a strong magnetic field and the classical two-dimensional one-component plasma (2DOCP). The many-body wave functions (ψ_m, $m=1,3,5,\ldots$) proposed by Laughlin[2] for the quantum system have, in the limit of large numbers of electrons, a uniform density $n_m(\vec{r}) = (2\pi a_L^2 m)^{-1}$, and a pair-correlation function identical to that of the 2DOCP with ion-disk radius $a_m = a_L\sqrt{2m}$ and plasma parameter $\Gamma_m = 2m$. [$a_L \equiv (\hbar c/eH)^{1/2}$ is the magnetic length.] The 2DOCP has a crystallization transition which occurs at $\Gamma \simeq 140$.[3] An analogous "transition" must occur for the quantum problem since for some value of $\Gamma = 2m$, the charge-density wave[4] (CDW) state of the quantum system is expected to become lower in energy than Laughlin's fluid state. In this Rapid Communication we report accurate values for the energies of both Laughlin's state and, in the Hartree-Fock approximation, for the CDW state. We find that for $m=3$, 5, 7, and 9 the fluid state has lower energy, while for $m \geq 11$ the CDW state is preferred. The "freezing transition" thus occurs for $\Gamma \sim 20$, well before the corresponding transition in the classical system.

The energy per electron for Laughlin's state ψ_m, ϵ_L^m, is related to the pair-correlation function of the 2DOCP by[2]

$$\epsilon_L^{\Gamma/2} = \frac{e^2}{a_L\sqrt{\Gamma}} \int_0^\infty dx\,[g^\Gamma(xa) - 1] \ , \tag{1}$$

where $g^\Gamma(r)$ is the pair-correlation function. We have evaluated $g^\Gamma(r)$ for a series of values of Γ in the relevant range using the methods described in Ref. 3 with up to 256 particles and generating as many as 5 million configurations. This function is plotted in Fig. 1 for $\Gamma=6$ and for $\Gamma=10$, corresponding to the two highest density nontrivial Laughlin states. The pair-correlation function for $\Gamma=2$ corresponds to a wave function consisting of the single Slater determination for a full Landau level in the quantum system. This connection makes the exact result[5] for $g^\Gamma(r)$ at $\Gamma=2$ in the 2DOCP obvious. Girvin[6] has recently suggested an approximation for $g^\Gamma(r)$ motivated by the form of ψ_m for other values of Γ and we have compared this with his expression in Fig. 1. The approximation he uses is expected to deteriorate at larger Γ values, but even for small Γ his pair correlation functions are not particularly good, underestimating the first peak height and for $\Gamma=10$, missing the second peak entirely. Nevertheless, they are constrained to satisfy the charge neutrality sum rule and, as we see below, they yield accurate values for $\epsilon_L^{\Gamma/2}$.

In the Hartree-Fock approximation, the energy of the CDW state may be written as[4]

$$\epsilon_{\rm CDW} = \frac{e^2}{2a_L} \sum_{\vec{G}} |\Delta(\vec{G})|^2 \left[\frac{\exp(-G^2a_L^2/2)}{Ga_L}(1-\delta_{\vec{G},0}) - \left(\frac{\pi}{2}\right)^{1/2} \exp(-G^2a_L^2/4) I_0(G^2a_L^2/4) \right], \tag{2}$$

where the sum over $\vec{G}$ is over the reciprocal-lattice vector of a hexagonal lattice CDW with one electron per unit cell. In Eq. (2) the $\Delta(\vec{G})$, which are related to the Fourier

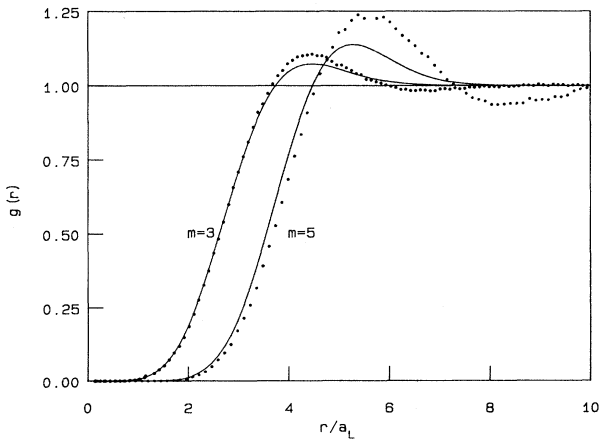

FIG. 1. Pair correlation function, $g(r)$, of the 2DOCP at $m=3$ ($\Gamma=6$, $\nu=\frac{1}{3}$) and $m=5$ ($\Gamma=10$, $\nu=\frac{1}{5}$). The magnetic length, a_L, is related to the ion-disk radius, a, of the plasma by $a=\sqrt{2m}\,a_L$. The solid lines are Girvin's approximate pair correlation functions and the dots are the result of our Monte Carlo study.

components of the electron density by $n(\vec{G}) = N_L \exp(-G^2 a_L^2/4)\Delta(\vec{G})$, where N_L is the number of states per Landau level, must be determined by a self-consistent calculation. For the interaction energy alone the preferred state is a lattice of point electrons for which the classical energy is[7]

$$\epsilon_{\text{lat}} = -0.782\,133 \frac{e^2}{a_L}\sqrt{\nu} \ , \tag{3}$$

where the Landau-level filling factor $\nu = 2\pi a_L^2 \bar{n}$ and $\bar{n}$ is the average electron density. In the CDW state the degree to which the electrons are localized around lattice sites is limited by the cost in kinetic energy which, in our strong magnetic field case, takes the form of limiting the electron density at a lattice site to the value $(2\pi a_L^2)^{-1} = \bar{n}/\nu$, corresponding to a full Landau level. To the accuracy of our calculations, which were performed using methods described elsewhere[8] and retained all $\vec{G}$ vectors with $G \lesssim 10 a_L^{-1}$, we find that the electron density on the lattice sites attains this value and thus ϵ_{CDW} approaches ϵ_{lat} for $n(\vec{R})/\bar{n} = \nu^{-1} >> 1$.

In Fig. 2 we compare the energies of fluid and lattice states of the system by plotting $\nu^{-1}(\epsilon_L - \epsilon_{\text{lat}})(e^2/a_L)$ and $\nu^{-1}(\epsilon_{\text{CDW}} - \epsilon_{\text{lat}})(e^2/a_L)$. (Note that the only element causing uncertainty in ϵ_{CDW} comes from the truncation of the sum over reciprocal lattice vectors.) These sums can readily be extended so that errors in $\epsilon_{\text{CDW}} - \epsilon_L$ are dominated by uncertainties in the latter quantity. Accurate values for ϵ_L were calculated at $\Gamma = 5$, 6, 10, 20, 30, and 40 and the estimated uncertainties in these quantities are indicated in Fig. 1.[9] The values obtained and the exact value available at $\Gamma = 2$ can be accurately fitted by the form[10]

$$\epsilon_L \simeq -0.782\,133 \frac{e^2}{a_L}\sqrt{\nu}(1 - 0.211\nu^{0.74} + 0.012\nu^{1.7}) \ . \tag{4}$$

(See Fig. 2.) The exponents appearing here, particularly that of the leading correction to the classical lattice energy, are suggested by fits to the OCP energies in two and three dimensions. The CDW state energies cannot be fitted accurately to a form similar to that of Eq. (4) for the full range $0 \leq \nu \leq 1$ but $\nu \leq \frac{1}{3}$ are faithfully represented by

$$\epsilon_{\text{CDW}} \simeq -0.782\,133 \frac{e^2}{a_L}\sqrt{\nu}(1 - 0.372\nu - 0.013\nu^2) \ . \tag{5}$$

(See Fig. 2.) It is the different exponents for the leading correction terms in Eqs. (4) and (5) which eventually lead to a preference for the CDW state and, in this respect, the situation is quite similar to that in the classical 2DOCP. For the quantum system, however, the crossover occurs for $\Gamma \simeq 20$ rather than for $\Gamma \simeq 140$.

We have also illustrated in Fig. 2 the estimates of the Laughlin state energy which result from Girvin's scheme for estimating the pair correlation of the 2DOCP. The accuracy of these estimates is impressive. For example at $\Gamma = 6$ and 10 ($\nu = \frac{1}{3}$ and $\frac{1}{5}$), his scheme gives $\epsilon/(e^2/a_L) = -0.4120$ and -0.3261, respectively. These values are considerably closer to the essentially exact results reported here, $\epsilon/(e^2/a_L) = -0.4100 \pm 0.0001$ at $\Gamma = 6$ and $\epsilon/(e^2/a_L) = -0.3277 \pm 0.0002$ at $\Gamma = 10$, than those determined by

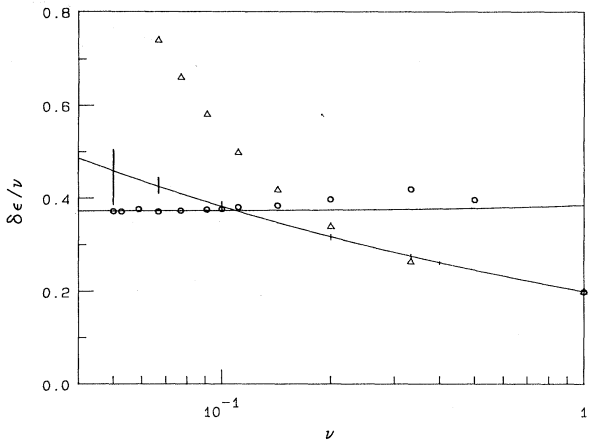

FIG. 2. Energy per electron in units of e^2/a_L for electrons of density $\bar{n} = \nu(2\pi a_L^2)^{-1}$. $\delta\epsilon$ is the energy difference between incompressible quantum fluid or CDW states and a classical triangular lattice of point electrons. The solid lines are the fits to our calculated values [Eqs. (3) and (4)]. For the fluid states the vertical lines indicate the uncertainty in the energy determined from the Monte Carlo study of the 2DOCP. The circles are values obtained from our CDW calculations. The triangles result from Girvin's scheme for estimating the energy of the fluid state.

Laughlin with a numerical solution of the hypernetted-chain approximation equations, -0.4156 ± 0.0012 at $\Gamma = 6$ and -0.3340 ± 0.0028 at $\Gamma = 10$. However, the estimates from Girvin's scheme do deteriorate sufficiently at larger values of Γ, so that they are not useful for determining the value of Γ at which "crystallization" occurs. [Using Girvin's estimates the CDW state would already be more stable for $\Gamma = 14$ ($\nu = \frac{1}{7}$).] Girvin's scheme can also be used to evaluate energies for generalizations of Laughlin's trial wave functions, and our results suggest that such estimates should be reliable.

In closing, we note that for both fluid and lattice states, the energy estimates of the quantum system are based on trial wave functions. For Laughlin's states numerical evidence from small system calculations suggests that the energies should be close to the true ground-state energies. For the CDW states, we have used a Hartree-Fock approximation. The variational basis of this approximation suggests that these energies should also be accurate. Furthermore, calculations[11] have shown that, at least near $\nu = \frac{1}{3}$, the ratio of crystal and fluid energies is changed little when the Coulomb interaction is replaced by a more realistic model of the effective interaction for inversion layer electrons. Thus we expect that the predicted value of ν for the transition to a CDW state is a realistic one. The best hope for confirmation of this prediction would seem to be in observing the absence of an otherwise expected fractional quantum Hall effect at very low electron density or very strong magnetic field.

Note added in proof: Lam and Girvin[12] have recently estimated correlation corrections to the lattice-state energy. If it is assumed that liquid-state energies obtained using Laughlin's trial wave function have negligible errors, this shifts the crystallization point to new $\sim \frac{1}{6}$.

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