Helicons, Doppler-Shifted Cyclotron Resonance, and Gantmakher-Kaner Oscillations. II. p-Fold Azimuthal Symmetry*

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The conductivity of a metal, for the case of the wave vector in the direction of a magnetic field which in turn lies along an axis of p-fold symmetry, is examined. General properties are adduced which explicitly exhibit *all* the singularities of the conductivity (it is shown that there are no others) and reduce the problem to the conductivity of a collection of cylindrically symmetric Fermi surfaces with progressively smaller weightings. A model, which allows analytic computation, is used for the purpose of examining the possible modes of the electromagnetic field and the Gantmakher-Kaner oscillations. Additional modes, arising from the lack of cylindrical symmetry, are all heavily damped. The Gantmakher-Kaner oscillations are discussed as to damping, amplitude, and position. A general relation between Fermi-surface properties and the optical mass is derived in the absence of phonon effects.

I. INTRODUCTION

In a previous paper,[1] the interaction between the electromagnetic field and a metal was discussed for the case where a slab of metal was subjected to a dc magnetic field normal to its surface and an rf electromagnetic field tangential to its surface. The discussion was restricted to those metals whose Fermi surfaces exhibited cylindrical symmetry about the direction of the magnetic field. The electromagnetic response of the system was discussed in terms of the singularities of, what was in fact, the Green's function for the electromagnetic field. These were of two types. There were poles; that is, roots of the electromagnetic dispersion relation corresponding to helicons and to Doppler-shifted cyclotron resonance (DSCR). In addition, there were branch-cut singularities, corresponding to Gantmakher-Kaner oscillations (GKO). The paper examined in detail how the shape of the Fermi surface determined the nature of the conductivity, and hence the singularities and the electromagnetic response.

In this paper, we wish to extend some of these considerations to the case where the Fermi surface is no longer cylindrically symmetric, but rather has p-fold azimuthal symmetry about the direction of the magnetic field. It is necessary to understand the effects that result from this azimuthal variation, as experimental situations usually correspond to this case and it is important to know whether any such effects can alter the interpretation of the experimental results.

Following I, we concern ourselves here primarily with the singularities in the conductivity as a function of q, the wave number. In Sec. II, we exhibit the conductivity as an infinite sum of terms, each of which looks like the conductivity for a cylindrical Fermi surface, but where the cylindrical case involved the cyclotron frequency ω_c in the singularity, the terms here involve $(pn+1)\omega_c$, where n is any (positive or negative) integer. In Sec. III, we use this expression to show that it is sufficient to look at the singularities of the conductivity term by term. That is, no singularities arise from the infinite series; the singularities of the series are the singularities of the individual terms. The expression for the conductivity also allows us, in Sec. IV, to derive (in the absence of phonon effects) a relation between the optical mass and the details of the Fermi surface. In order to exhibit details analytically and more explicitly, in Sec. V, as in I, we restrict ourselves to a specific model. Here we assume that the component of velocity in the direction of the magnetic field does not depend on the azimuthal angle and that the cyclotron frequency is a constant. This allows us to simplify the expression for the conductivity. Choice of a particular type of azimuthal variation enables us to present more explicit expressions, and allows us to expand the results in powers of the coefficient of the azimuthal variation of the Fermi surface. In Sec. VI, for a specific shape of Fermi surface, we plot the real and imaginary parts of the conductivity. Then, in Sec. VII, we discuss the additional roots of the electromagnetic dispersion relations that occur as a result of the azimuthal variation, and the possibility of detecting these additional modes. Finally, in Sec. VIII, we examine the GKO, their damping, amplitude, and position.

II. GENERAL EXPRESSION FOR $\sigma^{\pm}$ FOR p-FOLD AZIMUTHAL SYMMETRY

We will now exhibit $\sigma^{\pm}(\vec{q},\omega)$, the conductivity for circular polarization, in a form that will be convenient for us. The development is essentially that of Overhauser and Rodriguez[2] and of Alig and Rodriguez.[3] (Their $\vec{q}$ and ω are the negative of ours.) If the magnetic field $\vec{H}$ is directed along the z direction, and if we express the velocity of an electron on the Fermi surface $\vec{v}(k_z, u)$ in terms of the time variable u which describes the time it would take an electron with a z component of momentum k_z to move through an azimuthal angle ϕ in the absence of the rf field, then the conductivity tensor may be written as

$$\vec{\sigma}(\vec{q},\omega) = e^2 \frac{eH}{4\pi^3 c} \int_{FS} dk_z \int_0^T du\, \vec{v}(k_z,u) \int_{-\infty}^{u} du'\, \vec{v}(k_z,u') \times \exp\left(i\omega'(u-u') - i\vec{q}\cdot\int_{u'}^{u} du''\,\vec{v}(k_z,u'') \right), \tag{1}$$

where the k_z integral is over the Fermi surface, T is the period of an electron with k_z and in general depends on k_z, and $\omega' = \omega + i/\tau$, where τ is the relaxation time occurring in the Boltzmann equation. Following Ref. 2, we separate $\vec{v}(k_z, u)$ into an average part $\vec{v}_s(k_z)$ and a periodic part, which we write as the derivative of a position vector $\partial\vec{R}_p(k_z, u)/\partial u$. We also note that $\vec{v}$ may be written in terms of the momentum derivatives of the energy E as

$$v^z = \frac{\partial E}{\partial k_z}, \quad v^{\pm} = e^{\pm i\phi}\left(\frac{\partial E}{\partial \rho} \pm \frac{i}{\rho}\frac{\partial E}{\partial \phi} \right), \tag{2}$$

where ρ is the cylindrical radial component of momentum transverse to $\vec{H}$ and ϕ is the azimuthal angle, related to the time variable u by

$$\frac{du}{d\phi} = \frac{c}{eH}\frac{1}{2}\frac{\partial \rho_E^2(\phi)}{\partial E}\bigg|_{\phi} \quad \text{and } u(0) = 0 . \tag{3}$$

Here $\rho_E(\phi)$ is the radius of a surface of energy E and position ϕ and k_z (which we suppress).

If the Fermi surface has p-fold azimuthal symmetry about the direction of $\vec{H}$, and if $\vec{q}$ is directed along $\vec{H}$, then (2) implies that we may expand

$$v^{\pm}(k_z,u) e^{i\vec{q}\cdot\vec{R}_p(k_z,u)} = \sum_{n=-\infty}^{\infty} v_n^{\pm}(\vec{q},k_z) e^{i(pn\pm 1)\omega_c u}, \tag{4}$$

where

$$v_{-n}^{\mp}(-\vec{q},k_z) = [v_n^{\pm}(\vec{q},k_z)]^* . \tag{5}$$

Inserting all this in (1) one gets

$$\sigma^{\pm}(\vec{q},\omega) = \tfrac{1}{2} e^2 \frac{eH}{4\pi^3 c}\int_{FS} dk_z \frac{2\pi i}{\omega_c} \times \sum_{n=-\infty}^{\infty} \frac{[v_n^{+}(\mp\vec{q},k_z)]^* v_n^{+}(\mp\vec{q},k_z)}{\omega' - \vec{q}\cdot\vec{v}_s(k_z) \pm (pn+1)\omega_c}, \tag{6}$$

where we have introduced the cyclotron frequency $\omega_c = 2\pi/T$, which in general depends on k_z. The circular-polarization components diagonalize the conductivity provided $p > 2$.

III. SINGULARITIES OF $\sigma^{\pm}(\vec{q},\omega)$

The expression (6) is an infinite sum of terms, each of which looks like the conductivity for a cylindrical surface [as in I, Eq. (22a)], with ω_c replaced by $(pn+1)\omega_c$ in the "energy" denominators. We may expect then to get a sum of singularities of the type discussed in I. In addition, it may also be possible to have additional singularities due to the fact that the sum is infinite. We will show now that this is, in fact, not the case, and that the singularities of the infinite sum are only those of the individual terms.

We note that (4) implies that

$$v_n^{\pm}(\vec{q},k_z) = \frac{p}{T}\int_0^{T/p} du\, v^{\pm}(k_z,u) e^{i\vec{q}\cdot\vec{R}_p(k_z,u)} \times e^{-i(pn\pm 1)\omega_c u}. \tag{7}$$

Inserting this in (6), performing the summation, and using the periodicity, the conductivity may be written as

$$\sigma^{\pm}(\vec{q},\omega) = \tfrac{1}{2} i e^2 \frac{eH}{4\pi^3 c}\int_{FS} \frac{dk_z}{1-e^{-2\pi i\beta_{\pm}}} \times p\int_0^{T/p} du\, W^{\pm}(\vec{q},k_z,u) e^{iu\omega_c(1-p\beta_{\pm})}, \tag{8}$$

where

$$\beta_{\pm} = \frac{\omega' - \vec{q}\cdot\vec{v}_s(k_z) \pm \omega_c}{\pm p\omega_c} \tag{9}$$

and

$$W^{\pm}(\vec{q},k_z,u) = 2\pi i\,\frac{p}{T}\int_0^{T/p} du'\, e^{\pm i\vec{q}\cdot[\vec{R}_p(k_z,u') - \vec{R}_p(k_z,u'-u)]} \times [v^{+}(k_z,u')]^* v^{+}(k_z,u'-u) . \tag{10}$$

Remembering that $\vec{q}$ only has a z component, it is easily seen that $W^{\pm}(\vec{q},k_z,u)$ and its derivatives with respect to q_z are analytic functions of q_z.[4] From this it follows[4] that the function

$$F^{\pm}(\vec{q},k_z) = p\int_0^{T/p} du\, W^{\pm}(\vec{q},k_z,u) e^{iu\omega_c(1-p\beta_{\pm})} \tag{11}$$

and its derivatives are also analytic in q_z.

Now consider the function

$$g(\vec{q},k_z) = (1 - e^{-2\pi i\beta_{\pm}})^{-1} . \tag{12}$$

This has simple poles at $-\beta_{\pm} = n = 0, \pm 1, \pm 2, \ldots,$ or at

$$q_z = q_n(k_z) \equiv \frac{\omega' \pm (np+1)\omega_c}{v_s^z(k_z)} . \tag{13}$$

If we write

$$g(\vec{q},k_z) = A(\vec{q},k_z) - S(\vec{q},k_z), \tag{14}$$

where

$$S(\vec{q},k_z)=\pm\frac{ip\omega_c}{2\pi v_s^z(k_z)}\sum_{n=-N}^{N}\frac{1}{q_z-q_n(k_z)}, \tag{15}$$

then $A(\vec{q},k_z)$ is an analytic function for $q_z\in D_N$ where the domain D_N is a circle of radius smaller than the smallest magnitude of $q_{N+1}(k_z)$. Then writing the conductivity as

$$\sigma^{\pm}(\vec{q},\omega)=\tfrac{1}{2}ie^2\frac{eH}{4\pi^3c}\int_{FS}dk_z\,F^{\pm}(\vec{q},k_z)\times[A(\vec{q},k_z)-S(\vec{q},k_z)], \tag{16}$$

we see that the first term gives a contribution to $\sigma^{\pm}(q,\omega)$ that is analytic[4] in q_z for $q_z\in D_N$, while the second term gives only those singularities resulting from the $q_n(k_z)$. There can be no additional singularities as $S(\vec{q},k_z)$ is a finite sum. If we now increase N, the D_N get progressively larger, and we thus see that the only singularities that $\sigma^{\pm}(q,\omega)$ has for any region in the complex q_z plane, are those resulting from the $q_n(k_z)$. That is, only the singularities explicitly exhibited term by term in (6) exist. No new singularities occur due to the infinite sum, and we can consider the singularities as being simply a sum of those kinds already familiar to us from our study of cylindrical Fermi surfaces in I.

IV. OPTICAL MASS

If we return to (6) and examine this in the $\vec{q}=0$, $H=0$ limit, we should have the classical Drude form for the conductivity. Were it not for phonon effects, the parameters entering in this form would be the same as those obtained from the intraband optical absorption spectrum. Because of the high frequencies involved in the optical measurements, the phonon effects produce different lifetimes and masses than at the lower frequencies of interest here.[5] We shall thus be able to relate the optical mass to the Fermi-surface characteristics only in the absence of phonon effects, which must be treated separately.

Using (7) one can write

$$\sum_{n=-\infty}^{\infty}|v_n^{+}(0,k_z)|^2=\frac{1}{T}\int_0^T du\,|v^{+}(k_z,u)|^2=\frac{1}{T}\frac{c}{eH}\frac{1}{2}\int_0^{2\pi}d\phi\,\frac{\partial\rho_E^2(\phi)}{\partial E}\bigg|_{\phi}\,|v^{+}(k_z,\phi)|^2, \tag{17}$$

where we have used (3) to change from the time variable u to the angular variable ϕ. Hence in the $\vec{q}=0$, $H=0$ limit, the conductivity (6) becomes

$$\sigma^{\pm}(0,\omega)=\frac{i}{\omega+i/\tau}\frac{e^2}{16\pi^3}\int_{FS}dk_z\times\int_0^{2\pi}d\phi\,\frac{\partial\rho_E^2(\phi)}{\partial E}\bigg|_{\phi}\,|v^{+}(k_z,\phi)|^2. \tag{18}$$

This should be equal to the Drude conductivity

$$\sigma_D(\omega)=\frac{i}{\omega+i/\tau}\frac{ne^2}{m_D}, \tag{19}$$

where n is the electron density in the conduction band

$$n=\frac{2}{(2\pi)^3}\frac{1}{2}\int_{FS}dk_z\int_0^{2\pi}d\phi\,\rho_E^2(\phi), \tag{20}$$

and m_D is the optical mass.

Noting that (2) can be written as

$$v^{\pm}(k_z,\phi)=e^{\pm i\phi}\,2\left(\frac{\partial\rho_E^2(\phi)}{\partial E}\bigg|_{\phi}\right)^{-1}\left(\rho_E(\phi)\mp i\frac{\partial\rho_E(\phi)}{\partial\phi}\bigg|_E\right), \tag{21}$$

we equate (18) and (19) (this is equivalent to defining m_D by a conductivity sum rule for the conduction electrons) to get

$$\frac{1}{m_D}=\int_{FS}dk_z\int_0^{2\pi}d\phi\,2\left(\frac{\partial\rho_E^2(\phi)}{\partial E}\bigg|_{\phi}\right)^{-1}\left[\rho_E^2(\phi)+\left(\frac{\partial\rho_E(\phi)}{\partial\phi}\bigg|_E\right)^2\right]\left(\int_{FS}dk_z\int_0^{2\pi}d\phi\,\rho_E^2(\phi)\right)^{-1}. \tag{22}$$

We have suppressed reference to k_z, which is, of course, held constant in the derivatives. For comparison we note that, at a given k_z, the cyclotron mass is

$$m_c(k_z)=\frac{1}{4\pi}\int_0^{2\pi}d\phi\,\frac{\partial\rho_E^2(\phi)}{\partial E}\bigg|_{\phi}. \tag{23}$$

In both (22) and (23), E is to be evaluated at the Fermi energy.

V. SIMPLE MODEL

In order to examine some quantitative implications of (6), we shall, for the rest of this paper, restrict ourselves to a simple model. We assume that the surfaces of constant energy, in the vicinity of the Fermi surface, have the form

$$E(\vec{k})=E(\rho,\phi)+g(k_z). \tag{24}$$

This implies, by (2), that v^z is independent of ϕ. [Another simple model, where $v^z(k_z,u)=v_s(k_z)+\omega(k_z)\cos(p\omega_cu)$ and $v^{\pm}(k_z,u)=v(k_z)\,e^{\pm i\omega_cu}$, allows one to obtain the summation in (6) in closed form. However, it can be shown that no model with $v^{\pm}$ in this form is consistent with any analytic Fermi surface except one with cylindrical symmetry.] Since (24) implies v^z has no ϕ dependence, and since $\vec{q}$ is in the z direction, we have

$$\vec{q}\cdot\vec{R}_p(k_z,u)=0 \tag{25}$$

and hence

$$v_n^{\pm}(\vec{q},k_z)=v_n^{\pm}(0,k_z)\equiv u_n^{\pm}(k_z). \tag{26}$$

If the Fermi surface has a plane of mirror symmetry containing the z axis, we may take

$$E(\rho, \phi) = E(\rho, -\phi) \tag{27}$$

and then the $u_n^\pm$ are real.

The conductivity (6) then becomes

$$\sigma^\pm(\vec{q}, \omega) = \tfrac{1}{2} e^2 \frac{eH}{4\pi^3 c} \int_{FS} dk_z K^\pm(k_z) , \tag{28}$$

where

$$K^\pm(k_z) = \frac{2\pi i}{\omega_c} \sum_{n=-\infty}^{\infty} \frac{[u_n^\pm(k_z)]^2}{\omega' - \vec{q}\cdot\vec{v}_s(k_z) \pm (pn+1)\omega_c} . \tag{29}$$

We now specify $E(\rho, \phi)$ so that

$$E(\vec{k}) = \rho^2/2m + g(k_z) + f(\phi) . \tag{30}$$

According to (23), this $E(\vec{k})$ gives a constant cyclotron mass, equal to m, and (3) implies that

$$\phi = \omega_c u, \quad \omega_c = eH/mc \text{ (independent of } k_z). \tag{31}$$

Combining (4), (25), (26), (21), and (31), and integrating by parts, we get

$$u_n^\pm(k_z) = \frac{1 \pm pn}{2\pi m} \int_0^{2\pi} d\phi\, \rho_E(\phi) e^{-ipn\phi} . \tag{32}$$

For the case $f(\phi) = \alpha \cos p\phi$, so

$$E(\vec{k}) = \rho^2/2m + g(k_z) + \alpha \cos p\phi, \tag{33}$$

[with

$$\alpha < E - g(k_z) \tag{34}$$

at the Fermi energy, so that $\rho_E^2(\phi)$ is positive], the integral (32) can be performed to give

$$u_0^\pm(k_z) = (2\alpha/m\eta)^{1/2} F(-\tfrac{1}{4}, \tfrac{1}{4}; 1; \eta^2), \tag{35}$$

where

$$\eta(k_z) = \alpha/[E - g(k_z)] < 1, \tag{36}$$

and $F(a, b; c; z)$ is the Gauss hypergeometric series.[6] For $n \neq 0$, (32) gives

$$u_n^\pm(k_z) = -(1 \pm np)\left(\frac{2\alpha}{m\eta}\right)^{1/2} \frac{1}{2^{3n-1}} \frac{(2n-2)!}{(n-1)!n!}\, \eta^2 \times F(\tfrac{1}{2}n - \tfrac{1}{4}, \tfrac{1}{2}n + \tfrac{1}{4}; n+1; \eta^2). \tag{37}$$

These results may be written as

$$u_n^\pm(k_z) = (1 \pm np)\left(\frac{2\alpha}{m\eta}\right)^{1/2} \left(\delta_{n,0} - 2(\tfrac{1}{8}\eta)^n \times \sum_{k=0}^{\infty} \frac{(4k+2n-2)!(1-\delta_{k,0}\delta_{n,0})}{(2k+n-1)!(k+n)!k!} (\tfrac{1}{8}\eta)^{2k}\right) . \tag{38}$$

Thus $u_n^\pm(k_z)$ begins like η^n and, therefore, the coefficient of the nth singularity in (29) begins like η^{2n}. Since $\eta < 1$, we see that the successively higher singularities have smaller and smaller coefficients.

The surface (33) contains p ridges running parallel to k_z and spaced at equal azimuthal angles around the surface. If the ridges are not too deep, η will be sufficiently small that we need keep only the first several terms in (38). That is, in this case we can neglect all but the smallest n values in (38) and, therefore, in (29). We then get

$$K^\pm(k_z) = \frac{2\pi i}{\omega_c}\frac{2\alpha}{m\eta} \left(\frac{1 - \frac{1}{8}\eta^2}{\omega' - \vec{q}\cdot\vec{v}_s \pm \omega_c} + \frac{\frac{1}{16}\eta^2(1+p)^2}{\omega' - \vec{q}\cdot\vec{v}_s \pm (1+p)\omega_c} + \frac{\frac{1}{16}\eta^2(1-p)^2}{\omega' - \vec{q}\cdot\vec{v}_s \pm (1-p)\omega_c} + 0(\eta^4)\right) . \tag{39}$$

VI. SPECIFIC EXAMPLE

The integral over k_z required by (28) can be performed approximately, with the result depending on the nature of $\vec{v}_s(k_z)$, as discussed by Gantmakher and Kaner[7] and in I. However, by choosing $g(k_z)$ as given in Fig. 2 of I,

$$g(k_z) = (2kv/\pi)\sin^2(\pi k_z/2k), \tag{40}$$

the integral may be performed exactly to give

$$\sigma^\pm(\vec{q}, \omega) = \frac{ie^2}{\pi^2} \frac{k(\mu - kv/\pi)}{\Omega_0} \left\{ \frac{1}{(1 - z_0^2)^{1/2}} + \frac{\alpha^2}{\mu(\mu - 2kv/\pi)} \sum_{n=-1}^{1} c_n \frac{\Omega_0}{\Omega_n} \frac{1}{\gamma^2 + z_n^2} \times \left[\frac{\gamma^2}{(1+\gamma^2)^{1/2}} + \frac{z_n^2}{(1-z_n^2)^{1/2}} \right] \right\} , \tag{41}$$

where

$$\Omega_n = \omega' \pm (pn+1)\omega_c , \quad z_n = qv/\Omega_n ,$$

$$\gamma^2 = \frac{(kv/\pi)^2}{\mu(\mu - 2kv/\pi)} ,$$

and

$$c_0 = -\tfrac{1}{8} , \quad c_{\pm 1} = \tfrac{1}{16}(1 \pm p)^2 . \tag{42}$$

In (41) the square roots are defined so that, for real q, the real part of $\sigma^\pm(q, \omega)$ is always nonnegative (see I). Of course, there is no singularity at $z_n^2 = -\gamma^2$, only at $z_n^2 = 1$.

In Fig. 1, we plot $(\omega_c/\omega_p^2)\, 4\pi i \sigma^\pm(\vec{q}, 0)$, as a function of qv/ω_c, for the collisionless limit $(\tau \to \infty)$ and the parameters

$$p = 4, \quad \gamma^2 = 3, \quad \alpha^2/\mu(\mu - 2kv/\pi) = \tfrac{1}{100} . \tag{43}$$

[$\gamma^2 = 3$ corresponds to an average neck radius about 0.28 times the average belly radius. The value of α^2 is about $\frac{1}{7}$ of the maximum given by (34).] Here

$$\omega_p^2 = 4\pi n e^2/m_D , \tag{44}$$

and (22) gives a 1% difference between m_D and m. This correction only enters because, following I, we have chosen to express our results in terms of ω_p^2. At low frequencies $(\omega \ll \omega_c)$, the term multiplying the brackets in (41) is independent of all masses.[5]

As in I, the intersections of the curve in Fig. 1(a), with the straight lines given by

$\mp(\omega_c^3 c^2/\omega\, \omega_p^2 v^2)(qv/\omega_c)^2$, correspond to roots of the electromagnetic field dispersion relation

$$q^2 = (4\pi i\omega/c^2)\sigma^{\pm}(\vec{q}, \omega) \tag{45}$$

for $\omega \ll \omega_c$, providing that Re $\sigma^{\pm}(\vec{q}, \omega)$ is sufficiently small.

VII. POLES

The pole contributions to the electromagnetic field, or the surface impedance, corresponding to the modes of the electromagnetic field, occur at the roots of (45). It is apparent from Fig. 1 that there will always be such roots [intersections of the curve in Fig. 1(a) with straight lines from the origin] in the vicinity of

$$(qv^2/\omega_c)^2 = (\pm 4n + 1)^2 \tag{46}$$

for this case of $p = 4$. Experimentally, however, these harmonic DSCR are not likely to be detectable, for, like the DSCR at $n = 0$, lifetime effects are most important near the values (46) (see I). Hence, if the pole contributions resulting from the azimuthal variation of the Fermi surface are to be detectable, it can only be at roots of (45) away from the values (46). However, the singular parts of (41) only dominate quite close to the values (46), and Im$\sigma^{\pm}$ is quite small between these values. A glance at Fig. 1(b) shows that Re$\sigma^{\pm}$ is appreciable there, so that even though the real part of (45) vanishes, the imaginary part stays sizable. Thus, rather than getting well-defined modes, as one does for the helicon, where $(qv/\omega_c)^2 \ll 1$ for $(qv/\omega_c)^2 > 1$, one gets appreciably damped modes (in terms of magnetic field, ΔH will be comparable to H) and it is not likely that one can detect any such modes resulting from the azimuthal variations. The qualitative aspects of this result do not appear to depend on the particular choice of parameters; they are the same even for α as large as (34) allows. [8]

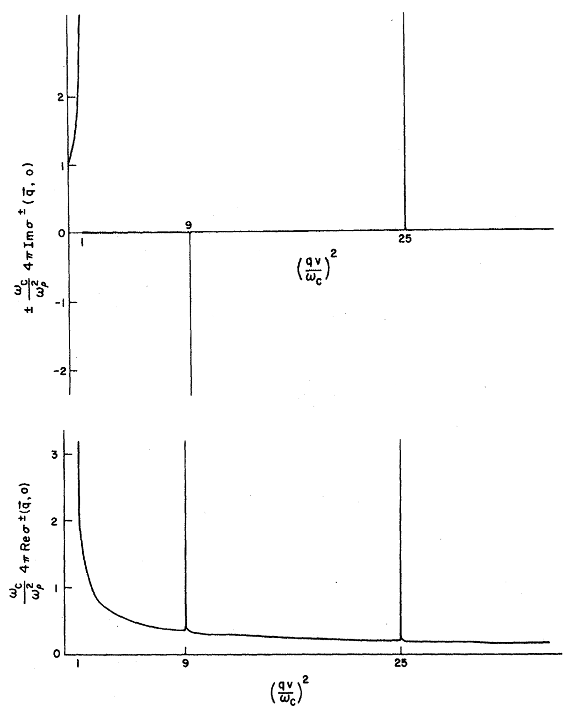

FIG. 1. (a) $\pm(\omega_c/\omega_p^2)\, 4\pi\, \mathrm{Im}\, \sigma^{\pm}(\vec{q}, 0)$ and (b) $(\omega_c/\omega_p^2)\, 4\pi \times \mathrm{Re}\sigma^{\pm}(\vec{q}, 0)$ as a function of $(qv/\omega_c)^2$ for the parameters (43) in the collisionless limit.

VIII. GANTMAKHER-KANER OSCILLATIONS

We now turn to the GKO. The presence of additional branch cuts in the conductivity ($n \neq 0$) implies additional GKO.

A. Damping of Higher GKO

Reference to I shows that the leading GKO contribution to the surface impedance for the nth singularity, for a slab of metal of thickness L, behaves like $\exp\{i(L/v)[\omega \pm (pn+1)\omega_c + i/\tau]\}$, and hence the damping $e^{-L/l}$ depends only on how many mean free paths l the electrons must travel through the sample, and not at all on the value of n. The GKO's from all the singularities have the same damping.

B. Amplitude of Higher GKO

The amplitudes of the GKO are determined by the strength of the singularities, and, as we have seen from (38), the higher singularities have coefficients which get progressively smaller. Following I, we can find the leading GKO contribution of the nth singularity as a function of thickness of the sample and its leading dependence on the parameter α, which measures the magnitude of the azimuthal variation, to be

$$[e^{i\lambda_n'}/(\lambda_n')^{3/2}]\,\alpha^{2|n|}\, F(\alpha^{2|n|}\lambda_n'), \tag{47}$$

where

$$\lambda_n' = \frac{[\omega \pm (pn+1)\omega_c + i/\tau]L}{v}, \tag{48}$$

and $F(x)$ is a slowly varying well-behaved function. Thus the behavior of the different GKO is essentially the same, but the amplitudes get progressively smaller for $n \neq 0$. [Since the dimensionless combination in which α appears is, by (34), always less than unity, higher powers of α^2 correspond to small coefficients.]

C. Position of Branch Cuts

Recently Baraff and Phillips[9] have shown that the branch cut that produces the GKO need not begin at the usual place [the minimum or extremum of $\Omega_n(k_z)/v_z^z(k_z)$], but rather may begin at a point where the "topological effectiveness" (essentially the coefficient of the singularity) changes rapidly. This observation does not change our results (in fact, model D of I can be considered as an extreme example of the effect), and we can see the effect with a simple model.

The contribution of the nth singularity to the conductivity is, according to (6), of the form

$$\sigma_n^{\pm} \sim \int_{FS} dk_z \frac{A(k_z)}{q - B(k_z)} , \tag{49}$$

where

$$B(k_z) = \Omega_n(k_z)/v_s^z(k_z) \tag{50}$$

and $A(k_z)$ is the topological effectiveness. Let us consider the case where $A(k_z)$ rises suddenly at $k_z = k_A$, but is otherwise slowly varying, so that we may write, approximately,

$$\frac{dA(k_z)}{dk_z} \simeq A\delta(k_z - k_A). \tag{51}$$

We will only consider the contribution from $k_z > 0$. [For a Fermi surface with reflection symmetry, the contribution from $k_z < 0$ is the same except for a reversal of sign of $B(k_z)$.] Then, integrating by parts, we get (if the Brillouin-zone boundary is at $k_z = k$)

$$\int_0^k dk_z \frac{A(k_z)}{q - B(k_z)} \simeq [A(k) - A] \int_0^k \frac{dk_z}{q - B(k_z)} + A \int_{k_A}^k \frac{dk_z}{q - B(k_z)} . \tag{52}$$

Let us suppose, without loss of generality, that $B(k_z)$ has one minimum value which is at $k_z = k_B < k_A$. Then the first integral in (52) has the usual branch cut at $q = B(k_B)$. But the second integral has a branch cut at $q = B(k_A)$ (as in model D of I). The point, however, is that (according to I), the branch cut at $q = B(k_A)$ is logarithmic and gives a contribution to the surface impedance that behaves with the sample thickness L like

$$e^{iLB(k_A)}/L \ln^2 L , \tag{53}$$

while any other branch cut, coming from $q = B(k_B)$, produces a contribution whose denominator increases with L at least as fast as the denominator in (53). Since the damping for any GKO goes like $e^{-L/l}$, the dependence on L of this denominator will dominate. Hence, for reasonably thick samples, the sudden change in the topological effectiveness at $k_z = k_A$ should produce GKO that are more detectable than those produced by most kinds of minima or extrema in $B(k_z)$.

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