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Onset of Superconductivity in One-Dimensional Systems*

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Time-dependent Ginzburg-Landau theory is used to investigate the resistive transition in a "one-dimensional" superconductor as a function of temperature and current through the wire. A diagrammatic expansion, in powers of the interaction between fluctuations, is described for the electrical conductivity. The conductivity is then calculated using a Hartree-Fock approximation for the interaction. When large currents flow through the wire, an unstable region is found near the depressed critical temperature, suggesting possible hysteresis effects which may have been observed in recent experiments.

I. INTRODUCTION

There has been a great deal of interest in investigating fluctuation effects in superconductors, and recently some of this interest has centered on attempts to obtain a detailed understanding of the resistive transition in samples which are effectively one dimensional. Such systems consist of tiny whisker crystals of superconducting material with cross-sectional dimension d much smaller than $\xi(T)$, the Ginzburg-Landau coherence length.

The original microscopic calculations by Aslamazov and Larkin[1] described the additional conductivity due to the presence of fluctuating pairs at temperatures $T > T_c$, and in the limit of small electric field, for systems of one, two, and three dimensions. Later several authors[2] obtained essentially the same results from linearized time-dependent Ginzburg-Landau (TDGL) theory, and succeeded in generalizing expressions for the excess conductivity to include the case of finite electric field.[3] All of these calculations were intended to describe the effects of thermally fluctuating pairs on the properties of the normal state at temperatures sufficiently above T_c that interaction between fluctuations could be neglected. The TDGL equation in this case is linear in the order parameter.

A theory of intrinsic resistance in one-dimensional systems below the superconducting transition, in which the (repulsive) interaction between Cooper pairs plays a dominant role, was proposed by Langer and Ambegaokar[4] (LA) and developed in detail by McCumber and Halperin[5] (MH). These theories assume the system to be initially in one of the metastable current-carrying states obtained as solutions to the equilibrium Ginzburg-Landau equations. Resistance is then due to thermal fluctuations which cause a transition from this metastable state to one with a smaller current, and depends primarily on the free-energy barrier for such a transition. In MH the time-dependent theory was used to obtain a detailed prediction for these transition rates. The LA-MH theory is based on the assumption of a large free-energy barrier to such transitions, and is therefore inapplicable at or above $T_c(I)$, the Ginzburg-Landau transition temperature at the given current I.

An interesting new idea was proposed by Marčelja[6] and elaborated by Masker, Marčelja, and Parks[7] (MMP) in an attempt to extend the applicability of linearized TDGL theory throughout the entire transition region. The essence of their scheme was to include the fourth-order term describing interaction between fluctuations in what may be characterized as a self-consistent Hartree approximation. The resulting theory proposed a prediction for the entire resistive transition in the limit of small current for systems of one, two, and three dimensions and was in essential agreement with the earlier theories in the "Aslamazov-Larkin" region above T_c. However, although the first experiments on films[7] and whiskers[8] indicated the MMP result to be of roughly the correct shape and width, the one-dimensional version of the theory was in fundamental disagreement with the LA-MH prediction in the asymptotic region below T_c. The most recent experimental evidence on whiskers[9] strongly supports the general form of the LA-MH result in this tail region, and we must therefore conclude that MMP theory at best becomes invalid at a temperature somewhat below T_c in one dimension.

The present paper is an attempt to extend the linearized TDGL theory describing the onset of the transition to lower temperature on a more systematic basis than MMP. In Sec. II a nonequilibrium perturbation theory based on the TDGL equation is developed to describe the initial effects of interaction between fluctuations in the presence of a finite electric field. The results are then utilized in Sec. III to formulate a self-consistent Hartree-Fock (HF) theory of the resistive transition in one-dimensional systems carrying finite current. In Sec. IV this HF theory is compared with the MH result and with MMP and its finite current generalization for various values of mean free path and current.

Throughout this paper we shall assume the validity of (nonlinear) TDGL theory as a basis for calculating the electrical conductivity of a superconductor. We shall not consider here the difficult theoretical questions involved in this assumption.

II. NONEQUILIBRIUM PERTURBATION THEORY

In this section we present a perturbation theory, based on the TDGL equation, for a system of arbitrary "dimensionality" in the presence of a uniform electric field.

The standard Ginzburg-Landau expressions for the free energy and supercurrent density of a superconductor are

$$F(\Psi, T)=\int d^3r\left[a\left|\Psi(\vec{r})\right|^2+\frac{b}{2}\left|\Psi(\vec{r})\right|^4+\delta\left|\left(\frac{1}{i}\vec{\nabla}+\frac{2e\vec{A}}{\hbar c}\right)\Psi(\vec{r})\right|^2\right] , \quad (2.1)$$

$$\vec{j}_s(\vec{r})=-\frac{2e\delta}{\hbar}\left[\Psi^*(\vec{r})\left(\frac{1}{i}\vec{\nabla}+\frac{2e\vec{A}}{\hbar c}\right)\Psi(\vec{r})+\text{c.c.}\right] , \quad (2.2)$$

where $a=a_0(T-T_c)/T_c$ and a_0, b, and δ are temperature independent. These parameters may be related to the properties of the conduction band through the microscopic derivation of Ginzburg-Landau theory.[10] The results, together with expressions for the various scaling parameters which we shall introduce, are summarized in Appendix A.

In the present paper, we interpret (2.1) as the free energy of the conduction electrons taken over the restricted ensemble of states which possess a given configuration of the complex order parameter $\Psi(\vec{r})$. In the superconductor, this order parameter corresponds to the wave function for the condensed pairs. We assume that $\Psi(\vec{r})$ can be treated as a classical field, and that (2.1) and (2.2) are valid for any instantaneous configuration of the order parameter. The probability density for finding a particular configuration $\Psi(\vec{r})$ in a system in thermal equilibrium is then given by

$$\rho=(1/Z)e^{-F(\Psi,T)/k_BT} , \quad (2.3)$$

where the partition function is $Z=\text{Tr}\,e^{-F/k_BT}$. [The trace here is over the configurations of the order parameter, and is realized as a functional integration over the real and imaginary parts of $\Psi(\vec{r})$.] All properties of a superconductor in equilibrium near its transition temperature may in principle be determined from the probability density (2.3).

In a one-dimensional model we do not consider fluctuations across the sample; Ψ is a function of only a single spatial variable denoting position along the wire. The equilibrium properties of the one-dimensional Ginzburg-Landau model constitute an "exactly soluble" problem. Using well-known techniques developed for other one-dimensional systems,[11] one can express the partition function Z, and all equal time correlation functions such as $\langle\Psi^*(r)\Psi(r')\rangle$, $\langle\Psi^*(r)\Psi^*(r')\Psi(r'')\Psi(r''')\rangle$, etc., in terms of the eigenvalues and eigenfunctions of the Schrödinger equation for a two-dimensional quartic oscillator with potential $aR^2+\frac{1}{2}bR^4$ and kinetic energy $-\frac{1}{4}\delta(k_BT/\delta d^2)^2\nabla^2_R$. In particular, for a long sample the free energy per unit volume $(-\ln Z/\Omega k_BT)$ is just equal to the ground-state energy of the oscillator, and $\langle|\Psi(r)|^2\rangle$ is the derivative of this quantity with respect to a, or the expectation value of R^2 in the ground state of the oscillator. We shall make no use of these exact solutions here, but we remark that there is no true phase transition in one dimension: The thermodynamic functions are analytic at all temperatures above zero, and there is no true long-range order at any temperature.

If the position variable $\vec{r}$ is treated as two- or three-dimensional, the properties of the equilibrum model are no longer exactly soluble. In addition, the model contains ultraviolet divergences which must be removed by introducing a short-wavelength cutoff and then renormalizing the Ginzburg-Landau parameters as one lets the cutoff wavelength tend towards zero.[7]

In order to study nonequilibrium phenomena we must have a description of the way in which the system evolves in time. TDGL theory predicts that, for small and slowly varying fluctuations of the order parameter, relaxation toward the configuration of minimum free energy is described by the equation

$$\hbar\gamma\left(\frac{\partial}{\partial t}-\frac{2ieV}{\hbar}\right)\Psi(\vec{r},t) = -\left[a+b|\Psi(\vec{r},t)|^2+\delta\left(\frac{1}{i}\vec{\nabla}+\frac{2e\vec{A}}{\hbar c}\right)^2\right]\Psi(\vec{r},t)\,, \tag{2.4}$$

where $V(\vec{r},t)$ is the electrochemical potential.[2,3,5,7,12] The result of microscopic theory for the additional parameter γ is given by (A8). In the absence of a scalar potential, (2.4) assumes the purely dissipative form:

$$\hbar\gamma\frac{\partial\Psi(\vec{r},t)}{\partial t}=-\frac{\delta F(\Psi,T)}{\delta\Psi^*(\vec{r},t)}\,. \tag{2.5}$$

We must add to this equation a Langevin noise term $f(\vec{r},t)$ to describe the effect of spontaneous thermal fluctuations associated with this dissipation. The simplest assumption is that $f(\vec{r},t)$ is a complex Gaussian stochastic variable with autocorrelation function:

$$\begin{aligned}&\langle f(\vec{r},t)f^*(\vec{r}\,',t')\rangle=2\hbar\gamma k_BT\delta(\vec{r}-\vec{r}\,')\delta(t-t')\,,\\&\langle f(\vec{r},t)f(\vec{r}\,',t')\rangle=0\,.\end{aligned} \tag{2.6}$$

This choice of noise source assures that the equilibrium distribution of Ψ, for $V=0$ and $\vec{A}$ independent of time, obtained from the TDGL equation will coincide with the canonical distribution (2.3).

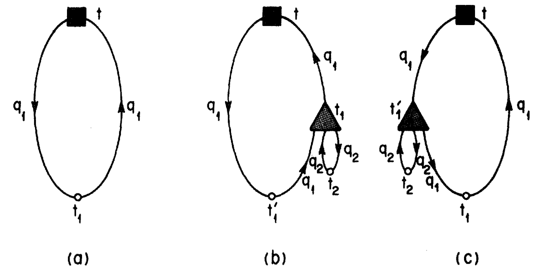


FIG. 1. Diagrams representing the zero- and first-order contributions to $\langle|\Psi_{\vec{q}_1}(t)|^2\rangle$ in powers of the interaction between fluctuations.

Consider now a superconductor in the presence of a constant electric field, and choose the gauge

$$V=0\,,\quad \vec{A}=-c\vec{E}t\,. \tag{2.7}$$

In this paper, we shall neglect the possibility of fluctuations in these potentials.[5] The full nonlinear TDGL equation with Langevin noise term added then becomes, after Fourier transformation,

$$\hbar\gamma\frac{d}{dt}\Psi_{\vec{q}_1}(t)=-\left[\left(a+\delta\left|\vec{q}_1-\frac{2e\vec{E}}{\hbar}t\right|^2\right)\Psi_{\vec{q}_1}(t)+\frac{b}{\Omega}\sum_{\vec{q}_2\vec{q}_3\vec{q}_4}\Psi^*_{\vec{q}_2}(t)\Psi_{\vec{q}_3}(t)\Psi_{\vec{q}_4}(t)\delta_{\vec{q}_1+\vec{q}_2,\vec{q}_3+\vec{q}_4}\right]+f_{\vec{q}_1}(t)\,, \tag{2.8}$$

where Ω is the volume of the system, and we have taken

$$\Psi(\vec{r},t)=\Omega^{-1/2}\sum_{\vec{q}}\Psi_{\vec{q}}(t)e^{i\vec{q}\cdot\vec{r}}\,. \tag{2.9}$$

The transform of (2.6) then becomes

$$\begin{aligned}&\langle f_{\vec{q}}(t)f^*_{\vec{q}\,'}(t')\rangle=2\hbar\gamma k_BT\delta_{\vec{q},\vec{q}\,'}\delta(t-t')\,,\\&\langle f_{\vec{q}}(t)f_{\vec{q}\,'}(t')\rangle=0\,.\end{aligned} \tag{2.10}$$

In order to solve the TDGL equation (2.8) we first consider the Green's function K^0 for the linear part of the equation:

$$\left(\hbar\gamma\frac{\partial}{\partial t}+a+\delta\left|\vec{q}-\frac{2e\vec{E}}{\hbar}t\right|^2\right)K^0(\vec{q},t,t')=\delta(t-t')\,. \tag{2.11}$$

The solution of (2.11) is

$$K^0(\vec{q},t,t')=\begin{cases}0 & \text{for } t<t'\\ \dfrac{1}{\hbar\gamma}\exp\left[-\dfrac{1}{\hbar\gamma}\displaystyle\int_{t'}^{t}dt_1\left(a+\delta\left|\vec{q}-\frac{2e\vec{E}}{\hbar}t_1\right|^2\right)\right] & \text{for } t>t'\,.\end{cases} \tag{2.12}$$

From this form we see immediately that K^0 can be factored in a manner characteristic of a linear time development operator:

$$K^0(\vec{q}, t, t') = \hbar\gamma K^0(\vec{q}, t, t'')K^0(\vec{q}, t'', t') \tag{2.13}$$

for $t' < t'' < t$. This property will be of use in subsequent calculations. The time integration in (2.12) may be carried out explicitly, and we have finally

$$K^0(\vec{q}, t, t') = \begin{cases} 0 & \text{for } t < t' \\ \dfrac{1}{\hbar\gamma} \exp\left\{-\dfrac{(t-t')}{\hbar\gamma}\left[a + \delta\left|\left(\vec{q} - \dfrac{2e\vec{E}}{\hbar}t\right) + \dfrac{e\vec{E}}{\hbar}(t-t')\right|^2 + \dfrac{\delta}{3}\left(\dfrac{eE}{\hbar}(t-t')\right)^2\right]\right\} & \text{for } t > t' . \end{cases} \tag{2.14}$$

Notice that this Green's function depends on two time variables and not simply on their difference. This is the result of the time dependence arising from our choice of gauge (2.7), and will be eliminated at the end of the calculation.

In terms of the Green's function for the linear operator, a formal solution of (2.8) is given by

$$\Psi_{\vec{q}_1}(t) = \int_{-\infty}^{t} dt_1 K^0(\vec{q}_1, t, t_1)\left(f_{\vec{q}_1}(t_1) - \frac{b}{\Omega}\sum_{\vec{q}_2\vec{q}_3\vec{q}_4} \Psi^*_{\vec{q}_2}(t_1)\Psi_{\vec{q}_3}(t_1)\Psi_{\vec{q}_4}(t_1)\delta_{\vec{q}_1+\vec{q}_2, \vec{q}_3+\vec{q}_4}\right) . \tag{2.15}$$

A perturbation theory in powers of the interaction may now be generated by successive iteration of this equation. Iterating once yields

$$\Psi_{\vec{q}_1}(t) = \int_{-\infty}^{t} dt_1 K^0(\vec{q}_1, t, t_1)\left[f_{\vec{q}_1}(t_1) - \frac{b}{\Omega}\sum_{\vec{q}_2\vec{q}_3\vec{q}_4} \delta_{\vec{q}_1+\vec{q}_2, \vec{q}_3+\vec{q}_4} \prod_{i=2}^{4}\left(\int_{-\infty}^{t_1} dt_i K^0(\vec{q}_i, t_1, t_i)\right) f^*_{\vec{q}_2}(t_2) f_{\vec{q}_3}(t_3) f_{\vec{q}_4}(t_4)\right] + O(b^2) . \tag{2.16}$$

Multiplying this expression by its complex conjugate and retaining only terms to first order in the interaction gives

$$\begin{aligned} \langle|\Psi_{\vec{q}_1}(t)|^2\rangle = &\int_{-\infty}^{t} dt_1 K^0(\vec{q}_1, t, t_1)\int_{-\infty}^{t} dt_1' K^0(\vec{q}_1, t, t_1') \\ &\times\left[\langle f_{\vec{q}_1}(t_1) f^*_{\vec{q}_1}(t_1')\rangle - \frac{b}{\Omega}\sum_{\vec{q}_2\vec{q}_3\vec{q}_4} \delta_{\vec{q}_1+\vec{q}_2, \vec{q}_3+\vec{q}_4} \prod_{i=2}^{4}\left(\int_{-\infty}^{t_1} dt_i K^0(\vec{q}_i, t_1, t_i)\right)\langle f^*_{\vec{q}_1}(t_1') f^*_{\vec{q}_2}(t_2) f_{\vec{q}_3}(t_3) f_{\vec{q}_4}(t_4)\rangle\right. \\ &\left. - \frac{b}{\Omega}\sum_{\vec{q}_2\vec{q}_3\vec{q}_4} \delta_{\vec{q}_1+\vec{q}_2, \vec{q}_3+\vec{q}_4} \prod_{j=2}^{4}\left(\int_{-\infty}^{t_1'} dt_j K^0(\vec{q}_j, t_1', t_j)\right)\langle f_{\vec{q}_1}(t_1) f_{\vec{q}_2}(t_2) f^*_{\vec{q}_3}(t_3) f^*_{\vec{q}_4}(t_4)\rangle\right] + O(b^2) . \end{aligned} \tag{2.17}$$

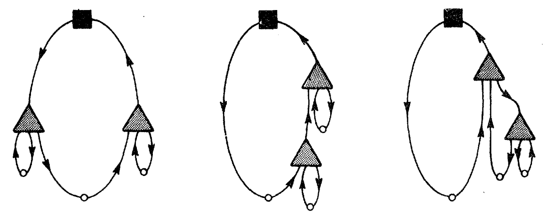
Using (2.10), the zero-order term in the above expression is seen to be

$$\langle|\Psi_{\vec{q}_1}(t)|^2\rangle^{(0)} = 2\hbar\gamma k_B T \int_{-\infty}^{t} dt_1 [K^0(\vec{q}_1, t, t_1)]^2 . \tag{2.18}$$

This term is represented graphically by Fig. 1(a), where the time scale is in the vertical direction. The open circle represents the action of the random forces at time t_1, the closed square the observation at time t, and the directed lines the Green's functions K^0 associated with the factors of f^* (forward going) and f (backward going).

Now consider the first-order terms in (2.17). Since the expectation value of a product of Gaussian random forces factors in pairs, these terms are easily reduced to the form

$$\begin{aligned} \langle|\Psi_{\vec{q}_1}(t)|^2\rangle^{(1)} = &\int_{-\infty}^{t} dt_1 K^0(\vec{q}_1, t, t_1)\int_{-\infty}^{t_1} dt_1' K^0(\vec{q}_1, t_1, t_1') K^0(\vec{q}_1, t, t_1')(2\hbar\gamma k_B T)^2\left(-\frac{2b}{\Omega}\right)\sum_{\vec{q}_2}\int_{-\infty}^{t_1} dt_2 [K^0(\vec{q}_2, t_1, t_2)]^2 \\ &+ \int_{-\infty}^{t} dt_1' K^0(\vec{q}_1, t, t_1')\int_{-\infty}^{t_1'} dt_1 K^0(\vec{q}_1, t, t_1) K^0(\vec{q}_1, t_1', t_1)(2\hbar\gamma k_B T)^2\left(-\frac{2b}{\Omega}\right)\sum_{\vec{q}_2}\int_{-\infty}^{t_1'} dt_2 [K^0(\vec{q}_2, t_1', t_2)]^2 . \end{aligned} \tag{2.19}$$



(a) (b) (c)

FIG. 2. Diagrams representing the second-order contributions to $\langle |\Psi_{\vec{q}_1}(t)|^2\rangle$ which consist of successive first-order corrections to the various propagation lines. Four additional distinct diagrams are obtained by reversing arrows in (b) and (c).

The corresponding diagrams are shown in Figs. 1(b) and 1(c).

Higher-order corrections to $\langle |\Psi_{\vec{q}_1}(t)|^2\rangle$ may be obtained by successive iteration of (2.15), but the procedure rapidly becomes extremely tedious. The structure of this somewhat unusual perturbation theory may, however, be inferred by comparing the diagrams in Fig. 1 with their corresponding mathematical expressions. The nth-order contribution to $\langle |\Psi_{\vec{q}_1}(t)|^2\rangle$ may be obtained by drawing all distinct diagrams containing n interaction vertices (triangles), $n+1$ fluctuation vertices (open circles), and a single measurement vertex at time t (closed square). The vertices are labeled with time indices and connected by propagator lines such that four lines are attached to each interaction vertex and two to each fluctuation and measurement vertex. Three of the lines from each interaction vertex must connect it to vertices at earlier times, and the remaining one to a vertex at a later time. The two lines attached to each fluctuation vertex must connect it with vertices at later times, while the measurement vertex must be the latest time in the diagram. Each propagator has an arrow associated with it, which may point either forward or backward in time; but the number of lines entering any vertex must equal the number of lines leaving it. The lines of the diagram are labeled with wave-vector indices to conserve wave vector at all vertices. The value of such a diagram is then computed according to the following rules:

(i) Associate with each propagation line of wave vector $\vec{q}$ running between times t' and t'' a factor $K^0(\vec{q}, t'', t')$, where t'' is the later time, regardless of the direction of the arrow.

(ii) Associate with each fluctuation vertex a factor $2\hbar\gamma k_B T$.

(iii) Associate with each interaction vertex a factor $-2b/\Omega$.

(iv) Multiply by $1/2^m$, where m is the number of equivalent (identical) lines connecting pairs of interaction vertices.

(v) Sum over all internal wave vectors and integrate over all internal times from $-\infty$ to their upper limit as indicated in the diagram.

As an illustration, all topologically distinct diagrams corresponding to second-order corrections to $\langle |\Psi_{\vec{q}_1}(t)|^2\rangle$ are indicated in Figs. 2 and 3.

We shall next carry out a partial resummation of this perturbation expansion. Notice that all of the second-order diagrams in Fig. 2 result from the insertion of additional first-order corrections to the propagation lines of Figs. 1(b) and 1(c). All such diagrams may be eliminated by renormalizing the Green's function self-consistently to first order. This is accomplished by the solution of the integral equation represented graphically in Fig. 4. The procedure here is precisely analogous to the usual sort of Hartree-Fock approximation to the self-energy in an equilibrium perturbation theory. From Fig. 4 we have for the renormalized Green's function K^1

$$K^1(\vec{q}, t, t') = K^0(\vec{q}, t, t') + \int_{t'}^{t} dt_1 K^1(\vec{q}, t, t_1) K^0(\vec{q}, t_1, t')\; 2\hbar\gamma k_B T\left(-\frac{2b}{\Omega}\right)\sum_{\vec{q}_2}\int_{-\infty}^{t_1} dt_2 [K^1(\vec{q}_2, t_1, t_2)]^2$$

$$= K^0(\vec{q}, t, t') + [-2b\langle|\Psi|^2\rangle]\int_{t'}^{t} dt_1 K^1(\vec{q}, t, t_1) K^0(\vec{q}, t_1, t')\;, \tag{2.20}$$

where

$$\langle|\Psi|^2\rangle \equiv \langle|\Psi(\vec{r}, t)|^2\rangle = \frac{2\hbar\gamma k_B T}{\Omega}\sum_{\vec{q}_2}\int_{-\infty}^{t} dt_2 [K^1(\vec{q}_2, t, t_2)]^2 \tag{2.21}$$

is independent of position and time in the presence of a constant electric field. The integral equation (2.20) is easily solved by iterating and using the factorization property (2.13) of the noninteracting Green's function. The result is

$$K^1(\vec{q}, t, t') = K^0(\vec{q}, t, t') \exp[(-2b\langle|\Psi|^2\rangle/\hbar\gamma)(t-t')]\;. \tag{2.22}$$

Comparing this expression with the explicit form (2.14) for the noninteracting Green's function, we see that the Hartree-Fock approximation merely amounts to replacing the parameter a in the linear theory with

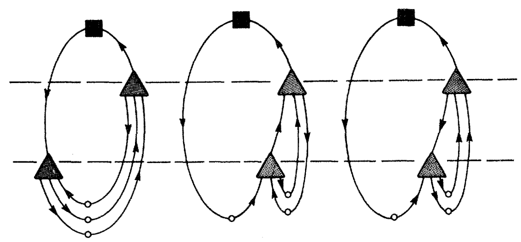


FIG. 3. Diagrams representing the second-order contributions to $\langle |\Psi_{\vec{q}_1}(t)|^2\rangle$ which are "irreducible" in the sense that they are not simply iterations of the first-order correction. Two additional distinct diagrams are obtained by reversing all arrows in (b) and (c).

$$\bar{a} = a + 2b\langle|\Psi|^2\rangle \ . \tag{2.23}$$

In two and three dimensions there is an ultraviolet divergence in the expression (2.21) for $\langle|\Psi|^2\rangle$ which must be removed from (2.23) by a renormalization of the transition temperature.

The average occupation of the wave-vector states may now be determined in this Hartree-Fock approximation from Fig. 1(a), using the renormalized Green's function (2.22):

$$\langle|\Psi_{\vec{q}}(t)|^2\rangle_{HF} = (2\hbar\gamma k_B T)\int_{-\infty}^{t} dt'[K^1(\vec{q},t,t')]^2$$

$$= \frac{2k_B T}{\hbar\gamma}\int_{-\infty}^{t} dt' \exp\left\{\frac{-2(t-t')}{\hbar\gamma}\right.$$

$$\left.\times\left[\bar{a} + \delta\left|\vec{q} - \frac{e\vec{E}}{\hbar}(t+t')\right|^2 + \frac{\delta}{3}\left(\frac{eE}{\hbar}(t-t')\right)^2\right]\right\} \ . \tag{2.24}$$

This result now includes the contributions from all diagrams in Figs. 1 and 2, as well as a whole hierarchy of higher-order terms. The first corrections to this approximation correspond to the diagrams in Fig. 3. The Hartree-Fock approximation presumably remains valid so long as the density of fluctuations in the system is sufficiently small that these second-order corrections are negligible.

The rather inconvenient time dependence, due to our choice of gauge, in the result (2.24) may be easily eliminated. The average population of pairs with a given kinetic momentum $\hbar\vec{k}$ is independent of time, for a constant electric field, and given in the HF approximation by

$$\langle|\Psi_{\vec{k}}|^2\rangle_{HF} \equiv \langle|\Psi_{\vec{q}}(t)|^2\rangle_{HF}\Big|_{\vec{q}=\vec{k}+2e\vec{E}t/\hbar}$$

$$= \frac{k_B T}{|\bar{a}|}\int_0^{\infty} dy \exp\left\{-y\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}\right.\right.$$

$$\left.\left.+\left|\bar{\xi}\vec{k} + \frac{1}{2}\frac{\vec{E}}{\bar{E}_0}y\right|^2 + \frac{1}{12}\left(\frac{E}{\bar{E}_0}\right)^2 y^2\right]\right\} \ . \tag{2.25}$$

Here we have defined a fictitious "effective temperature" parameter $\bar{T}$ by

$$\bar{a} \equiv a_0\frac{(\bar{T}-T_c)}{T_c} = a_0\frac{\Delta\bar{T}}{T_c} \tag{2.26}$$

and a "temperature-dependent" coherence length

$$\bar{\xi} \equiv \left(\frac{\delta}{|\bar{a}|}\right)^{1/2} = \xi(0)\left(\frac{T_c}{|\Delta\bar{T}|}\right)^{1/2} \ . \tag{2.27}$$

The scaling parameter for the electric field is

$$\bar{E}_0 \equiv \frac{|\bar{a}|^{3/2}}{\delta^{1/2}\gamma e} = E_0(0)\left(\frac{|\Delta\bar{T}|}{T_c}\right)^{3/2} , \tag{2.28}$$

where

$$E_0(0) \equiv \frac{a_0^{3/2}}{\delta^{1/2}\gamma e} \ . \tag{2.29}$$

All properties of the superconductor in the presence of a constant electric field may now be determined in the Hartree-Fock approximation from the result (2.25). In particular, the supercurrent density in this representation is given by[13]

$$\langle\vec{j}_s\rangle = -\frac{4e\delta}{\hbar\Omega}\sum_{\vec{k}}\vec{k}\langle|\Psi_{\vec{k}}|^2\rangle_{HF} \ . \tag{2.30}$$

III. HARTREE-FOCK THEORY

In this section we shall develop in detail the predictions of Hartree-Fock theory for the onset of the resistive transition in one-dimensional superconductors. We begin by obtaining an expression for the expectation value of the order parameter in the presence of the electric field. Using (2.25), we obtain for a one-dimensional system

$$\langle|\Psi|^2\rangle = \frac{1}{d^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\langle|\Psi_{\vec{k}}|^2\rangle_{HF}$$

$$= \frac{k_B T}{|\bar{a}|}\frac{1}{2(\sqrt{\pi})d^2\bar{\xi}}\int_0^{\infty} dy\, y^{-1/2}$$

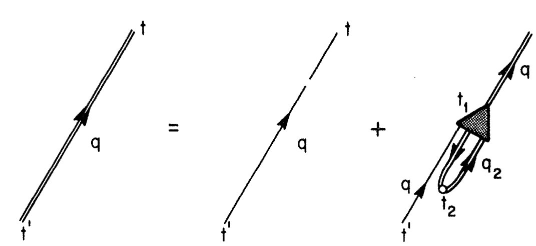


FIG. 4. Diagrammatic representation of the first-order self-consistent renormalization for the Green's function $K^1(q, t, t')$.

$$\times \exp\left\{-y\left[\frac{\Delta \overline{T}}{|\Delta \overline{T}|}+\frac{1}{12}\left(\frac{E}{\overline{E}_0}\right)^2 y^2\right]\right\} , \quad (3.1)$$

where d^2 is the cross-sectional area. The integral involved in this expression has previously been evaluated in terms of the tabulated Airy functions by one of us in a different context.[14] We therefore define the function

$$F_1(x) \equiv (1/\sqrt{\pi}) \int_0^\infty du\, u^{-1/2} \exp[-(xu+\tfrac{1}{12}u^3)]$$
$$= \pi\{[\mathrm{Ai}(-x)]^2+[\mathrm{Bi}(-x)]^2\}$$
$$\sim x^{-1/2} \qquad \text{as } x \to +\infty$$
$$\sim |x|^{-1/2}\exp(\tfrac{4}{3}|x|^{3/2}) \qquad \text{as } x \to -\infty . \quad (3.2)$$

The result (3.1) may now be written in the form

$$\langle|\Psi|^2\rangle = \frac{1}{2}\frac{a_0}{b}\left(\frac{\xi^3(0)}{\eta d^2}\right)\left(\frac{E_0(0)}{E}\right)^{1/3} F_1\left(\left(\frac{E_0(0)}{E}\right)^{2/3}\frac{\Delta\overline{T}}{T_c}\right) , \quad (3.3)$$

where we have defined the length

$$\eta \equiv \delta^2/bk_BT_c . \quad (3.4)$$

Now that $\langle|\Psi|^2\rangle$ has been determined as a function of the "effective temperature" $\overline{T}$, we return to (2.23) in order to establish the relationship between $\overline{T}$ and the actual temperature T. Using (3.3), this relationship may be written

$$\frac{\Delta T}{T_c} = \frac{\Delta\overline{T}}{T_c} - \left(\frac{\xi^3(0)}{\eta d^2}\right)\left(\frac{E_0(0)}{E}\right)^{1/3} F_1\left(\left(\frac{E_0(0)}{E}\right)^{2/3}\frac{\Delta\overline{T}}{T_c}\right) . \quad (3.5)$$

Next we wish to determine the magnitude of the current due to fluctuations. From (2.30) and (2.25) we find for the supercurrent density in a one-dimensional system

$$\langle j_s\rangle = \frac{-4e\delta}{\hbar d^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,k\langle|\Psi_k|^2\rangle_{\mathrm{HF}}$$
$$= \frac{ek_BT}{(\sqrt{\pi})\hbar d^2}\frac{E}{\overline{E}_0}\int_0^\infty dy\, y^{1/2}$$
$$\times \exp\left\{-y\left[\frac{\Delta\overline{T}}{|\Delta\overline{T}|}+\frac{1}{12}\left(\frac{E}{\overline{E}_0}\right)^2 y^2\right]\right\} . \quad (3.6)$$

Therefore the Hartree-Fock prediction for the total supercurrent carried by the wire becomes

$$I_s = d^2\langle j_s\rangle$$
$$= \frac{ek_BT}{\hbar}\frac{1}{\sqrt{\pi}}\int_0^\infty du\, u^{1/2}$$
$$\times \exp\left\{-\left[\left(\frac{E_0(0)}{E}\right)^{2/3}\frac{\Delta\overline{T}}{T_c}u+\tfrac{1}{12}u^3\right]\right\}$$
$$= \pi I_1 F_2\left(\left(\frac{E_0(0)}{E}\right)^{2/3}\frac{\Delta\overline{T}}{T_c}\right) . \quad (3.7)$$

The function defined here is simply the derivative of (3.2):

$$F_2(x) \equiv (1/\sqrt{\pi}) \int_0^\infty du\, u^{1/2} \exp[-(xu+\tfrac{1}{12}u^3)]$$
$$= 2\pi[\mathrm{Ai}(-x)\mathrm{Ai}'(-x)+\mathrm{Bi}(-x)\mathrm{Bi}'(-x)]$$
$$\sim \tfrac{1}{2}x^{-3/2} \qquad \text{as } x \to +\infty$$
$$\sim 2\exp(\tfrac{4}{3}|x|^{3/2}) \qquad \text{as } x \to -\infty . \quad (3.8)$$

We also adopt here the standard notation:

$$I_1 = ek_BT_c/\pi\hbar . \quad (3.9)$$

Equations (3.5) and (3.7) constitute our HF theory of the onset of the resistive transition in one-dimensional systems. The familiar result in the Aslamazov-Larkin region may be easily recovered using the asymptotic forms given in (3.2) and (3.8). In the limit of vanishing electric field and for temperatures sufficiently above the transition

$$(T-T_c)/T_c \gg [\xi^3(0)/\eta d^2]^{2/3} ,$$

we have

$$I_s = \frac{\pi e^2}{16\hbar}\,\xi(0)\left(\frac{T_c}{\Delta T}\right)^{3/2} E , \quad (3.10)$$

so that the excess conductivity due to fluctuations in a one-dimensional superconductor is given in this region by

$$\sigma_s^{(1)} = \frac{\pi e^2}{16\hbar d^2}\,\xi(0)\left(\frac{T_c}{\Delta T}\right)^{3/2} . \quad (3.11)$$

In general, however, our equations are quite complicated, and it will be advantageous to introduce a more convenient set of parameters.

In most experiments, the total current carried by the system

$$I = I_n + I_s = \sigma_n d^2 E + \pi I_1 F_2\left(\left(\frac{E_0(0)}{E}\right)^{2/3}\frac{\Delta\overline{T}}{T_c}\right) \quad (3.12)$$

is held constant, and the resistance is measured as a function of temperature. We assume that the normal conductivity σ_n is temperature independent near the transition. The natural scale for the electric field is then the field E_n which is present when the system is in the normal state, so that

$$E \equiv \alpha E_n = \alpha I/\sigma_n d^2 , \quad (3.13)$$

and α is the ratio of the measured resistance to the normal resistance:

$$\alpha = R/R_n \ . \tag{3.14}$$

We will define a dimensionless current by

$$j \equiv I/\pi I_1 \ . \tag{3.15}$$

The reduced temperature is

$$\epsilon \equiv (T - T_c)/T_c \ , \tag{3.16}$$

and the natural scale for this quantity turns out to be

$$\epsilon_c \equiv [\xi^3(0)/\eta d^2]^{2/3} \ . \tag{3.17}$$

The prediction of microscopic theory for ϵ_c is given by (A12). Using this set of parameters, Eqs. (3.5) and (3.7) assume the form

$$(1-\alpha)j = F_2\left(\left(\frac{\alpha j}{\alpha_0}\right)^{-2/3}\frac{\bar{\epsilon}}{\epsilon_c}\right) \ , \tag{3.18}$$

$$\frac{\epsilon}{\epsilon_c} = \frac{\bar{\epsilon}}{\epsilon_c} - \left(\frac{\alpha j}{\alpha_0}\right)^{-1/3} F_1\left(\left(\frac{\alpha j}{\alpha_0}\right)^{-2/3}\frac{\bar{\epsilon}}{\epsilon_c}\right) \ , \tag{3.19}$$

where we have defined an additional dimensionless parameter:

$$\alpha_0 \equiv \sigma_n d^2 E_0(0)\epsilon_c^{3/2}/\pi I_1 \ . \tag{3.20}$$

The microscopic expression for this quantity is given by (A13). It is found to depend only on the mean free path and to be of the order of unity for most systems of experimental interest.

The final step in formulating this theory is to obtain some sort of estimate of its region of validity. There are several ways this can be done. One method is to calculate the contributions of the lowest-order corrections to the HF expressions for the order parameter and supercurrent, given in Fig. 3, and then to assume the theory to be valid down to temperatures at which these corrections become significant. Explicit expressions for these contributions are given in Appendix B. The resulting integrals are extremely complicated, however, and we have not attempted to evaluate them exactly. One can readily establish an upper bound to the second-order correction in the form

$$\langle |\Psi_k|^2\rangle^{(2)} < C\langle |\Psi_k|^2\rangle_{\rm HF} \ , \tag{3.21}$$

where

$$C = \pi[ZF_1(x)]^3 F_2(x) \ , \qquad Z \equiv \left(\frac{\alpha j}{\alpha_0}\right)^{-2/3} , \qquad X \equiv Z\frac{\bar{\epsilon}}{\epsilon_c} \ . \tag{3.22}$$

The upper bound (3.21) appears to be too crude to be very useful in practice, however, particularly at large currents.

Another way of estimating where the theory may break down is simply to see where the results of the HF approximation deviate substantially from those obtained from the linearized theory, which neglects interaction between fluctuations altogether. We would not expect the HF theory to remain valid as a perturbation expansion much beyond the point at which interaction effects first become important. Both of these ideas will be utilized in the detailed interpretation of several numerical examples presented in Sec. IV.

IV. COMPARISON WITH PREVIOUS THEORIES

The HF theory developed in the preceding sections will now be compared with the previous theories of superconducting resistance in one-dimensional systems.

The present theory is basically similar to that proposed by MMP with the following exceptions: (i) A Hartree-Fock instead of a Hartree approximation is used to include the effect of interaction between fluctuations in a self-consistent manner. (ii) The first correction terms to the HF theory in powers of the interaction are used to estimate its region of validity as a formal perturbation expansion, while MMP assumed their theory to cover the entire transition region. (iii) The present theory is generalized to include the case of finite current.

The HF theory is contained in Eqs. (3.18) and (3.19) above.

All that need be done to employ a Hartree instead of a Hartree-Fock approximation is to remove the factor of 2 in (2.23). This corresponds to replacing (3.19) with

$$\frac{\epsilon}{\epsilon_c} = \frac{\bar{\epsilon}}{\epsilon_c} - \frac{1}{2}\left(\frac{\alpha j}{\alpha_0}\right)^{-1/3} F_1\left(\left(\frac{\alpha j}{\alpha_0}\right)^{-2/3}\frac{\bar{\epsilon}}{\epsilon_c}\right) \ . \tag{4.1}$$

Equations (4.1) and (3.18) taken together then constitute a self-consistent Hartree theory, of which the one-dimensional version of MMP is the zero-current limit. The Hartree theory does not correctly give the first correction to the linear TDGL theory as an expansion in powers of the interaction, that is, for a low density of Cooper pairs. However, Marčelja has argued in favor of the Hartree theory on the grounds that it gives the correct limiting value for $\langle|\Psi|^2\rangle$ in zero electric field for T well below T_c, whereas the HF theory does not. The HF theory for the case of zero field has previously been derived by Schmid[15] using a variational approach. More recently, Kajimura *et al.*[13] have written down expressions for the supercurrent at finite fields in the Hartree-Fock approximation, but have not discussed the case of one-dimensional systems.

The LA-MH theory of intrinsic superconducting resistance in one-dimension is in a sense complimentary to the present work. The LA calculation is based on the idea that well below the transition

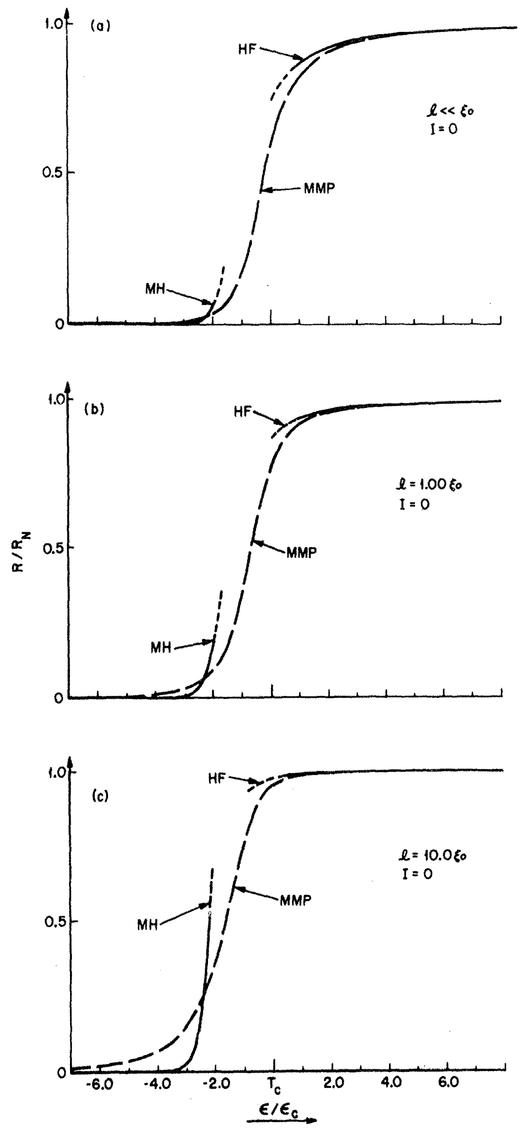


FIG. 5. Graphs of resistance vs temperature for dirty, intermediate, and clean one-dimensional systems carrying infinitesimal current according to the HE, MMP (Hartree), and MH theories. The scaling factor ϵ_c for the reduced temperature $\epsilon = (T - T_c)/T_c$ is mean free path dependent and given by (A12). Solid portions of curves indicate estimated regions of validity.

the system is essentially superconducting and can be found in one of the metastable current-carrying states which are obtained as plane-wave solutions to the equilibrium Ginzburg-Landau equations. A solution carrying a current I, and corresponding to a local minimum of the free energy, exists below the mean-field critical temperature

$$T_c(I) = T_c(1 - \tfrac{3}{4} j^{2/3} \epsilon_c) \ . \qquad (4.2)$$

The rate at which thermal fluctuations carry the system over the free-energy barrier to the next metastable state of lower current is proportional to $e^{-\Delta F_- / k_B T}$, where ΔF_- is the barrier height; this rate is then proportional to the resistance. MH succeeded in doing a detailed calculation, based on TDGL theory, of the "attempt frequency." These theories are valid only for temperatures below $T_c(I)$, where the free-energy barrier $\Delta F_- \gg k_B T$ so that the current-carrying states involved are truly metastable. The present HF theory, by contrast, ap-

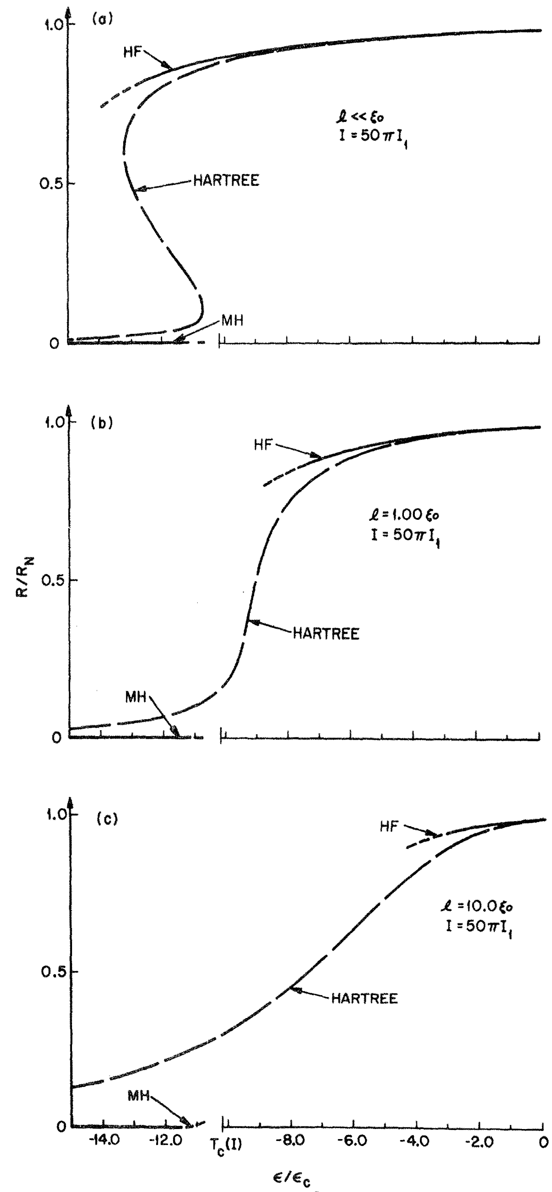


FIG. 6. Graphs of resistance vs temperature for dirty, intermediate, and clean one-dimensional systems in the presence of a fixed current $I = 50\pi I_1$, [$I \approx 1.05 T_c$ μA (T_c in °K)], as predicted by the HF, Hartree, and MH theories.

proaches the transition from above by assuming the system to be essentially normal, but containing a low density of thermally fluctuating pairs resulting in a small reduction in resistance. This theory ceases to be valid when the density of pairs becomes so large that their interactions play a dominant role. By combining these two pictures, we can obtain a complete description of the superconducting transition in one dimension everywhere but in a small region near the mean-field critical temperature. In the present notation, the MH results may be written as

$$\alpha = \left(\frac{\sqrt{2}}{\pi}\right)^{1/2} \alpha_0 \frac{1}{j}\left[1 - e^{-\pi j}\right]\left|\frac{\epsilon}{\epsilon_c}\right|^{9/4} e^{-\Delta F_- / k_B T_c} \times (1 - \sqrt{3}\,\kappa)^{15/4}(1 + \tfrac{1}{4}\kappa^2)\,, \quad (4.3)$$

where κ is determined from

$$j = 4\left|(\epsilon/\epsilon_c)\right|^{3/2}\kappa(1 - \kappa^2) \quad (4.4)$$

and the free-energy barrier is given by

$$\frac{\Delta F_-}{k_B T_c} = \frac{1}{2}\left|\frac{\epsilon}{\epsilon_c}\right|^{3/2}\left[\frac{8\sqrt{2}}{3}(1 - 3\kappa^2)^{1/2} - 8\kappa(1 - \kappa^2)\tan^{-1}\left(\frac{(1 - 3\kappa^2)^{1/2}}{\kappa\sqrt{2}}\right)\right]. \quad (4.5)$$

Numerical solutions depicting the predictions of HF, Hartree, and MH theories for several values of current and mean free path are shown in Figs. 5–7. Aside from determining the temperature scale through ϵ_c, the important mean-free-path dependence of these theories is contained in the parameter α_0. In developing these data, we have assumed the relation (A13) between α_0 and l predicted by microscopic theory.

Consider first the results for dirty, intermediate, and clean samples in the limit of zero current shown in Fig. 5. All the theories simplify considerably in this case, and their limiting forms are summarized in Appendix C. In each of these figures it appears that the HF theory, coming down from above T_c, may be joined smoothly to the MH prediction approaching the transition from below. If such an interpolation were made, we see that in the dirty limit the over-all shape of the transition would be very close to that predicted by MMP. In fact with a slight shift in critical temperature, MMP would be essentially indistinguishable from the combined HF and MH picture, except far down into the tail region. For the cleaner samples, however, the shape of the MMP prediction becomes noticeably different from the combined HF and MH results. In the first experiments on one-dimensional systems[8] the data obtained on dirty samples were fit remarkably well by the MMP form, while in cleaner specimens the fit was not especially good.

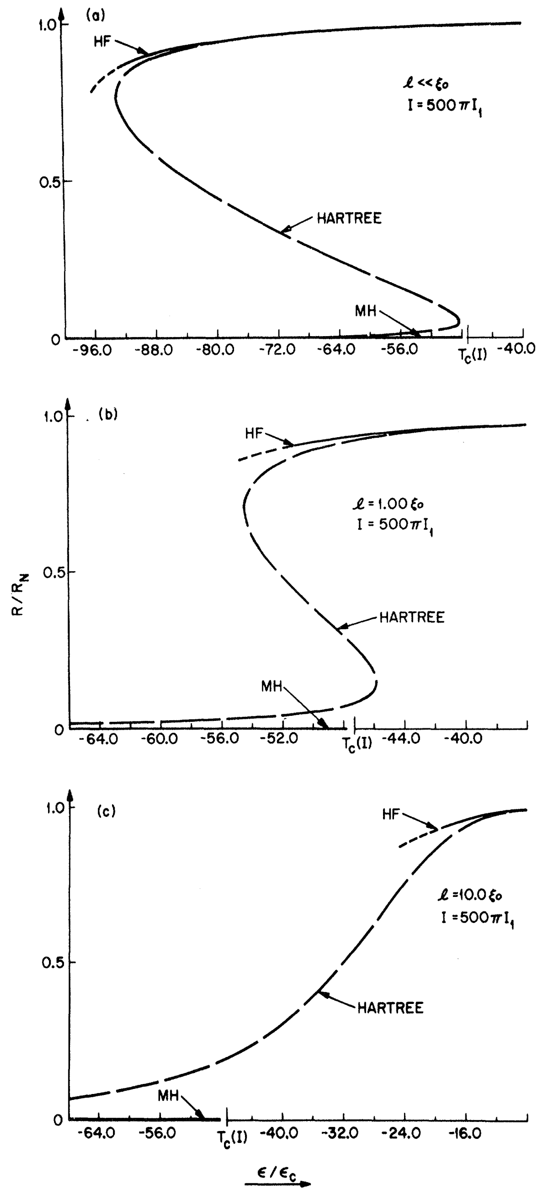


FIG. 7. Graphs of resistance vs temperature for dirty, intermediate, and clean one-dimensional systems in the presence of a fixed current $I = 500\pi I_1$ [$I \approx 10.5 T_c$ μA (T_c in °K)], as predicted by the HF, Hartree, and MH theories.

Later measurements[9] strongly support the general form of the MH result over MMP in the tail region, and we suspect that the initial success in fitting the MMP theory to data on dirty samples was due to the coincidence noted here. In addition we remark that possible discrepancies in the Azlamazov-Larkin theory for the linear region above T_c in clean samples have been investigated both experimentally[8,16] and theoretically.[17] These considerations on the validity of TDGL theory are beyond the scope of the present paper.

Graphs of the resistance versus temperature at a fixed current $I = 50\pi I_1$ [$I \approx 1.05 T_c$ μA (T_c in °K)], are shown in Fig. 6. The most important feature here is that in the dirty limit the HF prediction appears to remain valid down to temperatures at or below the mean-field critical temperature $T_c(I)$. In Fig. 6(a) we see from the separation of the HF from the Hartree result that the effects of interaction between fluctuations are just becoming significant in the neighborhood of $T_c(I)$; and our upper bound (3.22) on the lowest-order corrections to HF theory also becomes non-negligible in this region. The prediction of a basically normal-state solution to the TDGL equation existing below the mean-field transition temperature for this relatively small current in dirty samples is therefore marginal. On the other hand, MH theory predicts a basically superconducting state, with resistance $R/R_n \approx 0$ on the scale of the diagram, up to temperatures in the immediate neighborhood of $T_c(I)$. This raises the possibility of multiple solutions to the TDGL equation under these conditions. For the cleaner samples in Figs. 6(b) and 6(c), the HF approximation breaks down as a formal perturbation theory well above $T_c(I)$, and there is no evidence indicating multiple solutions.

The results for a relatively large current $I = 500\pi I_1$, [$I \approx 10.5 T_c$ μA (T_c in °K)], are shown in Fig. 7. The possible overlap in the regions of validity of the HF and MH theories noted above in the dirty limit may now be firmly established. In Fig. 7(a) we see that the effect of interaction between fluctuations remains negligible down to temperatures far below $T_c(I)$. Even the gross overestimate (3.22) for the magnitude of the lowest-order corrections to the HF prediction is found to be of order 10^{-2} near $T_c(I)$, and indicates that the HF approximation is an excellent one well down into the region in which MH theory predicts that the system will be essentially superconducting. For the intermediate system in Fig. 7(b), we would begin to make a reasonably confident prediction of such an overlap near this value of the current, while in the clean system of Fig. 7(c) the HF approximation once again breaks down well above $T_c(I)$.

The existence of two distinct solutions to the TDGL equation over a certain temperature range, for sufficiently large values of the current, opens up several interesting possibilities. First, we note that the existence of a third (unstable) solution is required by the fact that there must be a unique solution of the TDGL model for every value of temperature and electric field, and the resulting current must be a continuous function of these variables. Notice that the curves obtained from the Hartree theory in Figs. 6(a), 7(a), and 7(b) have an S shape, with a region of multiple values for the resistance in a range of temperature near $T_c(I)$.

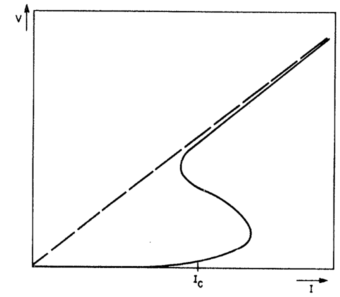


FIG. 8. Qualitative features of the I-V characteristic for a one-dimensional superconductor at a temperature sufficiently low that multiple solutions to the TDGL equation exist for a range of current values.

(The HF curves would have a similar shape, but have not been plotted outside their expected region of validity.) Although neither the Hartree nor the HF approximation is expected to be quantitatively valid in this region of multiple values, we nevertheless expect that the qualitative shape of the resistance curves is the same as the ones which would correspond to an exact solution of the TDGL model under these circumstances. In addition it may readily be shown that a plot of voltage versus current in the Hartree or HF approximation at a fixed temperature in these regions must also be S shaped, with an interval of negative differential resistivity dV/dI, as sketched in Fig. 8. A negative resistance region was noted independently by Gor'kov[18] for the case of a thin film at finite currents.

The possible physical consequences of such multiple solutions to the TDGL equation are quite interesting. If the temperature is varied at constant current, we might expect a vertical first-order transition between the low- and high-resistance states in the neighborhood of $T_c(I)$. It may also be possible for the system to remain metastably in either state over some temperature range, in which case a hysteresis would be observed in the experimental resistance-versus-temperature curves. Another possibility is that the system may become spatially inhomogeneous with some portions of the wire in each of the two states, or that portions of the system will oscillate rapidly in time between the two states with associated production of electromagnetic radiation. A more detailed investigation of these possibilities would undoubtedly require proper consideration of *voltage fluctuations*, which have been ignored in the present model.

The instabilities and hysteresis effects which one might predict on the basis of this analysis may have been observed in experiments on one-dimensional samples at finite currents by the Cornell group.[8] Some of their samples exhibit the hysteresis behavior discussed here, but with the superposition of small, curiously regular, resistance steps on the otherwise vertical hysteresis lobes of the transition. In many of their data at finite currents, however, these steps appear as the dominant feature, with a small amount of hysteresis showing only on the individual steps. The origin of these resistance steps is not at present understood, and is the subject of considerable interest. It is hoped that the theory presented here will provide an improved basis for understanding such effects.

V. CONCLUSION

In this paper we have developed a systemmatic perturbation theory for Ginzburg-Landau systems in the presence of a finite electric field, and have applied it to an investigation of the onset of the resistive transition in one-dimensional superconductors. In the limit of zero current, the HF theory derived here may be joined smoothly to the previous MH prediction of intrinsic superconducting resistance below the transition temperature. At sufficiently large currents, these two theories together predict the presence of an instability near the mean-field critical temperature and the possibility of associated hysteresis effects. Such instabilities are predicted to set in at smaller currents in dirtier samples.

ACKNOWLEDGMENTS

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APPENDIX A

The expressions obtained from microscopic theory for the Ginzburg-Landau coefficients depend on the normalization chosen for the order parameter. (The final results for all physical quantities, such as electrical conductivity, are of course normalization independent.) One choice which is fairly common leads to the expressions

$$a=\frac{\hbar^2}{2m\xi^2(0)}\,\frac{T-T_c}{T_c}=a_0\frac{\Delta T}{T_c}\ ,$$
$$b=\frac{\hbar^2}{2m\xi^2(0)}\,\frac{2}{N\chi(0.882\xi_0/l)}\ ,\quad \delta=\frac{\hbar^2}{2m}\ . \tag{A1}$$

These expressions are derived on the basis of a spherical band model for the conduction electrons with effective mass m. Here l is the mean free path and ξ_0 the BCS coherence length:

$$\xi_0=0.18\hbar v_f/k_B T_c\ , \tag{A2}$$

and the density of conduction electrons is

$$N=(1/3\pi^2)k_f^3\ . \tag{A3}$$

The Gor'kov function is defined by

$$\chi(\rho)=\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^2(2\nu+1+\rho)}\Big/\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^3}\ , \tag{A4}$$

so that in the clean and dirty limits we have

$$\chi(0.882\xi_0/l)=\begin{cases}1, & \xi_0\ll l\\ 1.33l/\xi_0\ , & \xi_0\gg l\ .\end{cases} \tag{A5}$$

The Ginzburg-Landau coherence length is defined by

$$\xi(T)=\left(\frac{\delta}{|a|}\right)^{1/2}=\xi(0)\left(\frac{T_c}{|\Delta T|}\right)^{1/2}\ , \tag{A6}$$

where

$$\xi(0)=0.74[\chi(0.882\xi_0/l)]^{1/2}\xi_0=\begin{cases}0.74\xi_0\ , & \xi_0\ll l\\ 0.85(\xi_0 l)^{1/2}\ , & \xi_0\gg l\ .\end{cases} \tag{A7}$$

The microscopic result for the additional parameter in the time-dependent theory is

$$\gamma=\frac{\pi a_0}{8k_B T_c}\ . \tag{A8}$$

We now list expressions for the various scaling parameters introduced in this paper both in terms of Ginzburg-Landau coefficients and the corresponding microscopic expressions:

$$E_0(0)=\frac{a_0^{3/2}}{\delta^{1/2}\gamma e}=\frac{8}{\pi}\,\frac{k_B T_c}{e\xi(0)}\ , \tag{A9}$$

$$\eta=\frac{\delta^2}{bk_B T_c}=\frac{1}{6\pi^2}\,\frac{E_F}{k_B T_c}\,\chi(0.882\xi_0/l)k_f\xi^2(0)\ , \tag{A10}$$

$$\pi I_1=ek_B T_c/\hbar=2.097T_c\times10^{-2}\,\mu\text{A}\ (T_c\text{ in }^\circ\text{K}), \tag{A11}$$

$$\epsilon_c^{3/2}=\frac{\xi^3(0)}{\eta d^2}=\frac{bk_B T_c}{\delta^{1/2}a_0^{3/2}d^2}=6\pi^2\left(\frac{k_B T_c}{E_F}\right)\left(\frac{\xi(0)}{d}\right)^2[\chi(0.882\xi_0/l)]^{-1}\frac{1}{k_F\xi(0)}\ , \tag{A12}$$

$$\alpha_0=\frac{\sigma_N d^2E_0(0)\epsilon_c^{3/2}}{\pi I_1}=1.84\frac{l}{\xi_0}[\chi(0.882\xi_0/l)]^{-1}\ . \tag{A13}$$

Several of these quantities may be expressed in a somewhat more convenient form by utilizing the BCS relation for the bulk critical field in the neighborhood of the transition temperature:

$$H_c(T)=1.74H_c(0)[1-T/T_c]\ . \tag{A14}$$

We then have

$$\eta = 0.24\xi(0)\left(\frac{H_c^2(0)\xi^3(0)}{k_B T_C}\right), \tag{A15}$$

$$\epsilon_c = 2.58\left(\frac{k_B T_C}{H_c^2(0)\xi(0)d^2}\right)^{2/3}, \tag{A16}$$

$$\alpha_0 = 10.57\left(\frac{\sigma_N}{e^2\xi(0)/\hbar d^2}\right)\left(\frac{k_B T_C}{H_c^2(0)\xi(0)d^2}\right). \tag{A17}$$

APPENDIX B

In this appendix we shall calculate an upper bound for the first correction in powers of the interaction to the HF theory developed in the text. The lowest-order diagrams not contained in our first-order resummation are shown in Fig. 3. Dashed lines have been placed on this figure to indicate the way in which we intend to break up the time integrations. Using the factorization property (2.13), the top $\frac{2}{3}$ of each of these diagrams is identical and we can write them together as

$$\begin{aligned}\langle |\Psi_{\vec{q}_1}(t)|^2\rangle^{(2)} = &\int_{-\infty}^{t} dt_1 [K(\vec{q}_1, t, t_1)]^2 (\hbar\gamma)^4 (2\hbar\gamma k_B T)^3 \left(-\frac{2b}{\Omega}\right)^2 \sum_{\vec{q}_2\vec{q}_3\vec{q}_4} \delta_{\vec{q}_1+\vec{q}_2,\,\vec{q}_3+\vec{q}_4} \int_{-\infty}^{t_1} dt_1' \prod_{i=1}^{4} [K(\vec{q}_i, t_1, t_1')] \\ &\times\left(\int_{-\infty}^{t_1'} dt_2 [K(\vec{q}_2, t_1', t_2)]^2 \int_{-\infty}^{t_1'} dt_3 [K(\vec{q}_3, t_1', t_3)]^2 \int_{-\infty}^{t_1'} dt_4 [K(\vec{q}_4, t_1', t_4)]^2\right. \\ &+\int_{-\infty}^{t_1'} dt_1'' [K(\vec{q}_1, t_1', t_1'')]^2 \int_{-\infty}^{t_1'} dt_3 [K(\vec{q}_3, t_1', t_3)]^2 \int_{-\infty}^{t_1'} dt_4 [K(\vec{q}_4, t_1', t_4)]^2 \\ &\left.+2\int_{-\infty}^{t_1'} dt_1'' [K(\vec{q}_1, t_1', t_1'')]^2 \int_{-\infty}^{t_1'} dt_2 [K(\vec{q}_2, t_1', t_2)]^2 \int_{-\infty}^{t_1'} dt_4 [K(\vec{q}_4, t_1', t_4)]^2\right).\end{aligned} \tag{B1}$$

The terms corresponding to Figs. 3(b) and 3(c) carry an extra factor of 2 because the diagrams with all lines reversed are not shown explicitly as in Figs. 1(b) and 1(c). The term corresponding to Fig. 3(a) also carries an extra factor of 2 here because of the other possible relative time ordering of the two interaction vertices. This factor of 2 is, however, canceled in the terms corresponding to Figs. 3(a) and 3(b) due to the presence of equivalent lines q_3 and q_4. Eliminating the explicit gauge dependence as in Sec. II, this expression may be reduced to the form

$$\begin{aligned}\langle |\Psi_{\vec{k}_1}|^2\rangle^{(2)} = &\frac{k_B T}{|\overline{a}|}\int_0^\infty dy \exp\left\{-y\left[\frac{\Delta\overline{T}}{|\Delta\overline{T}|} + \left|\overline{\xi}\vec{k}_1 + \frac{1}{2}\frac{\vec{E}}{\overline{E}_0}y\right|^2 + \frac{1}{12}\left(\frac{E}{\overline{E}_0}\right)^2 y^2\right]\right\} \\ &\times\int_0^\infty dy_1\, e^{-y_1(\overline{\xi}k_1)^2/4} \exp\left\{-y_1\left[\frac{\Delta\overline{T}}{|\Delta\overline{T}|} + \left(\frac{E}{\overline{E}_0}\right)^2 (y^2 + yy_1) + \frac{1}{12}\left(\frac{E}{\overline{E}_0}\right)^2 y_1^2\right]\right\} \\ &\times\int_0^\infty\int_0^\infty\int_0^\infty dy_2\,dy_3\,dy_4 \prod_{i=2}^{4}\left(\exp\left\{-y_i\left[\frac{\Delta\overline{T}}{|\Delta\overline{T}|} + \frac{1}{12}\left(\frac{E}{\overline{E}_0}\right)^2 y_i^2\right]\right\}\right) S,\end{aligned} \tag{B2}$$

where

$$\begin{aligned}S = &\frac{1}{2}\left(\frac{bk_B T}{|\overline{a}|^2\Omega}\right)^2 \sum_{\vec{k}_2,\vec{k}_3,\vec{k}_4} \delta_{\vec{k}_1+\vec{k}_2,\,\vec{k}_3+\vec{k}_4} \prod_{i=2}^{4} \left(e^{-y_1(\overline{\xi}k_i)^2/4}\right) \left(e^{-y_2|\overline{\xi}\vec{k}_2+\vec{\delta}_2|^2} e^{-y_3|\overline{\xi}\vec{k}_3+\vec{\delta}_3|^2} e^{-y_4|\overline{\xi}\vec{k}_4+\vec{\delta}_4|^2}\right. \\ &\left.+e^{-y_2|\overline{\xi}\vec{k}_1+\vec{\delta}_2|^2} e^{-y_3|\overline{\xi}\vec{k}_3+\vec{\delta}_3|^2} e^{-y_4|\overline{\xi}\vec{k}_4+\vec{\delta}_4|^2} + 2e^{-y_2|\overline{\xi}\vec{k}_2+\vec{\delta}_2|^2} e^{-y_3|\overline{\xi}\vec{k}_1+\vec{\delta}_3|^2} e^{-y_4|\overline{\xi}\vec{k}_4+\vec{\delta}_4|^2}\right), \quad \vec{\delta}_i \equiv \frac{1}{2}\left(\frac{\vec{E}}{\overline{E}_0}\right)(2y + y_1 + y_i).\end{aligned} \tag{B3}$$

To satisfy the wave-vector conservation condition we take

$$\vec{k}_2 = \vec{P}_1 - \vec{k}_1, \qquad \vec{k}_3 = \vec{P}_1 - \vec{P}_2, \qquad \vec{k}_4 = \vec{P}_2. \tag{B4}$$

In one dimension, the wave-vector summation then becomes

$$S = \frac{1}{8\pi^2}\left(\frac{\overline{\xi}}{\eta d^2}\right)\int_{-\infty}^{\infty} d\overline{P}_1 \int_{-\infty}^{\infty} d\overline{P}_2\, e^{-y_1[(\overline{P}_1-\overline{k}_1)^2+(\overline{P}_1-\overline{P}_2)^2+\overline{P}_2^2]/4} \left(e^{-y_2(\overline{P}_1-\overline{k}_1+\delta_2)^2} e^{-y_3(\overline{P}_1-\overline{P}_2+\delta_3)^2} e^{-y_4(\overline{P}_2+\delta_4)^2}\right.$$

$$+e^{-y_2(\bar{k}_1+\delta_2)^2}e^{-y_3(\bar{P}_1-\bar{P}_2+\delta_3)^2}e^{-y_4(\bar{P}_2+\delta_4)^2}+2e^{-y_2(\bar{P}_1-\bar{k}_1+\delta_2)^2}e^{-y_3(\bar{k}_1+\delta_3)^2}e^{-y_4(\bar{P}_2+\delta_4)^2}), \quad \bar{k}_1\equiv\bar{\xi}k_1 . \tag{B5}$$

It would, in principle, be possible to carry out the integration in (B5) exactly, since all that is required is a convolution of Gaussians, but the result would be hopelessly complicated. In order to obtain a viable result we make the grossest possible simplification, which gives as an upper bound to (B5):

$$S\le\frac{1}{2\pi^2}\left(\frac{\bar{\xi}}{\eta d^2}\right)\int_{-\infty}^{\infty}d\bar{P}_1\,e^{-y_1(\bar{P}_1-\bar{k}_1)^2/4}\int_{-\infty}^{\infty}d\bar{P}_2\,e^{-y_4(\bar{P}_2+\delta_4)^2}=\frac{1}{\pi}\left(\frac{\bar{\xi}}{\eta d^2}\right)^2(y_1y_4)^{-1/2} . \tag{B6}$$

Using this result in (B2) and making a similar simplification for various terms in the y_1 integrand we arrive at

$$\langle|\Psi_{k_1}|^2\rangle^{(2)}<\left(\frac{k_BT}{|\bar{a}|}\int_0^\infty dy\,\exp\left\{-y\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}+\left(\bar{\xi}k_1+\frac{1}{2}\frac{E}{\bar{E}_0}y\right)^2+\frac{1}{12}\left(\frac{E}{\bar{E}_0}\right)^2y^2\right]\right\}\right)$$
$$\times\frac{1}{\pi}\left(\frac{\bar{\xi}}{\eta d^2}\right)^2\left(\int_0^\infty dy_1\,y_1^{-1/2}\exp\left\{-y_1\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}+\frac{1}{12}\left(\frac{E}{\bar{E}_0}\right)^2y_1^2\right]\right\}\right)^2$$
$$\times\left(\int_0^\infty dy_2\exp\left\{-y_2\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}+\frac{1}{12}\left(\frac{E}{\bar{E}_0}\right)^2y_2^2\right]\right\}\right)^2 . \tag{B7}$$

Using the Schwartz inequality it may be shown that

$$\left(\int_0^\infty dy_2\exp\left\{-y_2\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}+\frac{1}{12}\left(\frac{E}{\bar{E}_0}\right)^2y_2^2\right]\right\}\right)^2\ge\left(\int_0^\infty dy_1\,y_1^{-1/2}\exp\left\{-y_1\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}+\frac{1}{12}\left(\frac{E}{E_0}\right)^2y_1^2\right]\right\}\right)$$
$$\times\left(\int_0^\infty dy_2\,y_2^{1/2}\exp\left\{-y_2\left[\frac{\Delta\bar{T}}{|\Delta\bar{T}|}+\frac{1}{12}\left(\frac{E}{\bar{E}_0}\right)^2y_2^2\right]\right\}\right) . \tag{B8}$$

Since the first factor on the right-hand side in (B7) is just $\langle|\Psi_{k_1}|^2\rangle_{HF}$, we readily establish the inequality (3.21).

APPENDIX C

In the limit of zero current, the various theories of superconducting resistance in one-dimensional systems may be reduced to the following forms:

HF theory:

$$\frac{\epsilon}{\epsilon_c}=\left(\frac{\alpha}{2\alpha_0(1-\alpha)}\right)^{2/3}\left(1-\frac{2\alpha_0(1-\alpha)}{\alpha}\right) \tag{C1}$$

valid for $\epsilon\gtrsim\epsilon_c$;

Hartree (MMP) theory:

$$\frac{\epsilon}{\epsilon_c}=\left(\frac{\alpha}{2\alpha_0(1-\alpha)}\right)^{2/3}\left(1-\frac{\alpha_0(1-\alpha)}{\alpha}\right); \tag{C2}$$

MH theory:

$$\alpha=(\sqrt{2}\pi)^{1/2}\alpha_0\left|\frac{\epsilon}{\epsilon_c}\right|^{9/4}\exp\left(-\frac{4\sqrt{2}}{3}\left|\frac{\epsilon}{\epsilon_c}\right|^{3/2}\right) \tag{C3}$$

valid for $-\epsilon\gtrsim\epsilon_c$.

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Influence of Dislocation Motion on the Ultrasonic-Velocity Change in Superconducting Indium

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The change in velocity of compressional sound waves propagating in superconducting and normal indium has been measured at 13 MHz as a function of temperature and amplitude of the cw ultrasonic signal. It is found that the velocity of sound is strongly amplitude dependent, especially in the superconducting state. At the highest amplitude utilized, the velocity is observed to have changed by about one part in 10^3. The results can be explained on the basis of a direct interaction between the conduction electrons and oscillating dislocations as proposed by Granato and Lücke and by Kravchenko.

INTRODUCTION

Recent publications by several authors[1,2] have questioned the existence of a direct interaction between the electron gas in metals and oscillating dislocations, a mechanism which has been postulated to explain amplitude-dependent effects observed in ultrasonic attenuation measurements.[3-5] The purpose of this paper is to report that compressional-wave ultrasonic-velocity measurements in normal and superconducting indium strongly confirm the existence of an electron-dislocation interaction, and the experimental observations agree qualitatively with the theory of Granato and Lücke.[6] The results also are in accord with several recent experimental and theoretical investigations of the effect of the conducting electrons on dislocation motion (both vibrational and translational).[7-9] It appears that this technique might be quite useful for investigating dislocation effects in pure metals, especially the phenomena which occur when the electron gas is shorted out by the superconducting electrons.

The model of Granato and Lücke[6] assumes that a network of dislocations of length L_N in a crystal is pinned by impurity points L_C apart and by the intersection of the network dislocation loops. When an external stress wave is applied, the dislocations oscillate in a manner similar to the forced-damped vibration of a string[10] and two loss mechanisms are predicted to change the velocity and the logarithmic decrement (i.e., the attenuation) of the impressed wave. These are (a) a frequency-dependent amplitude-independent effect caused by the interaction of the forced dislocation motion with some damping motion (in this case, the conduction electrons) and (b) an amplitude-dependent loss due to the fact that on the basis of this model, the same stress-strain law is not followed on the loading and unloading cycles. Granato and Lücke derived the following equations for cases (a) and (b), respectively:

$$\left(\frac{\Delta v}{v}\right)_\nu = \frac{C_1}{2\pi}\,\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2 d^2}\,, \tag{1a}$$