PHYSICAL REVIEW B VOLUME 29, NUMBER 2 15 JANUARY 1984

Theory of the surface acoustic soliton. I. Insulating solid

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(Received 18 April 1983)

Starting from the equation of motion for anharmonically interacting surface phonons, a theory of the surface acoustic soliton in an insulating solid is developed, based on the coherent-state representation. The two-dimensional nonlinear integro-differential equation for the classical displacement field is obtained. By using the reductive perturbation method, the equation can be reduced to the nonlinear Schrödinger equation, which allows the existence of the surface acoustic soliton of the envelope type. The possibility of observing the surface acoustic soliton is also discussed.

I. INTRODUCTION

As is well known, surface acoustic waves (SAW) are propagated along the solid surface by confining their energy within about one wavelength from the crystal surface. Since the elastic energy density of the SAW becomes much larger than that of the bulk elastic waves, especially in a high-frequency region, the effect of the crystal anharmonicity will be enhanced in dealing with the propagation of the SAW with high frequency. There also exists the dispersive effect due to the discreteness of the crystal lattice or to the layered structure, so we can naturally expect that the surface acoustic soliton (hereafter referred to as the SA soliton) may be realized under an appropriate condition.

As for the bulk phonon, the self-trapping of heat pulses has been found in a bulk crystal of NaF with high purity at low temperatures under certain conditions on the phonon dispersion and lattice anharmonicity.[1] This phenomenon essentially occurs since the release of a large amount of thermal energy into a confined spot of the crystal enhances the lattice anharmonicity which is balanced with the lattice dispersion. In the field of SAW devices, there recently appeared an attempt[2] to detect the soliton based on a simple one-dimensional model, which has not yet been successful in detecting it experimentally.

In this paper we try to develop a theory of the SA soliton in an insulating crystal on the basis of a surface-phonon picture in the description of physical properties of solids with a stress-free boundary. A soliton as a macroscopic excitation in the crystalline solids is assumed to be a state in which a large number of phonons are excited and consequently the quantum fluctuations may be neglected. Following the work by Ichikawa *et al.*,[3] we take the expectation value of the equation of motion for the interacting surface phonons in a coherent state which automatically ensures the stress-free boundary condition of the linear elasticity theory. As a result, we can derive a nonlinear integro-differential equation for the classical displacement field, which is reduced to the well-known nonlinear Schrödinger equation by applying the reductive perturbation method.

In Sec. II we briefly recapitulate the quantization of the SAW which results in the surface phonon, and we then define the coherent states of the surface phonons. In Sec. III the Hamiltonian of the interacting surface phonons is described up to the cubic term of the displacement field from which we derive the equation of motion for the interacting surface phonons. Taking the expectation value of the equation of motion in the surface-phonon coherent state, we obtain the nonlinear integro-differential equation as the temporal evolution equation for the classical displacement field. In Sec. IV the reductive perturbation method for the slow modulation of a rapidly oscillating phenomenon is applied to the evolution equation to derive the simple nonlinear equation, which in our case becomes the nonlinear Schrödinger equation. Discussion on the condition for observing the SA soliton experimentally is presented in Sec. V. The final section will be devoted to a brief summary of the result and further discussions.

II. SURFACE PHONONS AND COHERENT STATES

In an isotropic elastic medium occupying a half-space with a stress-free boundary, there exists the famous surface acoustic wave (Rayleigh wave), which is most characteristic for the solid surface. The quantization of the SAW can be performed as follows: Let us set the configuration where the insulating solid extends over the half-space $x_3 > 0$ and has the flat surface $x_3 = 0$ parallel to the x_1-x_2 plane. The displacement vector $u_1(\vec{r},t)$ $(i=1,2,3)$ at the space-time point $(\vec{r},t)$ of the medium in this configuration can be expanded in terms of the Rayleigh-mode eigenfunction $u_i^{\vec{k}}(\vec{r})$ as

$$u_i(\vec{r},t)=\sum_{\vec{k}}(2\rho\omega_k)^{-1/2}[a_{\vec{k}}(t)+a^{\dagger}_{-\vec{k}}(t)]u_i^{\vec{k}}(\vec{r}) , \qquad (2.1)$$

where

$$u_i^{\vec{k}}(\vec{r})=\left[\frac{k}{WS}\right]^{1/2} e^{i\vec{k}\cdot\vec{x}}\times\begin{cases} i\frac{k_i}{k}\left[\frac{1}{\kappa_l}e^{-k_l x_3}-\frac{2\kappa_t}{\kappa_t^2+1}e^{-k_t x_3}\right], & i=1,2 \\ \left[-e^{-k_l x_3}+\frac{2}{\kappa_t^2+1}e^{-k_t x_3}\right], & i=3 \end{cases} \quad (2.2)$$

$k_l=\kappa_l k$, $k_t=\kappa_t k$, $\kappa_l=[1-(v/v_l)^2]^{1/2}$, $\kappa_t=[1-(v/v_t)^2]^{1/2}$, $W=(\kappa_l-\kappa_t)(\kappa_l-\kappa_t+2\kappa_l\kappa_t^2)/(2\kappa_l^3\kappa_t)$, and v, v_t, and v_l are the velocities of the SAW, the transverse sound wave, and the longitudinal sound wave, respectively. ρ is the mass density, S is the surface area, $\vec{k}=(k_1,k_2)$ is the wave vector parallel to the surface, and $\vec{r}=(\vec{x},x_3)=(x_1,x_2,x_3)$. Throughout this paper, $\hbar$ and S are taken to be unity. $a_{\vec{k}}$ and its Hermitian conjugate $a_{\vec{k}}^\dagger$ are the annihilation and creation operators of the surface phonon (a quantum of the SAW) which satisfy the following commutation relation:

$$[a_{\vec{k}},a_{\vec{k}'}^\dagger]=\delta_{\vec{k},\vec{k}'} . \quad (2.3)$$

Next we define the coherent state of the surface phonons $|\alpha_{\vec{k}}\rangle$ as an eigenstate of the annihilation operator $a_{\vec{k}}$ by

$$a_{\vec{k}}|\alpha_{\vec{k}}\rangle=\alpha_{\vec{k}}|\alpha_{\vec{k}}\rangle . \quad (2.4)$$

In a standard manner,[4] $|\alpha_{\vec{k}}\rangle$ is written in terms of the eigenstate of the occupation number operator $N_{\vec{k}}\equiv a_{\vec{k}}^\dagger a_{\vec{k}}$, $N_{\vec{k}}|n_{\vec{k}}\rangle=n_{\vec{k}}|n_{\vec{k}}\rangle$, as

$$|\alpha_{\vec{k}}\rangle=e^{-|\alpha_{\vec{k}}|^2/2}\sum_{n_{\vec{k}}}\frac{(\alpha_{\vec{k}})^{n_{\vec{k}}}}{(n_{\vec{k}}!)^{1/2}}|n_{\vec{k}}\rangle . \quad (2.5)$$

The following identities may be useful to perform the calculation in the next section:

$$\frac{1}{\pi}\int d^2\alpha_{\vec{k}}|\alpha_{\vec{k}}\rangle\langle\alpha_{\vec{k}}|=1 , \quad (2.6a)$$

$$\frac{1}{\pi}\int d^2\alpha_{\vec{k}}f(\alpha_{\vec{k}}^*)e^{\alpha_{\vec{k}}\beta_{\vec{k}}^*-|\alpha_{\vec{k}}|^2}=f(\beta_{\vec{k}}^*) . \quad (2.6b)$$

Here $f(x)$ is an arbitrary real function of x which can be expanded in a power series.

III. HAMILTONIAN OF THE INTERACTING SURFACE PHONONS AND NONLINEAR WAVE EQUATION

The elastic energy density E up to the cubic term in deformation tensor $\eta_{ij}=\frac{1}{2}(\zeta_{ij}+\zeta_{ji}+\zeta_{ki}\zeta_{kj})$ is given by

$$E=\frac{\rho}{2}\sum_i \dot{u}_i^2+\frac{1}{2}\sum_{i,j,k,l}c_{ijkl}\eta_{ij}\eta_{kl} + \sum_{\substack{i,j,k\\l,m,n}}c_{ijklmn}\eta_{ij}\eta_{kl}\eta_{mn} , \quad (3.1)$$

where $\zeta_{ij}=\partial_j u_i$ and c_{ijkl} and c_{ijklmn} are the second- and third-order elastic constant tensors, respectively. For an isotropic crystal, the energy density which accounts for the cubic anharmonic interaction is given by

$$E_a=\sum_{\substack{i,j,k,\\l,m,n}}\hat{c}_{ijklmn}\zeta_{ij}\zeta_{kl}\zeta_{mn} , \quad (3.2)$$

where

$$\begin{aligned}\hat{c}_{ijklmn}&=\alpha\delta_{ij}\delta_{kl}\delta_{mn}+\beta(\delta_{ij}\delta_{kn}\delta_{lm}+\delta_{kl}\delta_{in}\delta_{jm}+\delta_{mn}\delta_{jk}\delta_{il})+\gamma(\delta_{in}\delta_{jk}\delta_{lm}+\delta_{il}\delta_{jm}\delta_{kn})\\&+(\lambda/6+\beta)(\delta_{ij}\delta_{km}\delta_{ln}+\delta_{kl}\delta_{im}\delta_{jn}+\delta_{mn}\delta_{ik}\delta_{jl})\\&+(\mu/6+\gamma)(\delta_{lm}\delta_{ik}\delta_{jn}+\delta_{km}\delta_{il}\delta_{jn}+\delta_{km}\delta_{in}\delta_{jl}+\delta_{ln}\delta_{ik}\delta_{jm}+\delta_{im}\delta_{ln}\delta_{jk}+\delta_{kn}\delta_{im}\delta_{jl}) .\end{aligned} \quad (3.3)$$

Here λ and μ are the Lamé coefficients and α, β, and γ are, respectively, defined by

$$\alpha=c_{111}-6c_{155}+4c_{456} ,$$

$$\beta=c_{155}-2c_{456} ,$$

and

$$\gamma=c_{456} ,$$

where the following replacements of indices were conventionally used:

$11\rightarrow1$, $22\rightarrow2$, $33\rightarrow3$, $23\rightarrow4$, $13\rightarrow5$, $12\rightarrow6$.

Substituting the expansion (2.1) into Eq. (3.1) and integrating over a whole volume, we can obtain the Hamiltonian for the interacting surface phonons in an insulating solid as

$$\begin{aligned}H&=H_0+H_I ,\\ H_0&=\sum_{\vec{k}}\omega_k(a_{\vec{k}}^\dagger a_{\vec{k}}+\tfrac{1}{2}) ,\\ H_I&=\sum_{\vec{k},\vec{k}',\vec{k}''}\Phi_{\vec{k},\vec{k}',\vec{k}''}(a_{\vec{k}}+a_{-\vec{k}}^\dagger)(a_{\vec{k}'}+a_{-\vec{k}'}^\dagger)\\&\quad\times(a_{\vec{k}''}+a_{-\vec{k}''}^\dagger)\delta_{\vec{k}+\vec{k}'+\vec{k}'',\vec{0}} ,\end{aligned} \quad (3.4)$$

where $\Phi_{\vec{k},\vec{k}',\vec{k}''}$ is the three-phonon vertex function defined by

$$\Phi_{\vec{k},\vec{k}',\vec{k}''}=(2\rho)^{-3/2}(\omega_k\omega_{k'}\omega_{k''})^{-1/2}\times\sum_{\substack{i,j,k,\\ l,m,n}}\hat{c}_{ijklmn}\int d\vec{r}(\partial_j u_i^{\vec{k}})(\partial_l u_k^{\vec{k}'})(\partial_n u_m^{\vec{k}''}) , \tag{3.5}$$

whose explicit expressions will be given in Appendix A. Then the equation of motion for the interacting surface phonons can be easily derived in the following form:

$$i\dot{a}_{\vec{k}}=[a_{\vec{k}},H]=\omega_k a_{\vec{k}}+3\sum_{\vec{k}'}\Phi_{-\vec{k},\vec{k}',\vec{k}-\vec{k}'}(a_{\vec{k}'}+a^\dagger_{-\vec{k}'})\times(a_{\vec{k}-\vec{k}'}+a^\dagger_{-\vec{k},\vec{k}'}) \tag{3.6a}$$

and similarly

$$i\dot{a}^\dagger_{-\vec{k}}=-\omega_k a^\dagger_{-\vec{k}}-3\sum_{\vec{k}'}\Phi_{-\vec{k},\vec{k}',\vec{k}-\vec{k}'}(a_{\vec{k}'}+a^\dagger_{-\vec{k}'})\times(a_{\vec{k}-\vec{k}'}+a^\dagger_{-\vec{k}+\vec{k}'}) . \tag{3.6b}$$

We assume here that the soliton is a macroscopic nonlinear entity in the SAW, if it really exists, which corresponds to a state in which a large number of surface phonons are excited, and consequently the quantum fluctuations may be neglected.[3] This prospect can be realized by taking the expectation value of the equations of motion (3.6a) and (3.6b) in a coherent state of the surface phonons $|\{\alpha_{\vec{k}}\}\rangle=\prod_{\vec{k}}|\alpha_{\vec{k}}\rangle$.

In the coherent representation of the interacting surface phonons, the temporal evolution equation for the expectation value of the annihilation operator $a_{\vec{k}}$ is given by

$$i\dot{\alpha}_{\vec{k}}=\omega_k\alpha_{\vec{k}}+3\sum_{\vec{k}'}\Phi_{-\vec{k},\vec{k}',\vec{k}-\vec{k}'}(\alpha_{\vec{k}'}+\alpha^*_{-\vec{k}'})\times(\alpha_{\vec{k}-\vec{k}'}+\alpha^*_{-\vec{k}+\vec{k}'}) \tag{3.7a}$$

and similarly for $\alpha^*_{-\vec{k}}$ by

$$i\dot{\alpha}^*_{-\vec{k}}=-\omega_k\alpha^*_{-\vec{k}}-3\sum_{\vec{k}'}\Phi_{-\vec{k},\vec{k}',\vec{k}-\vec{k}'}(\alpha_{\vec{k}'}+\alpha^*_{-\vec{k}'})\times(\alpha_{\vec{k}-\vec{k}'}+\alpha^*_{-\vec{k}+\vec{k}'}) . \tag{3.7b}$$

As the expectation value $\langle\{\alpha_{\vec{k}}\}|u_i(\vec{r},t)|\{\alpha_{\vec{k}}\}\rangle$ corresponds to the classical displacement field $u_i^c(\vec{r},t)$, we can express $u_i^c(\vec{r},t)$ in terms of $\alpha_{\vec{k}}$ and $\alpha^*_{-\vec{k}}$ as

$$u_i^c(\vec{r},t)=\langle\{\alpha_{\vec{k}}\}|u_i(\vec{r},t)|\{\alpha_{\vec{k}}\}\rangle=\sum_{\vec{k}}(2\rho\omega_k)^{-1/2}[\alpha_{\vec{k}}(t)+\alpha^*_{-\vec{k}}(t)]u_i^{\vec{k}}(\vec{r}) . \tag{3.8}$$

If we define

$$A_{\vec{k}}(t)\equiv(2\rho\omega_k)^{-1/2}[\alpha_{\vec{k}}(t)+\alpha^*_{-\vec{k}}(t)] , \tag{3.9}$$

then Eq. (3.8) leads to

$$u_i^c(\vec{r},t)=\sum_{\vec{k}}A_{\vec{k}}(t)u_i^{\vec{k}}(\vec{r}) . \tag{3.10}$$

Using Eqs. (3.7a), (3.7b), and (3.9), we obtain the temporal evolution equation for $A_{\vec{k}}(t)$ as

$$\ddot{A}_{\vec{k}}+\omega_k^2A_{\vec{k}}+6\sqrt{2\rho}\sum_{\vec{k}'}(\omega_k\omega_{k'}\omega_{\vec{k}-\vec{k}'})^{1/2}\times\Phi_{-\vec{k},\vec{k}',\vec{k}-\vec{k}'}A_{\vec{k}'}A_{\vec{k}-\vec{k}'}=0 . \tag{3.11}$$

Here we consider the third component of the classical displacement field at the surface ($x_3=0$) as

$$U(\vec{x},t)\equiv u_3^c(\vec{x},0,t)=\sum_{\vec{k}}B_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} , \tag{3.12}$$

where

$$B_{\vec{k}}(t)=\left[\frac{k}{W}\right]^{1/2}\frac{1-\kappa_t^2}{1+\kappa_t^2}A_{\vec{k}}(t) . \tag{3.13}$$

Then $B_{\vec{k}}(t)$ satisfies the following equation:

$$\ddot{B}_{\vec{k}}+\omega_{\vec{k}}^2B_{\vec{k}}+\sum_{\vec{k}'}F_{\vec{k},\vec{k}'}B_{\vec{k}'}B_{\vec{k}-\vec{k}'}=0 , \tag{3.14}$$

where $F_{\vec{k},\vec{k}'}$ is defined by

$$F_{\vec{k},\vec{k}'}=6\sqrt{2\rho}(\omega_k\omega_{k'}\omega_{\vec{k}-\vec{k}'})^{1/2}(k'|\vec{k}-\vec{k}'|)^{-1/2}\times\left[\frac{1+\kappa_t^2}{1-\kappa_t^2}\right]\Phi_{-\vec{k},\vec{k}',\vec{k}-\vec{k}'} . \tag{3.15}$$

Substituting the inverse transformation of Eq. (3.12) into Eq. (3.14), we obtain the following nonlinear wave equation for $U(\vec{x},t)$:

$$\partial_t^2U(\vec{x},t)+\int d\vec{x}_1W(\vec{x}_1)U(\vec{x}-\vec{x}_1,t)+\int\int d\vec{x}_1d\vec{x}_2F(\vec{x}_1,\vec{x}_2)U(\vec{x}-\vec{x}_1,t)\times U(\vec{x}-\vec{x}_1-\vec{x}_2,t)=0 , \tag{3.16}$$

where

$$W(\vec{x})=\sum_{\vec{k}}\omega_k^2e^{i\vec{k}\cdot\vec{x}} \tag{3.17}$$

and

$$F(\vec{x}_1,\vec{x}_2)=\sum_{\vec{k}_1,\vec{k}_2}F_{\vec{k}_1,\vec{k}_2}e^{i(\vec{k}_1\cdot\vec{x}_1+\vec{k}_2\cdot\vec{x}_2)} . \tag{3.18}$$

The second term of Eq. (3.16) gives the dispersive term. For instance, if we assume the intrinsic lattice dispersion as[5]

$$\omega_k^2=(vk)^2[1-(hk)^2] , \tag{3.19}$$

where h is the length parameter whose order is of lattice constant, we obtain

$$\int d\vec{x}_1 W(\vec{x}_1) U(\vec{x}-\vec{x}_1,t) = -[v^2\partial_x^2 + (vh)^2\partial_x^4] U(\vec{x},t) . \quad (3.20)$$

The third term of Eq. (3.16) provides the nonlinear term of the convolution type. Since Eq. (3.16) gives the rather complicated nonlinear integro-differential equation for the classical displacement field at the solid surface, we try, in the next section, to reduce it to the simpler nonlinear equation which allows the analytical treatment by means of the reductive perturbation method.

Here we note that the displacement field $U(\vec{x},t)$ satisfies the boundary condition at the free surface in the linear elasticity theory but not the nonlinear boundary condition. Therefore our theory should be considered as an approximate theory of the SA soliton which is valid for the weak nonlinearity. We emphasize, however, that even if the nonlinearity is weak enough for the above approximation to be valid, the soliton can still exist by keeping the balance with the weak dispersion. The generalization of the present theory to incorporate the nonlinear boundary condition seems to be very interesting and it will be left for a future problem.

IV. REDUCTION TO THE NONLINEAR SCHRÖDINGER EQUATION

In this section we show that the nonlinear wave equation (3.16) can be reduced to the nonlinear Schrödinger equation by applying the reductive perturbation method developed by Taniuti and Yajima.[6]

Now we expand the wave field $U(\vec{x},t)$ in terms of a small parameter ϵ and of harmonics $Z_l = \exp[il(\vec{k}\cdot\vec{x} - \omega_k t)]$,

$$U(\vec{x},t) = \sum_{\alpha=1}^{\infty} \sum_{l=-\infty}^{\infty} \epsilon^\alpha Z_l U_l^{(\alpha)}(\xi,\tau) , \quad (4.1)$$

where ξ and τ are the stretched coordinates defined by

$$\xi = \epsilon(\vec{k}\cdot\vec{x} - \lambda t) ,$$
$$\tau = \epsilon^2 t , \quad (4.2)$$
$$\lambda = \vec{k}\cdot\vec{v}_g ,$$

and $\vec{v}_g = \partial\omega_k/\partial\vec{k}$ is the group velocity. The above expansion (4.1) implies that the fast local oscillations are taken into account through the harmonics, while the dependence on ξ and τ guarantees the slowly varying property of the amplitude function $U_l^{(\alpha)}(\xi,\tau)$.

We first discuss the linear terms with dispersion. Substituting the expansion (4.1) into Eq. (3.16), we obtain for the first term

$$\partial_t^2 U(\vec{x},t) = \sum_{\alpha=1}^{\infty} \sum_{l=-\infty}^{\infty} \epsilon^\alpha Z_l [-(l\omega_k)^2 + 2i\epsilon l\omega_k \lambda\partial_\xi - 2i\epsilon^2 l\omega_k \partial_\tau + \epsilon^2\lambda^2\partial_\xi^2 + O(\epsilon^3)] U_l^{(\alpha)}(\xi,\tau) , \quad (4.3)$$

and for the second term

$$\int d\vec{x}_1 W(\vec{x}_1) U(\vec{x}-\vec{x}_1,t) = \sum_{\alpha=1}^{\infty} \sum_{l=-\infty}^{\infty} \epsilon^\alpha Z_l \int d\vec{x}_1 W(\vec{x}_1) e^{-il\vec{k}\cdot\vec{x}_1}$$
$$\times [U_l^{(\alpha)}(\xi,\tau) - \xi_1\partial_\xi U_l^{(\alpha)}(\xi,\tau) + \tfrac{1}{2}\xi_1^2\partial_\xi^2 U_l^{(\alpha)}(\xi,\tau) + O(\epsilon^3)]$$
$$= \sum_{\alpha=1}^{\infty} \sum_{l=-\infty}^{\infty} \epsilon^\alpha Z_l [\widetilde{W}_l - \epsilon\widetilde{W}_l^1 \partial_\xi + (\epsilon^2/2)\widetilde{W}_l^2 \partial_\xi^2 + O(\epsilon^3)] U_l^{(\alpha)}(\xi,\tau) , \quad (4.4)$$

where $\xi_1 = \epsilon(\vec{k}\cdot\vec{x}_1)$ and $\widetilde{W}_l$, $\widetilde{W}_l^1$, and $\widetilde{W}_l^2$ are respectively defined by

$$\widetilde{W}_l = \int d\vec{x}_1 W(\vec{x}_1) e^{-il\vec{k}\cdot\vec{x}_1} = \omega_{l\vec{k}}^2 , \quad (4.5a)$$

$$\widetilde{W}_l^1 = \int d\vec{x}_1 W(\vec{x}_1)(\vec{k}\cdot\vec{x}_1) e^{-il\vec{k}\cdot\vec{x}_1} = i\partial_l \omega_{l\vec{k}}^2 , \quad (4.5b)$$

and

$$\widetilde{W}_l^2 = \int d\vec{x}_1 W(\vec{x}_1)(\vec{k}\cdot\vec{x}_1)^2 e^{-il\vec{k}\cdot\vec{x}_1} = -\partial_l^2 \omega_{l\vec{k}}^2 . \quad (4.5c)$$

For the nonlinear term, the third term of Eq. (3.16), we have

$$\int\int d\vec{x}_1 d\vec{x}_2 F(\vec{x}_1,\vec{x}_2) U(\vec{x}-\vec{x}_1,t) U(\vec{x}-\vec{x}_1-\vec{x}_2,t)$$
$$= \sum_{\alpha,\alpha'=1}^{\infty} \sum_{l,l'=-\infty}^{\infty} \epsilon^{\alpha+\alpha'} Z_{l+l'} \int\int d\vec{x}_1 d\vec{x}_2 e^{-i(l+l')\vec{k}\cdot\vec{x}_1 - il'\vec{k}\cdot\vec{x}_2} F(\vec{x}_1,\vec{x}_2) U_l^{(\alpha)}(\xi-\xi_1,\tau) U_{l'}^{(\alpha')}(\xi-\xi_1-\xi_2,\tau) , \quad (4.6)$$

where the relation

$$Z_l(\vec{x}-\vec{x}_1,t) Z_{l'}(\vec{x}-\vec{x}_1-\vec{x}_2,t) = Z_{l+l'} e^{-i(l+l')\vec{k}\cdot\vec{x}_1 - il'\vec{k}\cdot\vec{x}_2}$$

has been used and ξ_2 is defined by $\xi_2=\epsilon(\vec{k}\cdot\vec{x}_2)$. Making an expansion of $U_l^{(\alpha)}(\xi-\xi_1,\tau)$ and $U_{l'}^{(\alpha')}(\xi-\xi_1-\xi_2,\tau)$ with respect to the small variables ξ_1 and ξ_2, and integrating over both x_1 and x_2, we obtain

$$\int\int d\vec{x}_1 d\vec{x}_2 F(\vec{x}_1,\vec{x}_2)U(\vec{x}-\vec{x}_1,t)U(\vec{x}-\vec{x}_1-\vec{x}_2,t)$$
$$=\sum_{\alpha,\alpha'=1}^{\infty}\sum_{l,l'=-\infty}^{\infty}\epsilon^{\alpha+\alpha'}Z_{l+l'}\{\tilde{F}_{l+l',l'}U_l^{(\alpha)}U_{l'}^{(\alpha')}$$
$$-\epsilon[\tilde{F}^1_{l+l',l'}(\partial_\xi U_l^{(\alpha)}U_{l'}^{(\alpha')}+U_l^{(\alpha)}\partial_\xi U_{l'}^{(\alpha')})+\tilde{F}^2_{l+l',l'}U_l^{(\alpha)}\partial_\xi U_{l'}^{(\alpha')}]+O(\epsilon^2)\}\ , \qquad (4.7)$$

where $\tilde{F}$'s are defined by

$$\tilde{F}_{m,n}=\int\int d\vec{x}_1 d\vec{x}_2 F(\vec{x}_1,\vec{x}_2)e^{-im\vec{k}\cdot\vec{x}_1-in\vec{k}\cdot\vec{x}_2}\equiv F_{m\vec{k},n\vec{k}}\ , \qquad (4.8a)$$

$$\tilde{F}^1_{m,n}=\int\int d\vec{x}_1 d\vec{x}_2 F(\vec{x}_1,\vec{x}_2)(\vec{k}\cdot\vec{x}_1)e^{-im\vec{k}\cdot\vec{x}_1-in\vec{k}\cdot\vec{x}_2}=i\partial_m F_{m\vec{k},n\vec{k}}\ , \qquad (4.8b)$$

and

$$\tilde{F}^2_{m,n}=\int\int d\vec{x}_1 d\vec{x}_2 F(\vec{x}_1,\vec{x}_2)(\vec{k}\cdot\vec{x}_1)^2 e^{-im\vec{k}\cdot\vec{x}_1-in\vec{k}\cdot\vec{x}_2}=-\partial_m^2 F_{m\vec{k},n\vec{k}}\ . \qquad (4.8c)$$

Combining these results (4.3), (4.4), and (4.7), we obtain

$$\sum_{\alpha=1}^{\infty}\sum_{l=-\infty}^{\infty}\epsilon^\alpha Z_l\Bigg[(-W_l+\epsilon H_l\partial_\xi+\epsilon^2 J_l\partial_\xi^2-2i\epsilon^2 l\omega_k\partial_\tau)U_l^{(\alpha)}$$
$$+\sum_{\alpha'=1}^{\infty}\sum_{l'=-\infty}^{\infty}[\epsilon\tilde{F}_{l,l'}U_{l-l'}^{(\alpha-\alpha')}U_{l'}^{(\alpha')}-\epsilon^2\tilde{F}^1_{l,l'}(U_{l-l'}^{(\alpha-\alpha')}\partial_\xi U_{l'}^{(\alpha')}+\partial_\xi U_{l-l'}^{(\alpha-\alpha')}U_{l'}^{(\alpha')})$$
$$-\epsilon^2\tilde{F}^2_{l,l'}U_{l-l'}^{(\alpha-\alpha')}\partial_\xi U_{l'}^{(\alpha')}]+O(\epsilon^3)\Bigg]=0\ , \qquad (4.9)$$

where

$$W_l\equiv(l\omega_k)^2-\tilde{W}_l=(l\omega_k)^2-\omega^2_{l\vec{k}}\ , \qquad (4.10a)$$

$$H_l\equiv 2il\lambda\omega_k-\tilde{W}^1_l=2i\vec{k}\cdot\left[l\omega_k\frac{\partial\omega_k}{\partial\vec{k}}-\omega_{l\vec{k}}\frac{\partial\omega_{l\vec{k}}}{\partial(l\vec{k})}\right]\ , \qquad (4.10b)$$

and

$$J_l\equiv\lambda^2-\tfrac{1}{2}\tilde{W}^2_l=k^2\left[\left(\frac{\partial\omega_k}{\partial\vec{k}}\right)^2-\frac{1}{2}\frac{\partial^2(\omega_{l\vec{k}})^2}{\partial(l\vec{k})^2}\right]\ . \qquad (4.10c)$$

For the dispersive system, W_l, H_l, and J_l satisfy the following relations:

$$W_0=W_{\pm1}=0,\quad W_l\neq0 \text{ for } |l|\geq2\ , \qquad (4.11a)$$

$$H_0=H_{\pm1}=0,\quad H_l\neq0 \text{ for } |l|\geq2\ , \qquad (4.11b)$$

$$J_l\neq0 \text{ for all } l\ . \qquad (4.11c)$$

We show in Appendix B that the following relation holds true for the nonlinear coefficients $\tilde{F}$'s:

$$\tilde{F}_{m,n}=\tilde{F}^1_{m,n}=F^2_{m,n}=0\ , \qquad (4.11d)$$

if either m or n is equal to zero or m is equal to n.

Equating the coefficients of the terms of equal powers of ϵ in Eq. (4.9), we readily obtain the following equations for $U_l^{(\alpha)}$. The terms to first order of ϵ yield

$$W_l U_l^{(1)}=0\ , \qquad (4.12)$$

to second order of ϵ

$$W_l U_l^{(2)}-H_l\partial_\xi U_l^{(1)}-\sum_{l'}\tilde{F}_{l,l'}U_{l-l'}^{(1)}U_{l'}^{(1)}=0\ , \qquad (4.13)$$

and to third order of ϵ

$$W_l U_l^{(3)}-H_l\partial_\xi U_l^{(2)}-J_l\partial_\xi^2 U_l^{(1)}+2i\omega_k\partial_\tau U_l^{(1)}-\sum_{l'}[\tilde{F}_{l,l'}(U_{l-l'}^{(2)}U_{l'}^{(1)}+U_{l-l'}^{(1)}U_{l'}^{(2)})-\tilde{F}^1_{l,l'}(U_{l-l'}^{(1)}\partial_\xi U_{l'}^{(1)}+\partial_\xi U_{l-l'}^{(1)}U_{l'}^{(1)})$$
$$-\tilde{F}^2_{l,l'}U_{l-l'}^{(1)}\partial_\xi U_{l'}^{(1)}]=0\ . \qquad (4.14)$$

From Eqs. (4.11a) and (4.12) we have

$$U_l^{(1)}=0 \text{ for } |l|\geq2\ , \qquad (4.15)$$

and from Eq. (4.13),

$$U_{\pm2}^{(2)}=\tilde{F}_{\pm2,\pm1}(U_{\pm1}^{(1)})^2/W_{\pm2} \qquad (4.16)$$

and

$$U_l^{(2)}=0 \quad \text{for } |l| \geq 3 . \tag{4.17}$$

For $l=0$, Eq. (4.14) results in

$$\partial_\xi^2 U_0^{(1)}=0 , \tag{4.18}$$

which has the unique solution

$$U_0^{(1)}=0 , \tag{4.19}$$

if we require the following localized boundary condition:

$$U(\vec{x},t)=0 \quad \text{as } \xi \to \pm\infty .$$

Equation (4.14) can be reduced to the following equation for $l=\pm 1$:

$$-J_{\pm 1}\partial_\xi^2 U_{\pm 1}^{(1)} \pm 2i\omega_k \partial_\tau U_{\pm 1}^{(1)} - (\widetilde{F}_{\pm 1,\pm 2}+\widetilde{F}_{\pm 1,\mp 1})U_{\mp 1}^{(1)}U_{\pm 2}^{(2)}=0 . \tag{4.20}$$

Substituting Eq. (4.16) into Eq. (4.20) and using the reality condition $U_l^{(\alpha)*}=U_{-l}^{(\alpha)}$, we finally obtain the nonlinear Schrödinger equation for $U_{\pm 1}^{(1)}$ as

$$-J_{\pm 1}\partial_\xi^2 U_{\pm 1}^{(1)} \pm 2i\omega_k \partial_\tau U_{\pm 1}^{(1)} - W_{\pm 2}^{-1}\widetilde{F}_{\pm 2,\pm 1}(\widetilde{F}_{\pm 1,\pm 2}+\widetilde{F}_{\pm 1,\mp 1}) \, | U_{\pm 1}^{(1)} |^2 U_{\pm 1}^{(1)}=0 . \tag{4.21}$$

To discuss further, we take the wave vector $\vec{k}$ of the carrier wave parallel to the x_1 axis (hereafter we simply refer to x_1 as x) and rewrite Eq. (4.21) in terms of the original variables x and t. Defining

$$\psi(x,t)=2\epsilon U_{-1}^{(1)}(\xi,\tau) \tag{4.22}$$

and using the relations

$$\partial_\xi=(\epsilon k)^{-1}\partial_x$$

and

$$\partial_\tau=\epsilon^{-1}v_g\partial_x+\epsilon^{-2}\partial_t ,$$

we finally obtain the asymptotic equation of motion for the displacement field $U(\vec{x},t)$ as

$$i\partial_t\psi+ivg\partial_x\psi+P\partial_x^2\psi+Q\,|\psi|^2\psi=0 , \tag{4.23}$$

where

$$P=\frac{J_{-1}}{2\omega_k k^2}=-\frac{1}{2}\frac{\partial^2\omega_k}{\partial k^2} \tag{4.24a}$$

and

$$Q=\frac{\widetilde{F}_{-2,-1}(\widetilde{F}_{-1,-2}+\widetilde{F}_{-1,1})}{2\omega_k W_{-2}} = \frac{(F_{2\vec{k},\vec{k}})^2}{8\omega_k^3(1-\omega_{2k}^2/4\omega_k^2)} . \tag{4.24b}$$

The displacement field $U(x,t)$ is then given by

$$U(x,t) \simeq \mathrm{Re}\{\psi(x,t)\exp[-i(kx-\omega_k t)]\} .$$

Introducing the variable $y=x-v_g t$, we can rewrite Eq. (4.23) as

$$i\partial_t\psi+P\partial_y^2\psi+Q\,|\psi|^2\psi=0 . \tag{4.25}$$

It is worthy to note that the coefficient Q is always positive in spite of the details of the third-order elastic constants. Therefore the nonlinear Schrödinger equation (4.25) has an envelope soliton solution given by

$$\psi(y,t)=\bar{\alpha}\,\mathrm{sech}\left[\bar{\alpha}\left[\frac{Q}{2P}\right]^{1/2}(y-\bar{\beta}t)\right] \times\exp\left\{i\left[\frac{\bar{\beta}}{P}y-\left[\frac{\bar{\beta}^2}{2P}-Q\bar{\alpha}\right]t\right]\right\} , \tag{4.26}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are independent parameters. We can, in consequence, anticipate the existence of the SA soliton of the envelope type in the propagation of high-frequency SAW on the dispersive solid surface.

V. CONDITIONS TO OBSERVE THE SURFACE ACOUSTIC SOLITON

In the preceding section it has been shown that the two-dimensional displacement field $U(\vec{x},t)$ reveals the explicit soliton behavior whose profile is of the envelope type. In the actual experimental situation, we almost use a pulselike wave as an initial state, then under some appropriate condition the pulse wave achieves self-trapping and becomes the envelope soliton due to a competition between dispersion which tends to broaden the pulse and the nonlinearities which tend to narrow it. Therefore whether we can really observe the envelope soliton experimentally or not in the propagation of high-frequency SAW depends essentially upon the propagation distance to achieve maximal self-trapping. Of course, this distance should be smaller than the sample size so as to detect the envelope soliton.

In general, the inverse scattering method tells us that the bound state of the relevant eigenvalue equation with initial pulse as its potential has one-to-one correspondence to the soliton. Furthermore, the number of bound states does not change with time and is determined only by the property of the initial potential.

As an initial envelope, we assume the square pulse of height H and width Δ involving the carrier wave with the frequency k as

$$\psi(x,0)=\begin{cases} H, & 0<x<\Delta \\ 0 & \text{elsewhere} . \end{cases} \tag{5.1}$$

Applying the inverse scattering method and performing a somewhat lengthy calculation after Zakharov and Shabat,[7] we can obtain the condition for which the initial pulse (5.1) is regularized to be the envelope soliton due to the competition between the dispersion and the nonlinearity as

$$H\Delta > \frac{4\sqrt{3}}{9}\pi\left[\frac{2P}{Q}\right]^{1/2} . \tag{5.2}$$

The pulse height η of the induced envelope soliton starting from the initial condition (5.1) is determined by the intersection of the following two curves:

$$Y=\left[\frac{2P}{Q}\right]^{1/2}\frac{X}{H\Delta},\quad 0<X<H\Delta \tag{5.3a}$$

and

$$Y=\pm\sin X,\quad \cot X>0 \tag{5.3b}$$

where

$$X=\left[\frac{Q}{2P}\right]^{1/2}\Delta\left[H^2-\frac{\eta^2}{4}\right]^{1/2}. \tag{5.4}$$

In this case, the SA soliton can be written by

$$U(x,t)=\eta\,\mathrm{sech}\left[\eta\left[\frac{Q}{2P}\right]^{1/2}(x-v_g t)\right]\times\exp\left\{-i\left[kx-\left[\omega_k+\frac{Q}{2}\eta^2\right]t\right]\right\}. \tag{5.5}$$

Also, the propagation time t_M required to achieve maximum self-trapping can be estimated to be

$$t_M=\frac{\Delta}{4\eta}\left[\frac{2}{PQ}\right]^{1/2}. \tag{5.6}$$

To begin with, we give the limitations on the various experimental quantities such as ω_k, Δ, and H.

(1) Maximum frequency of the coherent SAW attainable experimentally at present is about 3 GHz:

$$\nu_k\lesssim 3\,. \tag{5.7}$$

(2) In order to detect the surface displacement by laser light reflection, minimum amplitude of the surface oscillation should be of the order of 1 Å:

$$H\gtrsim 1\,. \tag{5.8}$$

(3) To make the asymptotic method valid which was employed in Sec. IV, one envelope wave should involve at least several tens of wavelengths of the carrier waves:

$$\Delta\gtrsim 10k^{-1}\,. \tag{5.9}$$

(4) The propagation distance d of the envelope wave required to achieve maximal self-trapping should be smaller than the sample size L:

$$d\equiv v_g t_M<L\,. \tag{5.10}$$

From Eq. (5.2), it is shown that the product of the width and height of the envelope soliton is constant and its order of magnitude is $\sqrt{2P/Q}$. Therefore, combining the above limitations (5.8) and (5.9), we obtain

$$2P/Q\sim(H\Delta)^2\gtrsim 7.3\times10^{-22} \tag{5.11}$$

(measured in cm^4). Also, the t_M can be estimated to be $t_M\sim H^{-2}Q$ if we consider the soliton with the same height as that of the initial pulse ($\eta\sim H$). This determines the magnitude of the constant Q to be

$$Q\gtrsim v_g/H^2L\sim 1.6\times10^{21} \tag{5.12}$$

(measured in $cm^{-2}\,sec^{-1}$), where we have assumed that $L=1$ cm and the group velocity $v_g=4.9\times10^5$ cm/sec for Si.

To discuss the competing effect between the nonlinearity and the dispersion quantitatively, we introduce the following three quantities:

$$N=\frac{(F_{2\vec{k},\vec{k}})^2}{8k^6}\,, \tag{5.13a}$$

$$D_1(k)=-\frac{k^2}{\omega_k}\frac{\partial^2\omega_k}{\partial k^2}\,, \tag{5.13b}$$

and

$$D_2(k)=1-\frac{\omega_{2k}^2}{4\omega_k^2}\,. \tag{5.13c}$$

Using the above expression, we have

$$P=\frac{\omega_k D_1(k)}{2k^2}\,, \tag{5.14a}$$

$$Q=\frac{k^6 N}{\omega_k^3 D_2(k)}\,, \tag{5.14b}$$

and then

$$\frac{2P}{Q}=\frac{\omega_k^4}{k^8}\frac{D_1(k)D_2(k)}{N}\,. \tag{5.14c}$$

The value of N is tentatively estimated to be

$$N\sim 3.0\times10^{23} \tag{5.15}$$

(measured in $cm^{-4}\,sec^4$) for Si as an example.

For the dispersive effect, we consider the following two kinds of dispersive terms.

(a) The intrinsic dispersion due to the discreteness of the crystalline solid.

(b) The extrinsic dispersion due to the layered structure of the solid surface.

First we consider the case (a) where the dispersion relation can be approximately expressed by Eq. (3.19),

$$\omega_k^2=(vk)^2[1-(hk)^2]\,. \tag{3.19}$$

In this case, $D_1(k)$ and $D_2(k)$ are given by

$$D_1(k)=3(hk)^2[1-\tfrac{2}{3}(hk)^2][1-(hk)^2]^{-2} \tag{5.16a}$$

and

$$D_2(k)=3(hk)^2[1-(hk)^2]^{-1}\,. \tag{5.16b}$$

Substituting these results in Eq. (5.14c) and taking $h=3$ Å, we have

$$\frac{2P}{Q}=\frac{(vh)^4}{N}[1-\tfrac{2}{3}(hk)^2][1-(hk)^2]^{-1}$$
$$\sim 1.6\times10^{-31} \tag{5.17}$$

(measured in cm^4), which is almost independent of the frequency and is too small to satisfy the soliton formation condition (5.11).

Next we consider the case (b) whose configuration is, for example, a ZnO layer over the Si substrate. In this case, the effect of dispersion is essentially determined by the relative magnitude of the layer thickness and wavelength of the carrier wave.

Solving the boundary-value problem for the elastic wave equation in the ZnO-Si layered structure to evaluate the dispersion relation, we obtain the critical values of the ZnO film thickness h_l and the frequency of the carrier wave ν_k as

$$h_l \lesssim 2.4\times 10^{-5} \qquad (5.18)$$

(measured in cm) and

$$\nu_k \gtrsim 1.6 \qquad (5.19)$$

(measured in GHz) to satisfy the formation condition (5.12). Therefore we can conclude that the ZnO-Si layered structure with about 2000-Å layer thickness, for example, offers a good possibility of observing the SA soliton in the propagation of the SAW with the frequency about 2 GHz.

The introduction of the layered structure which ensures the soliton-generating condition (5.12) results in a new problem to be solved. The localized eigenmode of the elastic waves in the layered structure is no longer a simple Rayleigh wave but rather a complicated wave, i.e., Love wave, Sezawa wave, etc. This gives, however, no serious alteration in our results because the waves localized to the solid surface can always be described by the Rayleigh type wave, and the profile function of that wave toward the x_3 axis may only suffer a slight modification. Therefore the results obtained in this paper are not qualitatively altered.

Finally, it would be worthwhile to note the sign of the product PQ. In order for the SA soliton to be stable as an envelope soliton, the sign of the PQ should always be positive. This condition will be experimentally satisfied by adopting the layered structure with the elastically "soft" layer on the "hard" substrate which implied the dispersion relation satisfying the inequality $\partial^2\omega_k/\partial k^2<0$. The ZnO-Si layered structure which we have taken in this paper just satisfies this characteristic property.

VI. SUMMARY AND DISCUSSION

Starting from the equation of motion for the anharmonically interacting surface phonons, we have formulated the theory of the SA soliton in an insulating solid based on the coherent-state representation. The two-dimensional wave equation with a nonlinear term of the convolution type has been obtained which can be reduced to the simple nonlinear Schrödinger equation by means of the reductive perturbation method. We can therefore anticipate the possible existence of the SA soliton in the ballistic propagation of the high-frequency surface phonons on the dispersive crystal surface.

The experimental possibility of observing the SA soliton has been also discussed with the aid of the inverse scattering method. It has been shown that the layered structure, for example, ZnO-Si, with the layer-thickness around 2000 Å , may offer the necessary condition to generate the SA soliton in the propagation of high-frequency SAW (about 2 GHz).

In the field of solid-state devices, the SAW have been normally applied for use as a delay line, convolver, or SAW filter. Since the SA soliton has been shown to be of the envelope type involving high-frequency carrier waves and, in addition, is remarkably stable against the small perturbations existing in the medium, we can naturally expect that the SA soliton may be able to carry the information over long distances without being distorted. There will, therefore, be possibilities of making use of it for the application to a transmission system.

When the SA soliton is considered to have a potential application to the communication system, we must amplify and reshape the SA soliton to compensate for an unavoidable deterioration of the pulse shape due to the several intrinsic defects or irregularities in the medium. Such an amplification can be achieved by using the electron-phonon interaction in a semiconducting solid surface as an amplifier. The generalization of our formalism to include the electron-phonon interaction in a semiconductor is now under investigation and the result is reported in the following paper.

ACKNOWLEDGMENT

This work is supported by a Scientific Research Fund from the Ministry of Education, Science, and Culture of Japan.

APPENDIX A

In this appendix we present the explicit expression for the three-phonon vertex function $\Phi_{\vec{k},\vec{k}',\vec{k}''}$. In an isotropic elastic medium with free surface, $\Phi_{\vec{k},\vec{k}',\vec{k}''}$ is given by

$$\Phi_{\vec{k},\vec{k}',\vec{k}''}=(2\rho W)^{-3/2}(\omega_k\omega_{k'}\omega_{k''})^{-1/2}kk'k''\phi_{\vec{k},\vec{k}',\vec{k}''}\,, \qquad (A1)$$

where

$$\phi_{\vec{k},\vec{k}',\vec{k}''}=\phi^{(\alpha)}_{\vec{k},\vec{k}',\vec{k}''}+\phi^{(\lambda,\beta)}_{\vec{k},\vec{k}',\vec{k}''}+\phi^{(\beta)}_{\vec{k},\vec{k}',\vec{k}''}+\phi^{(\gamma)}_{\vec{k},\vec{k}',\vec{k}''}+\phi^{(\mu,\gamma)}_{\vec{k},\vec{k}',\vec{k}''}\,. \qquad (A2)$$

ϕ's are written, respectively, as

$$\phi^{(\alpha)}_{\vec{k},\vec{k}',\vec{k}''}=\alpha\left[\kappa_l-\frac{1}{\kappa_l}\right]^3 AH_{LLL}\,, \qquad (A3)$$

$$\phi^{(\lambda,\beta)}_{\vec{k},\vec{k}',\vec{k}''}=\left[\frac{\lambda}{6}+\beta\right]\left[\kappa_l-\frac{1}{\kappa_l}\right]\Bigg[\left[3\kappa_l^2A+\frac{1}{\kappa_l^2}(B+B'+B'')-2(C+C'+C'')\right]H_{LLL}$$
$$+2\left[-\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}A-\frac{\kappa_t}{\kappa_l(\kappa_t^2+1)}(B+B')+(C+C')\right]H_{LLT}$$
$$+2\left[-\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}A-\frac{\kappa_t}{\kappa_l(\kappa_t^2+1)}(B+B'')+(C+C'')\right]H_{LTL}$$
$$+2\left[-\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}A-\frac{\kappa_t}{\kappa_l(\kappa_t^2+1)}(B'+B'')+(C'+C'')\right]H_{TLL}$$
$$+\left[\frac{2}{\kappa_t^2+1}\right]^2\{[\kappa_t^2(A+B)-(\kappa_t^4+1)C']H_{LTT}+[\kappa_t^2(A+B')-(\kappa_t^4+1)C']H_{TLT}$$
$$+[\kappa_t^2(A+B'')-(\kappa_t^4+1)C'']H_{TTL}\}\Bigg], \qquad (A4)$$

$$\phi^{(\beta)}_{\vec{k},\vec{k}',\vec{k}''}=\beta\left[\kappa_l-\frac{1}{\kappa_l}\right]\Bigg[\left[3\kappa_l^2A+\frac{1}{\kappa_l^2}(B+B'+B'')-2(C+C'+C'')\right]H_{LLL}$$
$$+2\left[-\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}A-\frac{\kappa_t}{\kappa_l(\kappa_t^2+1)}(B+B')+(C+C')\right]H_{LLT}$$
$$+2\left[-\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}A-\frac{\kappa_t}{\kappa_l(\kappa_t^2+1)}(B+B'')+(C+C'')\right]H_{LTL}$$
$$+2\left[-\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}A-\frac{\kappa_t}{\kappa_l(\kappa_t^2+1)}(B'+B'')+(C'+C'')\right]H_{TLL}$$
$$+\left[\frac{2\kappa_t}{\kappa_t^2+1}\right]^2[(A+B-2C)H_{LTT}+(A+B'-2C')H_{TLT}+(A+B''-2C'')H_{TTL}]\Bigg], \qquad (A5)$$

$$\phi^{(\gamma)}_{\vec{k},\vec{k}',\vec{k}''}=2\gamma\Bigg\{\left[\kappa_l^3A-\kappa_l(C+C'+C'')-\frac{1}{\kappa_l^3}D+\frac{1}{\kappa_l}(E+E'+E'')\right]H_{LLL}$$
$$+\left[-\frac{2\kappa_l^2\kappa_t}{\kappa_t^2+1}A+\kappa_l(C+C')+\frac{2\kappa_t}{\kappa_l^2(\kappa_t^2+1)}D-\frac{1}{\kappa_l}(E+E')-\frac{2\kappa_t}{\kappa_t^2+1}(C''+E'')\right]H_{LLT}$$
$$+\left[-\frac{2\kappa_l^2\kappa_t}{\kappa_t^2+1}A+\kappa_l(C+C'')+\frac{2\kappa_t}{\kappa_l^2(\kappa_t^2+1)}D-\frac{1}{\kappa_l}(E+E'')-\frac{2\kappa_t}{\kappa_t^2+1}(C'+E')\right]H_{LTL}$$
$$+\left[-\frac{2\kappa_l^2\kappa_t}{\kappa_t^2+1}A+\kappa_l(C'+C'')+\frac{2\kappa_t}{\kappa_l^2(\kappa_t^2+1)}D-\frac{1}{\kappa_l}(E'+E'')-\frac{2\kappa_t}{\kappa_t^2+1}(C+E)\right]H_{TLL}$$
$$+\frac{2\kappa_t}{\kappa_t^2+1}\Bigg[\left[\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}(A-C)-(C'+C''-E'-E'')-\frac{2\kappa_t}{\kappa_l(\kappa_t^2+1)}(D-E)\right]H_{LTT}$$
$$+\left[\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}(A-C')-(C+C''-E-E'')-\frac{2\kappa_t}{\kappa_l(\kappa_t^2+1)}(D-E')\right]H_{TLT}$$
$$+\left[\frac{2\kappa_l\kappa_t}{\kappa_t^2+1}(A-C'')-(C+C'-E-E')-\frac{2\kappa_t}{\kappa_l(\kappa_t^2+1)}(D-E'')H_{TTL}\right]$$
$$+\frac{8\kappa_t}{\kappa_t^2+1}[\kappa_t^2(D-A)+(\kappa_t^4+\kappa_t^2+1)(C+C'+C''-E-E'-E'')]H_{TTT}\Bigg\}, \qquad (A6)$$

and

$$\phi^{(\mu,\gamma)}_{\vec{k},\vec{k}',\vec{k}''}=2\left[\frac{\mu}{6}+\gamma\right]\Bigg[3\left[\kappa_l^3A-\kappa_l(C+C'+C'')-\frac{1}{\kappa_l^3}D+\frac{1}{\kappa_l}(E+E'+E'')\right]H_{LLL}$$

$$+3\left[-\frac{2\kappa_l^2\kappa_t}{\kappa_t^2+1}A+\kappa_l(C+C')+\frac{2\kappa_t}{\kappa_l^2(\kappa_t^2+1)}D-\frac{2\kappa_t}{\kappa_t^2+1}(C''+E'')-\frac{1}{\kappa_l}(E+E')\right]H_{LLT}$$

$$+3\left[-\frac{2\kappa_l^2\kappa_t}{\kappa_t^2+1}A+\kappa_l(C+C'')+\frac{2\kappa_t}{\kappa_l^2(\kappa_t^2+1)}D-\frac{2\kappa_t}{\kappa_t^2+1}(C'+E')-\frac{1}{\kappa_l}(E+E'')\right]H_{LTL}$$

$$+3\left[-\frac{2\kappa_l^2\kappa_t}{\kappa_t^2+1}A+\kappa_l(C'+C'')+\frac{2\kappa_t}{\kappa_l^2(\kappa_t^2+1)}D-\frac{2\kappa_t}{\kappa_t^2+1}(C+E)-\frac{1}{\kappa_l}(E'+E'')\right]H_{TLL}$$

$$+\frac{2}{\kappa_t^2+1}\left\{\left[\frac{6\kappa_l\kappa_t^2}{\kappa_t^2+1}\left[A-\frac{D}{\kappa_l^2}\right]+\frac{2(\kappa_t^4+\kappa_t^2+1)}{\kappa_t^2+1}(E-\kappa_lC)-3\kappa_t(C'+C''-E'-E'')\right]H_{LTT}\right.$$

$$+\left[\frac{6\kappa_l\kappa_t^2}{\kappa_t^2+1}\left[A-\frac{D}{\kappa_l^2}\right]+\frac{2(\kappa_t^4+\kappa_t^2+1)}{\kappa_t^2+1}(E'-\kappa_lC')-3\kappa_t(C+C''-E-E'')\right]H_{TLT}$$

$$+\left[\frac{6\kappa_l\kappa_t^2}{\kappa_t^2+1}\left[A-\frac{D}{\kappa_l^2}\right]+\frac{2(\kappa_t^4+\kappa_t^2+1)}{\kappa_t^2+1}(E''-\kappa_lC'')\right.$$

$$\left.\left.-\,3\kappa_t(C+C'-E-E')\right]H_{TTL}\right\}$$

$$+\frac{8\kappa_t}{(\kappa_t^2+1)^3}[3\kappa_t^2(D-A)+(\kappa_t^4+\kappa_t^2+1)(C+C'+C''-E-E'-E'')]H_{TTT}\Bigg], \tag{A7}$$

where

$$H_{LLL}=(k_l+k_l'+k_l'')^{-1},\quad H_{TLL}=(k_t+k_l'+k_l'')^{-1},\quad H_{LTL}=(k_l+k_t'+k_l'')^{-1},$$

$$H_{LLT}=(k_l+k_l'+k_t'')^{-1},\quad H_{LTT}=(k_l+k_t'+k_t'')^{-1},\quad H_{TLT}=(k_t+k_l'+k_t'')^{-1},$$

$$H_{TTL}=(k_t+k_t'+k_l'')^{-1},\quad H_{TTT}=(k_t+k_t'+k_t'')^{-1},$$

$$A=kk'k'',$$

$$B=k(\vec{k}'\cdot\vec{k}'')^2/k'k'',\quad B'=k'(\vec{k}\cdot\vec{k}'')^2/kk'',\quad B''=k''(\vec{k}\cdot\vec{k}')^2/kk',$$

$$C=k(\vec{k}'\cdot\vec{k}''),\quad C'=k'(\vec{k}\cdot\vec{k}''),\quad C''=k''(\vec{k}\cdot\vec{k}'),$$

$$D=(\vec{k}\cdot\vec{k}')(\vec{k}'\cdot\vec{k}'')(\vec{k}\cdot\vec{k}'')/kk'k'',$$

and

$$E=(\vec{k}\cdot\vec{k}')(\vec{k}\cdot\vec{k}'')/k,\quad E'=(\vec{k}\cdot\vec{k}')(\vec{k}'\cdot\vec{k}'')/k',\quad E''=(\vec{k}\cdot\vec{k}'')(\vec{k}'\cdot\vec{k}'')/k''.$$

As is clearly seen in the above expression, $\Phi_{\vec{k},\vec{k}',\vec{k}''}$ and also $\phi_{\vec{k},\vec{k}',\vec{k}''}$ are completely symmetric against the exchange of any two wave vectors and are invariant under the simultaneous inversion of wave vectors:

$$\begin{aligned}
&\Phi_{\vec{k},\vec{k}',\vec{k}''}=\Phi_{\vec{k}',\vec{k},\vec{k}''}=\Phi_{\vec{k}'',\vec{k},\vec{k}'}=\cdots ,\\
&\phi_{\vec{k},\vec{k}',\vec{k}''}=\phi_{\vec{k}',\vec{k},\vec{k}''}=\phi_{\vec{k}'',\vec{k},\vec{k}'}=\cdots ,\\
&\Phi_{\vec{k},\vec{k}',\vec{k}''}=\Phi_{-\vec{k},-\vec{k}',-\vec{k}''} ,\\
&\phi_{\vec{k},\vec{k}',\vec{k}''}=\phi_{-\vec{k},-\vec{k}',-\vec{k}''} .
\end{aligned} \tag{A8}$$

APPENDIX B

Here we show that the relation (4.11d) holds. From the definitions (3.15) and (A1), we have

$$\widetilde{F}_{m,n}=F_{m\vec{k},n\vec{k}}=\frac{3\,|\,m\,|\,k}{\rho W}\left[\frac{1+\kappa_t^2}{1-\kappa_t^2}\right]\phi_{-m\vec{k},n\vec{k},(m-n)\vec{k}} . \tag{B1}$$

Using (A8), we obtain

$$F_{m\vec{k},n\vec{k}}=F_{-m\vec{k},-n\vec{k}}=F_{m\vec{k},(m-n)\vec{k}} \tag{B2}$$

and

$$|\,m\,|\,F_{n\vec{k},m\vec{k}}=|\,n\,|\,F_{m\vec{k},n\vec{k}} . \tag{B3}$$

We now try to prove Eq. (4.11d) for the vertex function (A3). In this case we have

$$\phi^{(\alpha)}_{-m\vec{k},n\vec{k},(m-n)\vec{k}}=\alpha\left[\kappa_l-\frac{1}{\kappa_l}\right]^3 \times\frac{|\,m\,|\,|\,n\,|\,|\,m-n\,|\,k^2}{|\,m\,|+|\,n\,|+|\,m-n\,|} , \tag{B4}$$

which clearly vanishes if m or n is equal to zero or m is equal to n. Since the other vertex functions [(A4)–(A7)] also have the similar dependence on m and n to that of (A3) such as

$$\frac{|\,m\,|\,|\,n\,|\,|\,m-n\,|}{|\,m\,|+d\,|\,n\,|+d'\,|\,m-n\,|} ,$$

where d and d' are some constants, we see that Eq. (4.11d) holds for $\widetilde{F}_{m,n}$. Quite the same argument can be applied for $\widetilde{F}^1_{m,n}$ and $\widetilde{F}^2_{m,n}$ [Eq. (4.11d)], then we have completed the proof of the relation

$$\widetilde{F}_{m,n}=\widetilde{F}^1_{m,n}=\widetilde{F}^2_{m,n}=0$$

if either m or n is equal to unity or m is equal to n.

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