Ground state of the two-dimensional charged particles in a strong magnetic field and the fractional quantum Hall effect

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Finite-size two-dimensional systems are investigated by numerical diagonalization of the Hamiltonian. Both fermion systems and boson systems are investigated. The energy of the liquidlike ground state of the fermion system shows a downward cusp at $\frac{1}{3}$ filling of the lowest Landau level. This behavior is consistent with the observed fractional quantum Hall effect. The short-range behavior of the pair distribution function of the ground state is analyzed and good agreement with Laughlin's trial wave function is obtained.

I. INTRODUCTION

The discovery of the fractional quantum Hall effect by Tsui *et al.*[1-3] stimulated theoretical research on the ground state of the two-dimensional electrons in a strong magnetic field. The high mobility of the GaAs-GaAlAs heterojunction, where this effect is observed, indicates that the effect comes from the property of the ground state. However, when the effect was first reported, no existing theory for the ground state, such as the Wigner-crystal theory or the charge-density-wave (CDW) state theory,[4,5] could explain the experiment.[6]

The fractional quantization is first observed at about $\nu=\frac{1}{3}$ and $\frac{2}{3}$,[1] where ν is the filling of the Landau level defined by $\nu=2\pi l^2 n$, with n being the density of two-dimensional electrons and l being the Larmor radius, $l=(\hbar c/eB)^{1/2}$. In order to explain this effect, Yoshioka, Halperin, and Lee[7,8] made a numerical calculation for a finite-size system. They found that the ground state is not the CDW state, but rather a liquidlike state. They also found a downward cusp in the ground-state energy as a function of the filling factor ν at $\nu=\frac{1}{3}$. It was shown that the experiment can be explained by the presence of the cusp in energy. On the other hand, Laughlin[9] suggested a trial wave function at $\nu=1/p$ where p is an odd integer. He showed that the wave function, which represents a liquid state, has a lower energy than the CDW state. He further argued that the excitation from this state has a charge e/p with a finite excitation energy, and that this explains the experiment. A different theory was presented by Tao and Thouless[10] for the state at $\nu=1/p$.

After these theories were published, other fractional quantizations at about $\nu=\frac{2}{5}$, $\frac{2}{7}$, etc. were found.[3] These effects can be explained if there are downward cusps in the ground-state energy at $\nu=\frac{2}{5}$ and $\frac{2}{7}$. Our numerical results seem to suggest small downward cusps at $\nu=\frac{2}{5}$ and $\frac{2}{7}$. However, it is difficult to deny the possibility that the cusps are spurious, resulting from boundary effects. Moreover, such numerical calculations do not reveal the origin of the cusps. On the other hand, several attempts to extend Laughlin's theory to explain the experiment have been done.[11-14] However, estimates of the energy have not been performed in these theories, and hence it is difficult to tell which theory is a better approximation to the actual ground state.

The purpose of this paper is to give details on our previous letter,[7] and to report the results of our further numerical calculations. One of the aims of these further calculations is to establish a connection between our numerical results and Laughlin's trial wave function.[9] The difference in the boundary conditions does not allow us to make a direct comparison of the wave functions, however. Hence, here we compare only the short-range behavior of the wave functions: We investigate how the wave functions behave when two electrons come close to each other. The behavior of the amplitudes of the wave functions are compared by calculating the pair distribution function $g(\vec{r})$. The phases of the wave functions are also compared. We also investigate a system of bosons. Since statistics are important in Laughlin's wave function, the investigation of the boson system gives further indirect support to Laughlin's wave function.

Another aim of the present paper is to obtain some clues to the nature of the ground state at $\nu=\frac{2}{5}$ and $\frac{2}{7}$. The behavior of the $g(\vec{r})$ indicates that the correlation between quasiparticles, rather than that between the particles themselves, is important at such values of ν.

In Sec. II the system and the method of the calculation are explained. The results of the numerical calculation, the behavior of the $g(\vec{r})$, the ν dependence of the energy, and the behavior of the wave function are presented in Sec. III. The comparisons of our results with Laughlin's wave function is also given there. A discussion is presented in Sec. IV.

II. METHOD

We consider a two-dimensional fermion or boson system in a rectangular cell. The cell is in the x-y plane, the boundary of which is given by $x=0$, $x=a$, $y=0$, and $y=b$. We impose periodic boundary conditions in both the x and y directions. A strong magnetic field B is applied along the z axis. The magnetic field quantizes the energy spectrum of a charged particle into a series of Lan-

dau levels. We consider a system where the particle density n is smaller than the degeneracy per unit area of the lowest Landau level, $1/2\pi l^2$, namely the filling factor of the Landau levels $\nu \equiv 2\pi l^2 n$ is less than 1. We assume that the splitting of the Landau levels $\hbar\omega_c$ is much larger than the typical Coulomb interaction energy $e^2/\epsilon l$, where ω_c is the cyclotron frequency and ϵ is the dielectric constant. In such a case, we can consider only the lowest Landau level. In the actual experimental situation, however, the validity of this approximation is not obvious. However, we believe that the essence of the fractional quantum Hall effect is understood without considering the higher Landau levels. It is also assumed that the Zeeman splitting is larger than $e^2/\epsilon l$, and thus we consider the spins of all the particles to be aligned.

For an *infinite* system the wave function of the lowest Landau level in the Landau gauge, $\vec{A}=(0,Bx,0)$, is given as

$$\varphi_X(\vec{r})=\exp\left[i\frac{Xy}{l^2}-\frac{(X-x)^2}{2l^2}\right]. \tag{2.1}$$

In the *present* system the periodic boundary condition in the y direction requires that X be an integral multiple of $2\pi l^2/b$. Hence, allowed values for X are written as $X_j=2\pi l^2 j/b$ with an integer j. The periodic boundary condition in the x direction requires that at some integer m, X_m becomes equal to a. Hence the area of the cell ab must be $2\pi l^2 m$. This condition is equivalent to the condition that the total magnetic flux through the present cell is m times the unit magnetic flux $\Phi_0=hc/e$. In such a cell the wave function which satisfies the periodic boundary condition is given as

$$\varphi_j(\vec{r})=\left[\frac{1}{b\sqrt{\pi}l}\right]^{1/2}\sum_{k=-\infty}^{\infty}\exp\left[i\frac{(X_j+ka)y}{l^2}-\frac{(X_j+ka-x)^2}{2l^2}\right], \tag{2.2}$$

where $1\le j\le m$.

We consider a system of n ($\le m$) particles in the cell, mutually interacting through the Coulomb interaction. We also consider a uniform, neutralizing, positive background charge in order to eliminate the divergence due to the long-range nature of the Coulomb interaction. The boundary condition modifies the form of the Coulomb interaction, because a charge interacts with another charge as well as with all its images. Hence the Coulomb interaction is given as

$$V(\vec{r})=\sum_{k_1}\sum_{k_2}\frac{e^2}{\epsilon|\vec{r}+k_1a\hat{x}+k_2b\hat{y}|}, \tag{2.3}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the direction of the x and y axes, respectively. A particle also interacts with the images of itself. Therefore the Hamiltonian consists of single-particle terms, which result from the self-interaction and the interaction with the positive background, and a two-particle term, which results from an interaction with other particles. The kinetic energy part of the Hamiltonian is neglected since we consider only the lowest Landau level, and then it gives only a constant shift $(n/2)\hbar\omega_c$. Hence,

$$\mathcal{H}=\tfrac{1}{2}\sum_j\left[\left[V(\eta)-\frac{e^2}{\epsilon\eta}\right]_{\eta\to 0}-n\int_0^a dx\int_0^b dy\,V(\vec{r})\right]a_j^\dagger a_j$$
$$+\tfrac{1}{2}\sum_{j_i}\sum_{j_2}\sum_{j_3}\sum_{j_4}\int_0^a dx_1\int_0^b dy_1\int_0^a dx_2\int_0^b dy_2\varphi_{j_1}^*(\vec{r}_1)\varphi_{j_2}^*(\vec{r}_2)V(\vec{r}_1-\vec{r}_2)\varphi_{j_3}(\vec{r}_2)\varphi_{j_4}(\vec{r}_1)a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}a_{j_4}, \tag{2.4}$$

where $a_j^\dagger(a_j)$ is the creation (destruction) operator of the wave function $\varphi_j(\vec{r})$. When we substitute the Fourier transform of $V(\vec{r})$.

$$V(\vec{r})=\frac{1}{ab}\sum_q\frac{2\pi e^2}{\epsilon q}\exp(i\vec{q}\cdot\vec{r}), \tag{2.5}$$

where $\vec{q}=((2\pi/a)k_1,(2\pi/b)k_2)$ with k_1 and k_2 integers, then the terms with $\vec{q}=\vec{0}$ are canceled out and we have the following Hamiltonian:

$$\mathcal{H}=S\sum_j a_j^\dagger a_j+\tfrac{1}{2}\sum_{j_1}\sum_{j_2}\sum_{j_3}\sum_{j_4}A_{j_1j_2j_3j_4}a_{j_1}^\dagger a_{j_2}^\dagger a_{j_3}a_{j_4}. \tag{2.6}$$

Here S is the classical Coulomb energy of a Wigner crystal with a rectangular unit cell,[15] which is given by

$$S=-\frac{e^2}{\epsilon\sqrt{ab}}\left[2-\sum_{l_1,l_2}{}'\varphi_{-1/2}[\pi(\lambda l_1^2\lambda^{-1}l_2^2)]\right], \tag{2.7}$$

$$\varphi_n(z)=\int_1^\infty dt\,t^n e^{-zt}, \tag{2.8}$$

where $\lambda=a/b$, and the summation over l_1 and l_2 excludes $l_1=l_2=0$. On the other hand, A is given by

$$A_{j_1j_2j_3j_4}=\delta'_{j_1+j_2,j_3+j_4}\frac{1}{2ab}\sum_q\delta'_{j_1-j_4,q_yb/2\pi}\frac{2\pi e^2}{\epsilon q}\exp\left[-\frac{q^2l^2}{2}-i(j_1-j_3)\frac{q_xa}{m}\right], \tag{2.9}$$

where the Kronecker δ with a prime, δ'_{j_1,j_2}, becomes 1 when $j_1=j_2(\mathrm{mod}\ m)$.

For the system of n electrons in m sites, the filling factor ν is n/m. All of the possible ways to occupy m sites by n electrons form a basis, a component being given by

$$|j_1,j_2,\ldots,j_n\rangle\equiv a_{j_1}^\dagger a_{j_2}^\dagger\cdots a_{j_n}^\dagger|0\rangle,$$
$$1\le j_1\le j_2,\ldots,\le j_m\le m \tag{2.10}$$

where the equality for j's apply only for a boson system. The dimension of the basis is

$$\begin{bmatrix} m \\ n \end{bmatrix} \text{ for a fermion system}$$

and

$$\begin{bmatrix} m+n-1 \\ n \end{bmatrix} \text{ for a boson system .}$$

We calculate the matrix element of the Hamiltonian by this basis and diagonalize it numerically. In this process several symmetries are helpful: Owing to the translational symmetry along the y axis, the Hamiltonian conserves the total momentum along the y axis, $J \equiv \sum_{i=1}^{m} j_i \pmod m$. Hence we can make the dimension of the Hamiltonian about m^{-1} systems smaller than the dimension of the basis. Moreover, due to the translational symmetry along the x axis, two systems with values of J which differ by a multiple of n are equivalent, and due to the inversion symmetry, the system with $-J$ is equivalent to that with J. Hence, the number of values of J which give different results is very small.

For fixed ν and J, we must minimize the energy with respect to the aspect ratio a/b to find the ground state. Since we can interchange the x and y axes, two systems with aspect ratios of a/b and b/a are equivalent. Hence we can consider only systems with $a/b \geq 1$.

In the case of the fermion system, we have electron-hole symmetry. The system with n electrons in m sites is equivalent to that with the n holes or $m-n$ electrons in m sites. Actually, the off-diagonal matrix elements of a system of $\nu=n/m$ are the same as those of $\nu'=(m-n)/m$ with a corresponding J value. On the other hand, the diagonal matrix elements differ only by a constant value ΔE,

$$\Delta E = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} (A_{j_1 j_2 j_2 j_1} - A_{j_1 j_2 j_1 j_2}) - \sum_{j_1=1}^{m-n} \sum_{j_2=1}^{m-n} (A_{j_1 j_2 j_2 j_1} - A_{j_1 j_2 j_1 j_2}) \, . \tag{2.11}$$

Hence, for the case of the fermion system we can consider only the system with $\nu \leq \frac{1}{2}$. The results for $\nu > \frac{1}{2}$ shown below are obtained by this symmetry.

To investigate the property of an eigenstate of the Hamiltonian, especially of the ground state, we calculate the pair distribution function $g(\vec{r})$,

$$g(\vec{r}) \equiv \frac{ab}{n(n-1)} \left\langle \Psi \left| \sum_{i \neq j} \delta(\vec{r} + \vec{r}_i - \vec{r}_j) \right| \Psi \right\rangle = \frac{1}{n(n-1)} \sum_{q} \sum_{j_1} \cdots \sum_{j_4} \left[i\vec{q}\cdot\vec{r} - \frac{q^2 l^2}{2} - i(j_1 - j_3)\frac{q_x a}{m} \right] \delta'_{j_1 - j_4, q_y b/2\pi} \langle \Psi | a_{j_1}^{\dagger} a_{j_2}^{\dagger} a_{j_3} a_{j_4} | \Psi \rangle \, , \tag{2.12}$$

where $|\Psi\rangle$ is one of the eigenstates, and $\vec{q} = ((2\pi/a)k_1, (2\pi/b)k_2)$ with k_1 and k_2 being integers. This $g(\vec{r})$ is related to the energy per particle of the eigenstate E,

$$E = \frac{\langle \Psi | \mathscr{H} | \Psi \rangle}{n} = S + \frac{n-1}{2ab} \int_0^a dx \int_0^b dy \, V(\vec{r})[g(\vec{r}) - 1] \, . \tag{2.13}$$

III. RESULTS

We performed the numerical diagonalization of the Hamiltonian for $n=4$ to 8 for the fermion systems and $n=4$ to 7 for the boson systems. The maximum value of m for each n is chosen such that the dimension of the Hamiltonian matrix is smaller than 2000. The minimum of the energy is achieved at the aspect ratio a/b of the cell at about $n/4$. Although the position of the true minimum depends on n and m, and slightly deviates from $a/b = n/4$, the energy is not a sensitive function of a/b. Thus, in the following we fix the aspect ratio at $n/4$ in the discussion about the ground state.

When $\nu = q/p$ with p and q having no common measure but unity, the ground state is p-fold degenerate. Namely, the ground states are realized only at p different values of J, which differ by a multiple of n.

A. Pair distribution function

To investigate the nature of the ground state we calculate the pair distribution function $g(\vec{r})$. Figure 1(a) shows the $g(\vec{r})$ of the fermion ground state for $n=6$ and $m=18$, i.e., $\nu=\frac{1}{3}$. For the sake of comparison, in Fig. 1(b) we show the $g(\vec{r})$ of one of the higher-energy states. We identify this state as the CDW state obtained by the Hartree-Fock approximation,[5,6] because the energy becomes a minimum at $a/b = \sqrt{3}$, which is compatible with the CDW state of hexagonal symmetry, and the overall shape of $g(\vec{r})$ agrees with that of the Hartree-Fock CDW state. We find this kind of CDW state for $n=4$ and 6 systems at such values of m that m and n have a common factor. On the other hand, the $g(\vec{r})$ of the ground state, Fig. 1(a), has a very different form. There is no long-range order: The long-range order requires a peak in $g(\vec{r})$ at $(x,y) = (a/3, b/2)$, because we have peaks at $(x,y) = (a/3, 0)$ and $(0, b/2)$, but such a peak is not present in Fig. 1(a). Hence we conclude that the ground state is a liquidlike state.

The overall behavior of $g(\vec{r})$ for other values of m for $n=6$ are qualitatively the same. We have peaks at $(x,y) = (\pm a/3, 0)$ and $(x,y) = (0, \pm b/2)$. The $g(\vec{r})$ for the $n=4$ system shows similar behavior: If we remove the region $a/3 \leq x \leq \frac{2}{3}a$ from Fig. 1(b) we obtain qualitatively the same behavior as that of the $n=4$ system. Because

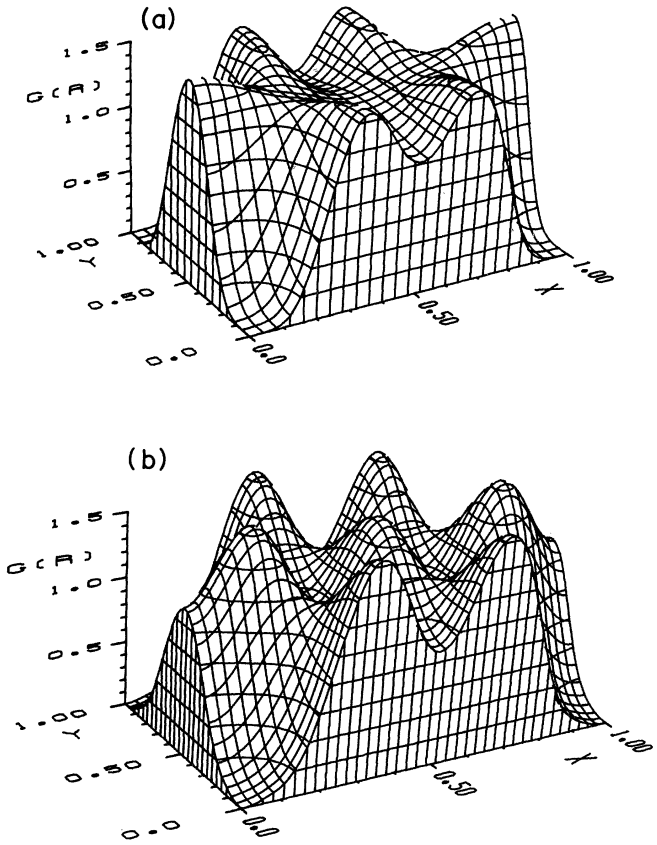

FIG. 1. (a) Perspective view of the pair correlation function $g(\vec{r})$ is plotted. The coordinates are normalized by the dimension of the cell: $(X,Y)=(x/a,y/b)$. The aspect ratio a/b is 1.5. (b) Perspective view of the $g(\vec{r})$ of the static identified as the CDW state. Here, the aspect ratio a/b is $\sqrt{3}$.

the aspect ratio of the $n=6$ system is 1.5 times greater than that of the $n=4$ system, this agreement is reasonable. For $\nu \leq \frac{1}{2}$, even the $g(\vec{r})$'s of the boson systems have overall behavior qualitatively similar to that of the fermion systems.

However, the behavior around the origin has a characteristic dependence on the filling factor ν and on the statistics of the system. This behavior is best seen when we expand $g(\vec{r})$ around the origin,[16]

$$g(\vec{r})=\sum_{i=0}^{\infty} c_i (r/l)^{2i} . \tag{3.1}$$

We least-squares-fit the calculated $g(\vec{r})$ in the region of $0 \leq r \leq 1.6l$ to obtain the c_i's. In this region of r, the dependence of $g(\vec{r})$ on the direction of $\vec{r}$ is very small. Thus we took the angular average and neglected the small angular dependence. The calculation was done for $n=4$ to 6 for fermion systems and for $n=4$ to 7 for boson systems. The results are shown in Figs. 2(a)–2(d). Although there is small scattering in the data, the c_i's are essentially independent of n, and they seem to vary continuously as functions of ν. When ν approaches 1, the c_i's of the fermion systems tend to values obtained from the following exact form of $g(\vec{r})$ at $\nu=1$:

$$g(\vec{r})=1-\exp\left[-\frac{r^2}{2l^2}\right] . \tag{3.2}$$

For the fermion system, c_0 is always zero because of the Fermi statistics. Moreover, c_1 and c_2 become almost zero for $\nu \leq \frac{1}{3}$ [Fig. 2(a)], as do c_3 and c_4 for $\nu \leq \frac{1}{5}$ [Fig. 2(b)]. For the boson systems, c_0 is not zero for $\nu > \frac{1}{2}$, and the sign of the c_i's are opposite of those of the fermion systems. The coefficients c_0 and c_1 become almost zero for $\nu \leq \frac{1}{2}$ [Fig. 2(c)], as do c_2 and c_3 for $\nu \leq \frac{1}{4}$ [Fig. 2(d)]. These behaviors are consistent with the $g(\vec{r})$ found by Laughlin's trial wave function:[9] His wave function is written as follows:

$$\Psi_p(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_n)=\prod_{j\neq k}(z_j-z_k)^p \exp\left[-\sum_j \frac{r_j^2}{4l^2}\right] ,$$
$$z_j=x_j-iy_j \tag{3.3}$$

where, for the fermion systems, p is odd, and for the boson systems, p is even. The filling factor for Ψ_p is $1/p$. Hence, at $\nu=1/p$, $g(\vec{r})$ should be proportional to r^{2p} for small r. Although our calculated c_i's with $i<p$ do not exactly vanish at $\nu=1/p$, the qualitative behavior of the c_i's is consistent with Laughlin's wave function. We also notice that for $\nu > 1/p$, c_{p-1} and c_{p-2} seem to increase proportionally to $\nu-1/p$, and that they also seem to be smooth functions of ν.

B. Ground-state energy

The ground-state energy per particle as a function of ν is shown in Figs. 3(a) and 3(b) for the fermion systems and the boson systems, respectively. For the fermion systems, we notice downward deviations of energy at $\nu=\frac{1}{3}$ and $\frac{2}{5}$, although the latter is much smaller than the former. We also notice large scattering of the data at $\nu=\frac{1}{2}$. The deviation at $\nu=\frac{1}{3}$ is clearly related to the behavior of $g(\vec{r})$ for small r. At $\nu=\frac{1}{3}$, c_1 and c_2 begin to increase, probably proportionally to $\nu-\frac{1}{3}$. Since the ground-state energy is related to $g(\vec{r})$ [Eq. (2.13)], this behavior of c_1 and c_2 should form a cusp at $\nu=\frac{1}{3}$ in the ground-state energy per particle. This cusp appears as a downward deviation in Fig. 3(a) because we can calculate only discrete points for finite systems. It is difficult to imagine that this behavior of c_1 and c_2 around $\nu=\frac{1}{3}$ disappears for an infinite system. Instead, we can expect the behavior of c_1 and c_2 to become sharper for an infinite system.

The anomalies at $\nu=\frac{2}{5}$ and $\frac{1}{2}$ are not related to the behavior of the c_i's. It is difficult to tell if the anomalies remain for an infinite system or not. However, if a downward cusp remains at $\nu=\frac{2}{5}$ and the scattered value at $\nu=\frac{1}{2}$ converges to give no cusp for an infinite system, this would be consistent with the experiments because a downward cusp in energy is required for the fractional quantum Hall effect.[7,8,11] That there is no cusp in the c_i's at $\nu=\frac{2}{5}$ indicates that the expected downward cusp at $\nu=\frac{2}{5}$ is a result of correlation among quasiparticles rather

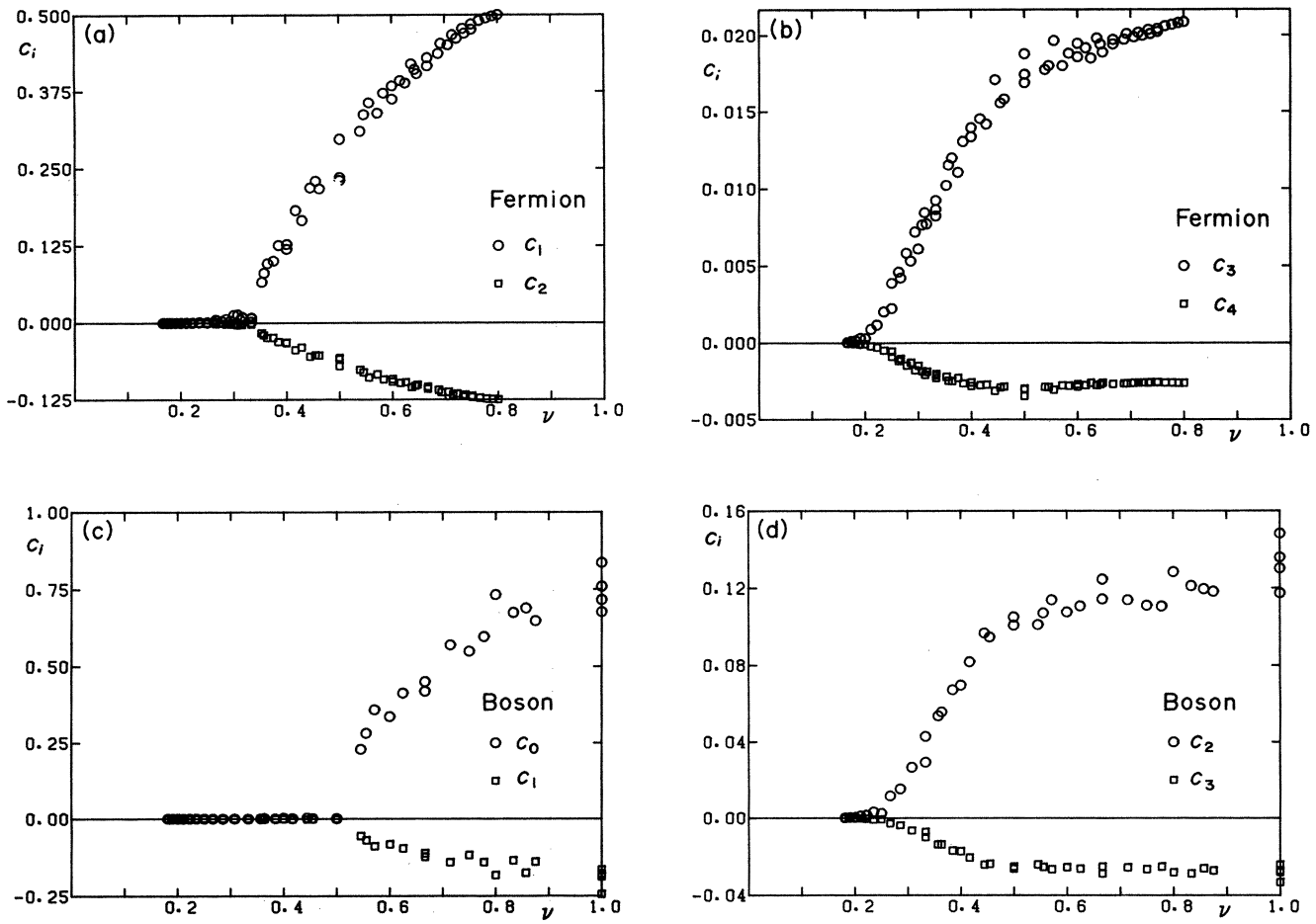

FIG. 2. Coefficients c_i defined by Eq. (3.1) are plotted as functions of ν. (a) c_1 and c_2 for the fermion system are plotted by circles and squares, respectively. (b) c_3 and c_4 for the fermion system are plotted by circles and squares, respectively. (c) c_0 and c_1 for the boson system are plotted by circles and squares, respectively. (d) c_2 and c_3 for the boson system are plotted by circles and squares, respectively.

than correlation among electrons.

For the boson systems we clearly see a downward cusp in energy at $\nu=\frac{1}{2}$, which can be explained in exactly the same way as the case of the fermion systems at $\nu=\frac{1}{3}$. We have another downward deviation at $\nu=\frac{2}{3}$. This observation is interesting, because recent theories[12,14] formulated to explain the $\frac{2}{5}$ and $\frac{2}{7}$ fractional quantizations of the fermion system require downward deviations at $\nu=\frac{2}{3}$ and $\frac{2}{5}$, when applied directly to the boson system. However, it seems difficult to explain the large downward deviation at $\nu=\frac{2}{3}$ using Anderson's theory.[13] According to his theory, the $\nu=\frac{2}{3}$ state should be explained as a superposition of four $\nu=\frac{1}{6}$ states. If the energy of such a state shows a large downward deviation, there should also be another downward deviation at $\nu=\frac{5}{6}$ with a magnitude similar to that at $\nu=\frac{2}{3}$, but we observe only a small, if any, deviation at $\nu=\frac{5}{6}$.

The ground-state energies of the fermion system and the boson system coincide within 0.03% at $\nu=\frac{4}{17}$, and the difference becomes much smaller for smaller ν. At first, this agreement seems strange because there is a qualitative difference in the behavior of $g(\vec{r})$ at small r between the fermion and boson systems. However, this is not so surprising, because at such a low value of ν there is little probability for two particles to come close to each other, and the ground-state energy is determined almost entirely by the long-range part of the Coulomb interaction. Interestingly, even at such a low value of ν, that the statistics are not important, and that the energy is determined by the long-range part of the Coulomb interaction, the ground state is still a liquidlike state. We calculated the ground-state energy and the energy of the CDW state down to $\nu=\frac{1}{9}$ for the $n=4$ fermion system, but the CDW state still has the higher energy. However, if we extrapolate the energies, the energy of the CDW state becomes the state with the lowest energy below about $\nu=0.075$.

Laughlin calculated the energy of his trial wave function, and his results agree quite well with ours. He obtained $-0.4156e^2/\epsilon l$ and $-0.3340e^2/\epsilon l$ at $\nu=\frac{1}{3}$ and $\frac{1}{5}$, respectively. On the other hand, we obtained $-0.4152e^2/\epsilon l$ for $n=4$, $-0.4127e^2/\epsilon l$ for $n=5$ and $=-0.4128e^2/\epsilon l$ for $n=6$ at $\nu=\frac{1}{3}$, and $-0.3322e^2/\epsilon l$ for $n=4$ at $\nu=\frac{1}{5}$. The difference between the two calculations is much smaller than the difference between the

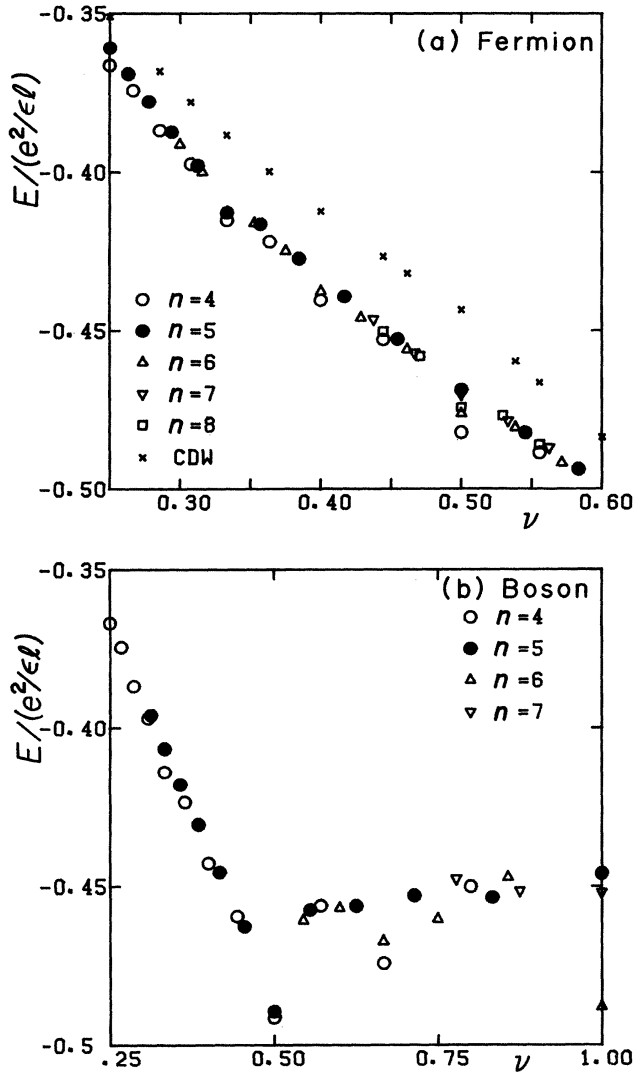

FIG. 3. (a) Ground-state energy per particle of the fermion system is plotted vs ν. Open circles, solid circles, triangles, inverted triangles, and squares show the results for $n=4$, 5, 6, 7, and 8 systems, respectively. Crosses show the energy of the Hartree-Fock CDW state of the infinite system (Ref. 6). (b) Plot similar to (a) for the boson system.

present liquidlike ground state and the Hartree-Fock CDW state, the energy of which is $-0.3885e^2/\epsilon l$ at $\nu=\frac{1}{3}$.

C. Wave function

Our wave function can be written by a linear combination of the basis as

$$\Psi=\sum_{\{j\}} w(j_1,j_2,\ldots,j_n)a^\dagger_{j_1}a^\dagger_{j_2}\cdots a^\dagger_{j_n}|0\rangle\ . \tag{3.4}$$

The eleven largest coefficients $w(j_1,j_2,j_3,j_4)$ of the ground state of the fermion system of $n=4$, $m=12$, ($\nu=\frac{1}{3}$), and $J=2$ are shown in Table I. The amplitude of the state where electrons occupy the Landau orbitals with equal spacings, in this case $w(2,5,8,11)$, is not very important. This state, $a^\dagger_2a^\dagger_5a^\dagger_8a^\dagger_{11}|0\rangle$, is the starting point

TABLE I. Eleven most important states in the ground state of the $n=4$ system at $\nu=\frac{1}{3}$ together with their amplitudes $w(j_1,j_2,j_3,j_4)$.

j_1	j_2	j_3	j_4	$w(j_1,j_2,j_3,j_4)$
1	6	7	12	0.4236
3	4	9	10	0.4236
2	5	8	11	0.3356
1	5	8	12	−0.3284
2	5	9	10	−0.3284
2	6	7	11	−0.3284
3	4	8	11	−0.3284
1	5	9	11	0.1228
2	4	8	12	0.1228
2	6	8	10	0.1228
3	5	7	11	0.1228

of the perturbation theory of Tao and Thouless.[10] Thus our calculation for a finite system does not support their theory, although our system may be too small to compare with their theory.

Because of the difference in the boundary conditions it is difficult to compare the present wave function with Laughlin's. As an indirect comparison of the wave function we have already calculated the pair distribution function $g(\vec{r})$, and have seen that the short-range behavior of the wave functions show qualitative agreement. Here, we examine the phase of the wave function, which does not appear in $g(\vec{r})$. For the comparison we fix the position of $n-1$ electrons and examine the phase of the wave function as a function of the position of the one remaining electron. Halperin[1] pointed out that Laughlin's wave function is characterized by the property that around each fixed electron the phase always changes by $2\pi p$ for a $\nu=1/p$ state, and there is no other "vortex," the point around which the phase changes more than or equal to 2π. That is, his wave function gives only $2\pi p$ vortices. For our wave function the distribution of the vortices depends on the configuration of the fixed electrons. For the case of $\nu=\frac{1}{3}$ we have a 6π vortex, but only for some special configurations. For other configurations, only one 2π vortex exists at the position of each fixed electron instead of a 6π vortex, although there are always two other 2π vortices near each fixed electron. The separation between the fixed electron and the accompanying vortices is usually less than l. Hence, when we examine the phase of the wave function on a scale larger than l, we have qualitative agreement between our wave function and Laughlin's. The discrepancies on a shorter scale are related to the fact that c_1 and c_2 in Eq. (3.1) do not exactly vanish at $\nu=\frac{1}{3}$ in our system. These discrepancies between our wave function and Laughlin's come from either the finite size of our system or from the fact that Laughlin's wave function is not an exact wave function of the ground state, or both.

IV. DISCUSSION

We have diagonalized the Hamiltonian for a finite system numerically, and found that the ground state is a

liquidlike state. The short-range behavior of the pair distribution function $g(\vec{r})$ shows a peculiar behavior, and it agrees very well with Laughlin's wave function. From the ν dependence of the c_i's, which are the coefficients in the expansion of $g(\vec{r})$, Eq. (3.1), we can confirm the presence of downward cusps in energy at $\nu=\frac{1}{3}$ for the fermion system and at $\nu=\frac{1}{2}$ for the boson system. We remark that the absence of special behavior in the c_i's at $\nu=\frac{2}{5}$ and $\frac{2}{7}$ does not deny the presence of downward cusps in energy at $\nu=\frac{2}{5}$ and $\frac{2}{7}$ for the fermion system. This is because we also do not observe speciality in the c_i's at $\nu=\frac{2}{3}$ for the fermion system, although the presence of a downward cusp there is guaranteed by electron-hole symmetry. An observed fractional quantum Hall effect at $\nu=\frac{2}{5}$ and $\frac{2}{7}$ indicates that, at these filling-factor values, correlation among quasiparticles, rather than correlation among electrons, is responsible for the downward cusp in energy, just as the correlation among holes is responsible for the downward cusp at $\nu=\frac{2}{3}$.

The downward cusps in energy are necessary to explain the fractional quantum Hall effect, as discussed elsewhere.[6,7,11] However, degeneracy of the ground state is also a requirement for fractional quantization. In our gauge, it is evident that when $\nu=q/p$ with mutually prime integers p and q, the ground state is at least p-fold degenerate, each state being specified by different values of the total momentum in the y direction, J. On the other hand, in Laughlin's gauge[9] the degeneracy is not evident. If we had no degeneracy of the ground state, Laughlin's gauge-invariance argument[17] for the integer quantum Hall effect would allow only integer quantum Hall values $\sigma_{xy}=(e^2/h)\times$integer, because we have an energy gap for the excitation from the $\nu=\frac{1}{3}$ ground state. When we examine how one of the degenerate ground states transforms to another ground state in the cylindrical geometry of Laughlin's gauge-invariance argument by increasing the central flux by a unit value, we find that the degeneracy is consistent with the fractional quantum Hall value $\sigma_{xy}=(q/p)(e^2/h)$ at $\nu=q/p$.

Thus we have shown that our numerical investigation supports Laughlin's wave function as a good approximation of the ground state of the present system at $\nu=1/p$. It is also shown that his wave function is also appreciable to the system of bosons. From the behavior of $g(\vec{r})$ we have confirmed the presence of a downward cusp at $\nu=\frac{1}{3}$ for the fermion system. This cusp and the threefold degeneracy of the ground state at $\nu=\frac{1}{3}$ is consistent with the fractional quantum Hall effect. The downward cusp in energy at $\nu=\frac{2}{3}$ for the boson system gives some clue to the understanding of the $\nu=\frac{2}{5}$ and $\frac{2}{7}$ fractional quantum Hall effect.

Note added in proof. In the text it is pointed out that for the fermion system, c_i's [Eq. (3.1) and Fig. 2] around $\nu=\frac{2}{3}$ are smooth functions of ν. However, it turns out that there actually is a cusp at $\nu=\frac{2}{3}$, which is obscured by scattering of the data from different values of ν. If the data are plotted for each n, the cusp shows up clearly. This cusp is necesary to have a downward cusp in energy at $\nu=\frac{2}{3}$, because the energy is given by Eq. (2.13) and the short-range part of the interaction is responsible for the cusp in energy. We can also expect cusps in c_i's at $\nu=\frac{2}{5}$ and $\frac{2}{7}$, which would be difficult to recognize because of the discreteness of the data. Hence I withdraw the discussion that the smooth behavior of c_i's at $\nu=\frac{2}{5}$ and $\frac{2}{7}$ indicates that the correlation among quasiparticles is responsible for the downward cusp in energy.

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