Fractional quantization of Hall conductance. II

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Our many-body theory for fractional quantized Hall effect is generalized to include $\nu=q/p$ filling. As the electrons of a two-dimensional system under a strong magnetic field are in a regular array of the Landau orbitals, correlation energy is enhanced and energy gaps are formed. Such a state can give rise to the Hall steps at fractional multiples of e^2/h. The motion equation of one-particle Green function provides a systematic theory for calculating energy gaps.

I. INTRODUCTION

In our preceding paper,[1] hereafter referred to as I, we presented a many-body theory for fractional quantized Hall effect. Since then, more Hall plateaus have been observed[2,3] at $\frac{2}{7}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{4}{3}$, and $\frac{5}{3}$. However, no anomaly has been found at filling factor $\frac{1}{2}$.

Laughlin[4] suggested an interesting wave function for the ground state and from it concluded that $1/m$ (m odd) fillings are the fractional quantized steps. Halperin[5] recently suggested a modification of this wave function in order to explain the Hall steps at $\nu=\frac{2}{5}$ and $\frac{2}{7}$ which cannot be explained by Laughlin. However, the proof that those wave functions yield the lowest-energy state is absent. Furthermore, the lack of explicit particle-hole symmetry is a problem in these wave functions.

Yoshioka, Halperin, and Lee's numerical calculation of the problem of a small number of electrons in a strong magnetic field[6] shows that the occupied Landau orbits are regularly arranged in the ground state of the system. Our many-body theory is a generalization: In an infinite two-dimensional system, if the electrons in a partially-filled Landau level can be arranged in a regular manner in the space of Landau orbitals, the correlation energy is enhanced and an energy gap is formed between the occupied sites (holes) and unoccupied sites (particles). This gap is essential to minimize the thermodynamic potential and to form a Hall step[1,5] with a value $\nu e^2/h$. The width of a Hall plateau and the low temperature necessary to observe the plateau are all determined by this gap. Our many-body theory always has particle-hole symmetry. For example, it gives a relation between the gaps at $\nu=\frac{1}{3}$ and $\nu=\frac{2}{3}$,

$$\frac{(\epsilon_p-\epsilon_h)_{2/3}}{(e^2/\epsilon)\sqrt{2\pi n}}=\frac{1}{\sqrt{2}}\frac{(\epsilon_p-\epsilon_h)_{1/3}}{(e^2/\epsilon)\sqrt{2\pi n}}=\text{const}\,, \qquad (1.1)$$

where n is the electron density of the sample and ϵ is the dielectric constant. Equation (1.1) explains Störmer's experiment:[2] the Hall plateau width of $\nu=\frac{2}{3}$ on a sample is the same as the Hall plateau width of $\nu=\frac{1}{3}$ on another sample which has only half the electron density of the first one.

In this paper I extend our theory to include $\nu=q/p$ fillings. This extension is a generalization of the $\nu=1/p$ situation. It enables us to calculate the gaps and correlation energy at $\nu=q/p$.

We start from a discussion about unperturbed states and the motion equation for a one-particle Green's function to calculate the screened (shielded) Coulomb interaction, and then establish self-consistent simultaneous equations for gaps. As Kadanoff and Baym discussed,[7] the self-energy and ground-state energy should be expanded in terms of the screened rather than the bare Coulomb interaction. This motion equation provides us with a systematic theory. From it we know that the ring approximation, which we used in I, is only the first-order expansion in the screened potential. Some results from this approximation are not good enough: The gap at $\nu=\frac{1}{2}$ was the biggest one, a finding which is not consistent with the experiment. This problem can be solved by calculating higher-order expansion terms in the screened potential. This reduces the gap at $\nu=\frac{1}{2}$, and the gap at $\nu=\frac{1}{3}$ is then the biggest.[8] Since a gap still exists at $\nu=\frac{1}{2}$, it would be interesting to see if a weak Hall step is observed eventually at $\nu=\frac{1}{2}$, but the details of such a calculation are beyond the scope of the present paper.

II. UNPERTURBED GROUND STATE

Let us still consider a standard two-dimensional electron-gas model, with interacting electrons moving in a uniform positive background to form an electrically neutral system. Fractional Hall conductance has been observed only in samples of very high mobility, so it is natural to suppose that the electron-electron interaction is more important than the impurity potential in lifting the degeneracy of the Landau level in those samples. Thus the properties of this two-dimensional electron-gas model will provide an explanation of the fractional Hall steps.

We take the magnetic field along the z direction, use the Landau gauge $\vec{A}=(0,Bx,0)$, and determine the system to have area L^2 in the x-y plane. The first Landau-level wave function periodic in the y direction may be written in the form

$$\phi_s(\vec{r})=\pi^{-1/4}(lL)^{-1/2} \times\exp\left[\frac{-2\pi isy}{L}-\frac{1}{2}\left[x-\frac{2\pi}{L}sl^2\right]^2\frac{1}{l^2}\right], \quad (2.1)$$

where s is an integer between 0 and $L^2/2\pi l^2$, and l is the magnetic length $l=(c\hbar/eB)^{1/2}$.

With the present experimental conditions[9] ($B=15$ T, $\epsilon\simeq 13$, $m^*=0.07m_e$) $e^2/\epsilon l<\hbar\omega_c$ still holds. The expected gaps are of the order $0.1e^2/\epsilon l$, so we may ignore the admixture of states above the first Landau level. In view of the strong magnetic fields involved, we assume that all the electron spins are parallel to the magnetic field; only one spin state needs to be considered. There is no difficulty in generalizing the discussion to include higher Landau levels and the antiparallel spin state.

The Coulomb interaction between the electrons in the lowest Landau level can be written in the form

$$\mathscr{H}_c=\tfrac{1}{2}\sum V(s_1-s_3,s_2-s_3)a^\dagger_{s_1}a^\dagger_{s_2}a_{s_4}a_{s_3}\delta_{s_1+s_2,s_3+s_4}, \quad (2.2)$$

where $a^\dagger$, a are creation and annihilation operators, and

$$V(s_1-s_3,s_2-s_3)=\frac{e^2}{\epsilon L}\int_{-\infty}^{\infty}dq\frac{1}{[q^2+4\pi^2(s_1-s_3)^2/L^2]^{1/2}}\exp\left[-\frac{l^2}{2}\left[q^2+\frac{4\pi^2(s_1-s_3)^2}{L^2}\right]+\frac{2\pi iql^2}{L}(s_2-s_3)\right]. \quad (2.3)$$

Because of the particle-hole symmetry, we only consider $\nu<\frac{1}{2}$. The behavior for $\nu>\frac{1}{2}$ can be deduced from the behavior for $\nu<\frac{1}{2}$.

Our many-body theory is not a standard perturbation theory. Once the higher Landau levels are ignored, the ground state Ψ_0 is independent of α when the $\mathscr{H}_c$ is replaced by $\alpha\mathscr{H}_c$ ($\alpha>0$); but without the Coulomb interaction the entire Hilbert space is degenerate and any state can be chosen as the ground state. Therefore, we cannot in the ordinary way define the unperturbed state as $\lim_{\alpha\to 0}\Psi_0$, which is unknown.[10] The Coulomb interaction is repulsive, so one may intuitively say that the ground state of $\mathscr{H}_c$, Ψ_0 is highly correlated in such a way that one electron avoids the others. This idea that electrons avoid each other suggests that the unperturbed ground state, Φ_0, be the state in which the occupied electrons are arranged in a regular manner and most uniformly distributed in the space. Since $\langle\Phi_0|\Psi_0\rangle\neq 0$,

$$\exp(-iHt)|\Phi_0\rangle/\langle\Phi_0|\exp(-iHt)|\Phi_0\rangle$$

tends to Ψ_0 as $t\to\infty(1-i0+)$.

In I the unperturbed ground state for $\nu=1/p$ is taken as the configuration in which the occupied electron states (hole sites) are equally spaced with an interval p. This assumption is justified by the fact that other configurations would reduce the gap and yield a higher ground-state energy.

This commensurate state has a broken symmetry. There are p equivalent states: the hole sites could be pj ($j=0,1,2,\ldots$), or $pj+1$, or $pj+2,\ldots,$ or $pj+p-1$. Owing to different boundary conditions for those p states, they cannot be mixed. As the system changes from one state to the next one in the sequence, e/p charge is transported across the system. Therefore, there are also p equivalent ground states of $\mathscr{H}_c$ adiabatically transformed from the above p unperturbed states and satisfying different boundary conditions.

The unperturbed state at $\nu=q/p$ should also be the state with period p and "most uniform distribution" of electrons. This most uniform distribution can be arranged in the following way. Since q and p have no common divisor, it is always possible to find two integers t_1, and t_2, satisfying

$$pt_2+qt_1=1 .$$

Let

$$\begin{aligned}0t_1&=0,\\1t_1&=\gamma_1 \pmod p,\\2t_1&=\gamma_2 \pmod p,\\&\cdots\\(q-1)t_1&=\gamma_{q-1} \pmod p.\end{aligned} \quad (2.4)$$

The configuration with hole sites $(0,\gamma_1,\gamma_2,\ldots,\gamma_{q-1})$ is the most uniform distribution. But it can be shown that for an infinite system this most uniform distribution has the same spectrum of particle and hole energies and the same ground-state energy as the configuration of hole sites $(0,1,2,\ldots,q-1)$.

Actually, let

$$s\in(0,1,2,\ldots,p-1) \quad (2.5)$$

(s could be a hole or a particle), and let

$$ts=z_s \pmod p, \quad (2.6)$$

where t is an arbitrary integer which has no common divisor with p. We construct a new configuration in which z_s is also a hole (particle) if s is a hole (particle) in the original configuration. The new set $(z_0,z_1,\ldots,z_{p-1})$ is $(0,1,2,\ldots,p-1)$ itself, arranged in a different order. Therefore, the new configuration has the same filling factor as the original one. The self-energy, the ground-state energy, and all other physical quantities can be expressed in Feynman diagrams. If we replace all quantum numbers s in one Feynman diagram by ts to get a new one, the conservation law still holds for this new diagram. Thus, hole (or particle) z_s must have the same energy in the new configuration as hole (or particle) s in the original configuration. The ground-state energies are also the same for both. Therefore, for example, at $\nu=\frac{2}{5}$ filling, the unperturbed state can be taken as shown in Fig. 1, where open circles represent holes and solid circles represent particles. Because of the symmetry, holes 0 and 1 have the same energy, particles 2 and 4 have the same energy, but particle 3 has a little higher energy. The spectrum is shown in Fig. 2, where ϵ_f is the Fermi energy and all particle and hole energies are effective energies. Of two gaps, the smaller

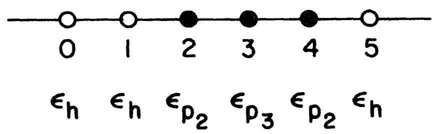


FIG. 1. Unperturbed ground state for $\nu=\frac{2}{5}$.

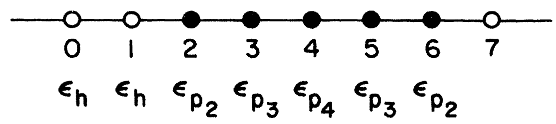


FIG. 3. Unperturbed ground state for $\nu=\frac{2}{7}$.

one, $\epsilon_{p_2}-\epsilon_h$, is more important because it will decide the temperature at which to observe the Hall step.

For $\nu=\frac{2}{7}$ filling, the unperturbed state and the spectrum are shown in Figs. 3 and 4. The most important gap is the smallest one, $\epsilon_{p_2}-\epsilon_h$. Generally, at $\nu=q/p$ $(q>1)$, the unperturbed ground state is as shown in Fig. 5; there are $1+[(q-1)/2]$ different hole energies and $1+[(p-q-1)/2]$ different particle energies where $[x]$ is the integer not exceeding but closest to x. The most important gap is $\epsilon_{p_q}-\epsilon_{h_0}$ (see Fig. 6).

In order to verify this unpertubed ground state, I have carried out some numerical calculations which confirm that a different configuration would yield smaller gaps and higher ground-state energy. For example, at $\nu=\frac{3}{7}$, the configuration with hole sites (0,1,3) is not equivalent to the configuration (0,1,2) but the (0,1,3) configuration has a higher ground-state energy than the (0,1,2) configuration. The smallest gap in the first configuration is also smaller than the latter one.

Figures 7 and 8 tell us that the configuration (0,1,3) makes all holes and particles have different energies. The unperturbed state we suggested makes the minimum breaking of the degeneracy of particle energies and the degeneracy of hole energies.

III. SCREENED POTENTIAL AND ENERGY GAPS

Let us introduce a small disturbance $w(\vec{r},t)$ into the Hamiltonian

$$H=H_0+H_1 ,$$

$$H_0=\int \psi^\dagger(\vec{r})h(\vec{r})\psi(\vec{r})d\vec{r} + \tfrac{1}{2}\int \psi^\dagger(\vec{r})\psi^\dagger(\vec{r}\,')v(\vec{r},\vec{r}\,')\psi(\vec{r}\,')\psi(\vec{r})d\vec{r}\,'d\vec{r} , \quad (3.1)$$

$$H_1=\int \rho(\vec{r})w(\vec{r},t)d\vec{r},\ \ \rho(\vec{r})=\psi^\dagger(\vec{r})\psi(\vec{r}) ,$$

where

$$h(\vec{r})=\frac{1}{2m}(-i\vec{\nabla}-e\vec{A})^2 , \quad (3.2)$$

$$v(\vec{r},\vec{r}\,')=\frac{e^2}{\epsilon\,|\vec{r}-\vec{r}\,'|} . \quad (3.3)$$

The potential $w(\vec{r},t)$ is to be put equal to zero in the final formulas.

We define the one-particle Green function as

$$G(1,2)=-i\langle T[\psi(1)\psi^\dagger(2)]\rangle , \quad (3.4)$$

where $(1)=(\vec{r}_1,t_1)$.

The motion equation for one-particle Green function gives the screened Coulomb interaction,[7,11] $W(1,2)$ as follows:

$$W(1,2)=v(1,2)+\int W(1,3)P(3,4)v(4,2)d(3)d(4) , \quad (3.5)$$

where

$$P(1,2)=-i\int G(2,3)G(4,2^+)\Gamma(3,4;1)d(3)d(4) \quad (3.6)$$

$$\Gamma(1,2,;3)=\delta(1,2)\delta(1,3)+\frac{\delta M(1,2)}{\delta V(3)} . \quad (3.7)$$

The self-energy is

$$M(1,2)=i\int W(1^+,3)G(1,4)\Gamma(4,2;3)d(3)d(4) , \quad (3.8)$$

where

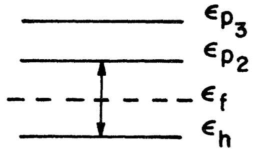


FIG. 2. Spectrum of effective particle and hole energies for $\nu=\frac{2}{5}$.

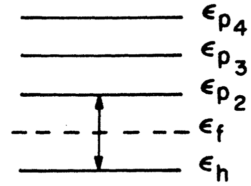


FIG. 4. Spectrum of effective particle and hole energies for $\nu=\frac{2}{7}$.

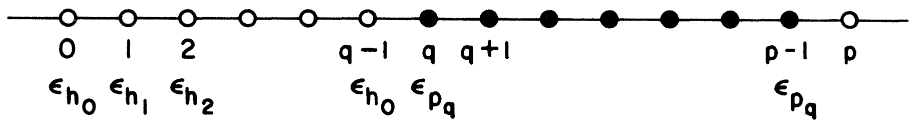


FIG. 5. Unperturbed ground state for $\nu=q/p$.

$$(1^+)=(\vec{r}_1,t_1+\Delta)\ , \tag{3.9}$$

$$v(1,2)=\frac{e^2}{\epsilon\,|\vec{r}_1-\vec{r}_2|}\delta(t_1-t_2)\ , \tag{3.10}$$

$$V(1)=w(1)-i\int v(1,3)G(3,3^+)d(3)\ . \tag{3.11}$$

The functional derivatives of G and W are given by

$$\frac{\delta G(1,2)}{\delta V(3)}=\int G(1,4)G(5,2)\Gamma(4,5;3)d(4)d(5)\ , \tag{3.12}$$

$$\frac{\delta W(1,2)}{\delta V(3)}=\int W(1,4)W(5,2)\frac{\delta P(4,5)}{\delta V(3)}d(4)d(5)\ . \tag{3.13}$$

Equations (3.5)—(3.13) provide a systematic theory: a selection law. A series approximation for W and M can be generated.

The first approximation for $W(1,2)$ and $M(1,2)$ is

$$W^{(1)}(1,2)=v(1,2)-iW(1,3)G(3,4)G(4,3)v(4,2)\ , \tag{3.14}$$

$$M^{(1)}(1,2)=iG(1,2)W(1^+,2)\ , \tag{3.15}$$

which is just the ring approximation. The next contribution to $W(1,2)$ and $M(1,2)$ is

$$W^{(2)}(1,2)=\int W(1,3)G(4,3)G(6,4)G(3,5)G(6,5)\times W(4,5)V(6,2)d(3)d(4)\ , \tag{3.16}$$

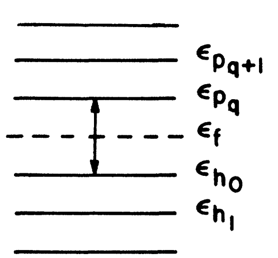


FIG. 6. Spectrum of effective particle and hole energies at $\nu=q/p$.

$$M^{(2)}=-\int W(1^+,3)G(1,4)G(4,3)G(3,2)\times W(4^+,2)d(3)d(4)d(5)d(6)\ , \tag{3.17}$$

and so on.

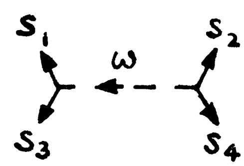
In this paper, we will still limit our calculation to the ring approximation, i.e., Eqs. (3.14) and (3.15). Let

$$W(1,2)=\int W(\vec{r}_1,\vec{r}_2,\omega)e^{-i\omega(t_1-t_2)}\frac{d\omega}{2\pi} \tag{3.18}$$

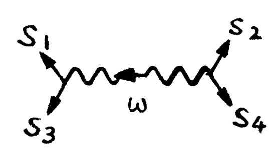
and

$$W_{s_1s_2s_3s_4}(\omega)=\int W(\vec{r}_1,\vec{r}_2,\omega)\phi^*_{s_1}(\vec{r}_1)\phi^*_{s_2}(\vec{r}_2)\times\phi_{s_4}(\vec{r}_2)\phi_{s_3}(\vec{r}_1)d\vec{r}_1d\vec{r}_2 = W_{s_1s_2s_3}(\omega)\delta_{s_1+s_2,s_3+s_4}\ . \tag{3.19}$$

Let us use



to represent $W_{s_1s_2s_3s_4}(\omega)$ and



the bare interaction. Equation (3.14) is

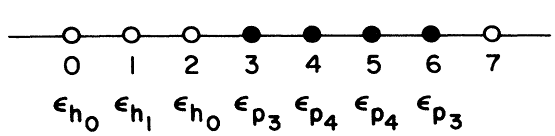
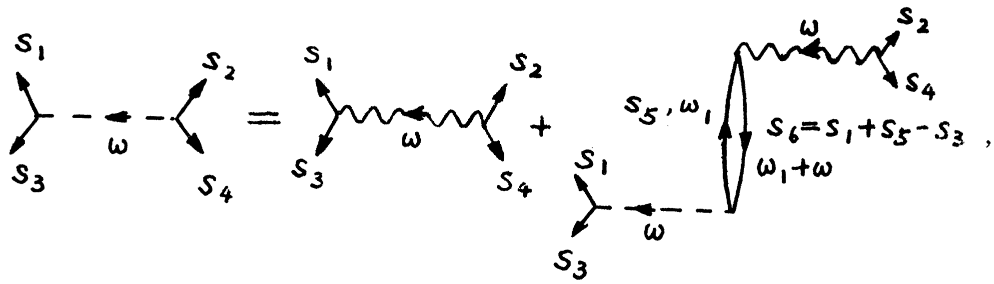


FIG. 7. Unperturbed ground state for $\nu=\frac{3}{7}$.



(3.20)

that is,

$$W_{s_1s_2s_3}(\omega)=V(s_1-s_3,s_2-s_3)-i\sum_{s_5}\int V(s_1-s_3,s_2-s_3)W_{s_1s_5s_3}(\omega)\frac{1}{\omega_1-\epsilon_{s_5}+i\delta_{s_5}}\frac{1}{\omega_1+\omega-\epsilon_{s_6}+i\delta_{s_6}}\frac{d\omega_1}{2\pi}\ , \tag{3.21}$$

where

$$\frac{1}{\omega-\epsilon_s+i\delta_s}=G_s(\omega)\ , \tag{3.22}$$

$$\delta_s=\begin{cases}\delta \ \text{ for particle},\\ -\delta \ \text{ for hole}\ .\end{cases}$$

Define a function $f(m,\omega)$,

$$\sum_{s=0}^{p-1}\int\frac{1}{\omega_1-\epsilon_s+i\delta_s}\frac{1}{\omega_1+\omega-\epsilon_{s+m}+i\delta_{s+m}}\frac{d\omega_1}{2\pi}=if(m,\omega)\ . \tag{3.23}$$

It is a periodic function,

$$f(p+m,\omega)=f(m,\omega)\ .$$

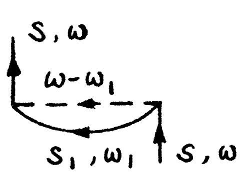
At filling $\nu=q/p$, with unperturbed state as in Fig. 5, we have

$$\begin{aligned}&f(0,\omega)=0\ ,\\ &f(1,\omega)=\frac{1}{\omega-(\epsilon_{p_q}-\epsilon_{h_0})+i\delta}-\frac{1}{\omega+(\epsilon_{p_q}-\epsilon_{h_0})-i\delta}\ ,\\ &f(2,\omega)=\frac{1}{\omega-(\epsilon_{p_{q+1}}-\epsilon_{h_0})+i\delta}-\frac{1}{\omega+(\epsilon_{p_{q+1}}-\epsilon_{h_0})-i\delta}+\frac{1}{\omega-(\epsilon_{p_q}-\epsilon_{h_1})+i\delta}-\frac{1}{\omega+(\epsilon_{p_q}-\epsilon_{h_1})-i\delta}\ ,\end{aligned} \tag{3.24}$$

From Eq. (2.3) and Eq. (3.21), we have

$$W_{s_1s_2s_3}(\omega)=\frac{e^2}{\epsilon L}\int_{-\infty}^{\infty}dk\frac{\exp\left[-\frac{l^2}{2}[k^2+4\pi^2(s_1-s_3)^2/L^2]+i\frac{2\pi}{L}l^2k(s_2-s_3)\right]}{[k^2+4\pi^2(s_1-s_3)^2/L^2]^{1/2}}\times\left[1-\frac{e^2}{\epsilon l^2p}\frac{\exp\left[-\frac{l^2}{2}[k^2+4\pi^2(s_1-s_3)^2/L^2]\right]}{[k^2+4\pi^2(s_1-s_3)^2/L^2]^{1/2}}f(s_1-s_3,\omega)\right]^{-1}\ . \tag{3.25}$$

Equation (3.15) gives the self-energy



That is,

$$\sum_{s_1} i\int W_{ss_1s_1}(\omega_1)\frac{1}{\omega-\omega_1-\epsilon_{s_1}+i\delta_{s_1}}\frac{d\omega_1}{2\pi}=-\frac{e^2q}{\epsilon lp}\left[\frac{\pi}{2}\right]^{1/2}+i\sum_{m=0}^{p-1}\frac{e^2}{\epsilon lp}\int_0^\infty\frac{e^{-\rho^2}d\rho}{\rho}\frac{\frac{e^2}{\epsilon lp}f(m,\omega_1)}{1-\frac{e^2}{\epsilon lp}\frac{1}{\rho}f(m,\omega_1)}\times\frac{d\omega_1/2\pi}{\omega-\omega_1-\epsilon_{s+m}+i\delta_{s+m}}\ . \tag{3.26}$$

The first term is just the exchange energy, which is the same for all particles and holes.

Substituting $s=0,\omega=\epsilon_{h_0}$, or $s=1,\omega=\epsilon_{h_1},\ldots,$ or $s=q,\omega=\epsilon_{p_q},\ldots,$ into (3.26), we have the hole energies ϵ_{h_0}, $\epsilon_{h_1},\ldots,$ and particle energies $\epsilon_{p_q},\ldots.$ All of these energies are expressed in functions of gaps. Thus we have self-consistent simultaneous equations to calculate these gaps and energies. For example, for $\nu=\frac{2}{5}$, we have

$$\Delta_0=-2\sqrt{\pi/2}-f_1(\Delta_{20},\Delta_{20})+f_1(\Delta_{20},0)-\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[f_2(\rho,\Delta_{20},\Delta_{30})\left[\frac{1}{\Delta_{30}+M(\rho,\Delta_{20},\Delta_{30})}+\frac{1}{\Delta_{20}+M(\rho,\Delta_{20},\Delta_{30})}\right]+f_3(\rho,\Delta_{20},\Delta_{30})\left[\frac{1}{\Delta_{30}+N(\rho,\Delta_{20},\Delta_{30})}+\frac{1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}\right]\right],$$

$$\Delta_2=-2\sqrt{\pi/2}-f_1(\Delta_{20},\Delta_{30}-\Delta_{20})+f_1(\Delta_{20},\Delta_{20})-\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[f_2(\rho,\Delta_{20},\Delta_{30})\left[\frac{1}{M(\rho,\Delta_{20},\Delta_{30})}-\frac{1}{M(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}\right]+f_3(\rho,\Delta_{20},\Delta_{30})\left[\frac{1}{N(\rho,\Delta_{20},\Delta_{30})}-\frac{1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}\right]\right],$$

$$\Delta_3=-2\sqrt{\pi/2}-2f_1(\Delta_{20},\Delta_{20}-\Delta_{30})+2\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[\frac{f_2(\rho,\Delta_{20},\Delta_{30})}{\Delta_{20}+M(\rho,\Delta_{20},\Delta_{30})}+\frac{f_3(\rho,\Delta_{20},\Delta_{30})}{\Delta_{30}+N(\rho,\Delta_{20},\Delta_{30})}\right],$$

where

$\Delta_0=\epsilon_{h_0}/(e^2/5\epsilon l)$,

$\Delta_2=\epsilon_{p_2}/(e^2/5\epsilon l)$,

$\Delta_3=\epsilon_{p_3}/(e^2/5\epsilon l)$,

$\Delta_{ij}=\Delta_i-\Delta_j$,

$$f_1(a,b)=\int_0^\infty d\rho\frac{ae^{-\rho^2}}{a^2\rho+2ae^{-\rho^2/2}+b(a^2\rho^2+2a\rho e^{-\rho^2/2})^{1/2}}\ , \tag{3.27}$$

$$M(\rho,a,b)=\frac{1}{\sqrt{2}}\left\{a^2+b^2+2\frac{e^{-\rho^2/2}}{\rho}(a+b)+\left[\left[a^2+b^2+2\frac{e^{-\rho^2/2}}{\rho}(a+b)\right]^2-4[a^2b^2+2e^{-\rho^2/2}ab(a+b)/\rho]\right]^{1/2}\right\}^{1/2},$$

$$N(\rho,a,b)=\frac{1}{\sqrt{2}}\left\{a^2+b^2+2\frac{e^{-\rho^2/2}}{\rho}(a+b)-\left[\left[a^2+b^2)+2\frac{e^{-\rho^2/2}}{\rho}(a+b)\right]^2-4[a^2b^2+2\frac{d^{-\rho^2/2}}{\rho}ab(a+b)]\right]^{1/2}\right\}^{1/2},$$

$$f_2(\rho,a,b)=\frac{b\{[M(\rho,a,b)]^2-a^2\}+a\{[M(\rho,a,b)]^2-b^2\}}{M(\rho,a,b)\{[M(\rho,a,b)]^2-[N(\rho,a,b)]^2\}}\ ,$$

$$f_3(\rho,a,b)=\frac{b\{a^2-[M(\rho,a,b)]^2\}+b\{a^2-[N(\rho,a,b)]^2\}}{N(\rho,a,b)\{[M(\rho,a,b)]^2-[N(\rho,a,b)]^2\}}\ . \tag{3.28}$$

TABLE I. Energy gap.

ν	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{5}$ [a]	$\frac{3}{5}$ [a]	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{1}{7}$	$\frac{2}{7}$ [a]
$\frac{\epsilon_p-\epsilon_h}{e^2/\epsilon l}$	0.183	0.183	0.0938	0.0938	0.075 33	0.075 33	0.034 18	0.0474
$\frac{\epsilon_p-\epsilon_h}{e^2\sqrt{2\pi n}/\epsilon}$	0.317	0.225	0.1483	0.121	0.168	0.084	0.0904	0.0883

[a]Only the smallest gap is listed.

From those equations we have

$$\epsilon_{p_2}-\epsilon_{h_0}=0.0938\frac{e^2}{\epsilon l}\,,$$
$$\epsilon_{p_3}-\epsilon_{h_0}=0.1023\frac{e^2}{\epsilon l}\,. \tag{3.29}$$

The expressions of energies for $\nu=\frac{2}{7}$ are in the Appendix. The numerical results are

$$\epsilon_{p_2}-\epsilon_{h_0}=0.047\,48\frac{e^2}{\epsilon l}\,,$$
$$\epsilon_{p_3}-\epsilon_{h_0}=0.051\,10\frac{e^2}{\epsilon l}\,, \tag{3.30}$$
$$\epsilon_{p_4}-\epsilon_{h_0}=0.051\,11\frac{e^2}{\epsilon l}\,.$$

IV. DISCUSSION

So far, all numerical results in Table I are from the ring approximation. Those results are expected to be improved by calculating higher-order expansion in screened potentials, and more calculations should be carried out for other filling factors.

In our theory, holes and particles are symmetric. Therefore, $\nu=1-q/p$ filling can be achieved from q/p filling by replacing holes with particles and vice versa. Thus, with the same density of electrons, we have a relation between two corresponding gaps at $\nu=1-q/p$ and $\nu=q/p$:

$$\left[\frac{\epsilon_p-\epsilon_h}{e^2/\epsilon l}\right]_{1-q/p}=\left[\frac{\epsilon_p-\epsilon_h}{e^2/\epsilon l}\right]_{q/p} \tag{4.1}$$

and

$$(\epsilon_p-\epsilon_h)_{1-q/p}=\left[\frac{q}{p-q}\right]^{1/2}(\epsilon_p-\epsilon_h)_{q/p}\,. \tag{4.2}$$

As we mentioned in I, until a more detailed theory is developed we cannot make strong statements about the temperature dependence of our model, but the analogy with superconductivity might suggest that the smallest energy gap $(\epsilon_p-\epsilon_h)_s$ at $\nu=q/p$ and the critical temperature at which the commensurate state becomes favorable are related as

$$T_c=\frac{(\epsilon_p-\epsilon_h)_s}{k_B c_0}\,, \tag{4.3}$$

where k_B is Boltzmann's constant and c_0 is a numerical constant. In superconductivity, $3.5\le c_0<5$. In this case, c_0 might vary from those numbers. But we cannot rule out another possibility that the quantum Hall system does not undergo a phase transition; there is no critical temperature T_c; upon heating the fractional quantized Hall effect disappears continuously.

As we mentioned before, the configuration with hole sites $(0,1,2,\ldots,q-1)$ at $\nu=q/p$ has the minimum breaking of the degeneracy of particle energies and the degeneracy of hole energies. This property is very important. From I, with the ring approximation, the solution of the equation,

$$\eta=f_1(\eta/\nu,\eta/\nu)-(1-2\nu)f_1(\eta/\nu,0)\,, \tag{4.4}$$

decides the gap at $\nu=1/p$:

$$\epsilon_p-\epsilon_h=\eta(e^2/\epsilon l)\,. \tag{4.5}$$

Taking $\nu=q/p$ [q/p is not $1/p$ or $(p-1)/p$] in Eq. (4.4), we calculate η from it, and define the value calculated from (4.5) as $(\epsilon_p-\epsilon_h)_0$. Define the ratio between $(\epsilon_p-\epsilon_h)_0$ and the smallest gap $(\epsilon_p-\epsilon_h)_s$ at a configuration of $\nu=q/p$ filling as λ,

$$\lambda=\frac{(\epsilon_p-\epsilon_h)_0}{(\epsilon_p-\epsilon_h)_s}=\lambda(j)\,, \tag{4.6}$$

where j is the total number of different hole energies and particle energies in that configuration. It turns out that $\lambda(j)$ is a monotonically increasing function. Therefore, the more breaking of the degeneracy of particle energies and the degeneracy of hole energies, the smaller will be the gap. Our ground state yields a bigger gap and lower ground-state energy than any other nonequivalent configuration of hole sites.

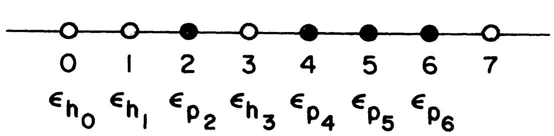


FIG. 8. Distribution of (0,1,3) makes all particle and hole energies different.

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APPENDIX

At $\nu=\frac{2}{7}$, the particle and hole energies are given by

$$\Delta_0=-2\sqrt{\pi/2}+f_1(\Delta_{20},0)-f_1(\Delta_{20},\Delta_{20})$$
$$+\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[f_2(\rho,\Delta_{20},\Delta_{30})\left[\frac{-1}{M(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}+\frac{-1}{M(\rho,\Delta_{20},\Delta_{30})+\Delta_{30}}\right]\right.$$
$$+f_3(\rho,\Delta_{20},\Delta_{30})\left[\frac{-1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}+\frac{-1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{30}}\right]$$
$$+f_2(\rho,\Delta_{30},\Delta_{40})\left[\frac{1}{M(\rho,\Delta_{30},\Delta_{40})+\Delta_{30}}+\frac{1}{M(\rho,\Delta_{30},\Delta_{40})+\Delta_{40}}\right]$$
$$\left.+f_3(\rho,\Delta_{30},\Delta_{40})\left[\frac{1}{N(\rho,\Delta_{30},\Delta_{40})+\Delta_{30}}+\frac{1}{N(\rho,\Delta_{30},\Delta_{40})+\Delta_{40}}\right]\right], \quad \text{(A1)}$$

$$\Delta_2=-2\sqrt{\pi/2}-f_1(\Delta_{20},\Delta_{30}-\Delta_{20})+f_1(\Delta_{20},\Delta_{20})$$
$$+\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[f_2(\rho,\Delta_{20},\Delta_{30})\left[\frac{-1}{M(\rho,\Delta_{20},\Delta_{30})+\Delta_{40}-\Delta_{20}}+\frac{1}{M(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}\right]\right.$$
$$+f_3(\rho,\Delta_{20},\Delta_{30})\left[\frac{-1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{40}-\Delta_{20}}+\frac{1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{20}}\right]$$
$$-f_2(\rho,\Delta_{30},\Delta_{40})\left[\frac{-1}{M(\rho,\Delta_{30},\Delta_{40})+\Delta_{30}-\Delta_{20}}+\frac{-1}{M(\rho,\Delta_{30},\Delta_{40})}\right]$$
$$\left.-f_3(\rho,\Delta_{30},\Delta_{40})\left[\frac{-1}{N(\rho,\Delta_{30},\Delta_{40})+\Delta_{30}-\Delta_{40}}+\frac{1}{N(\rho,\Delta_{30},\Delta_{40})}\right]\right], \quad \text{(A2)}$$

$$\Delta_3=-2\sqrt{\pi/2}-f_1(\Delta_{20},\Delta_{40}-\Delta_{30})-f_1(\Delta_{20},\Delta_{20}-\Delta_{30})$$
$$+\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[f_2(\rho,\Delta_{20},\Delta_{30})\left[\frac{-1}{M(\rho,\Delta_{20},\Delta_{30})}+\frac{1}{M(\rho,\Delta_{20},\Delta_{30})+\Delta_{30}}\right]\right.$$
$$+f_3(\rho,\Delta_{20},\Delta_{30})\left[\frac{-1}{N(\rho,\Delta_{20},\Delta_{30})}+\frac{1}{N(\rho,\Delta_{20},\Delta_{30})+\Delta_{30}}\right]$$
$$+f_2(\rho,\Delta_{30},\Delta_{40})\left[\frac{-1}{M(\rho,\Delta_{30},\Delta_{40})-\Delta_{30}+\Delta_{20}}+\frac{1}{M(\rho,\Delta_{30},\Delta_{40})+\Delta_{30}}\right]$$
$$\left.+f_3(\rho,\Delta_{30},\Delta_{40})\left[\frac{-1}{N(\rho,\Delta_{30},\Delta_{40})-\Delta_{30}+\Delta_{40}}+\frac{1}{N(\rho,\Delta_{30},\Delta_{40})+\Delta_{30}}\right]\right], \quad \text{(A3)}$$

$$\Delta_4=-2\sqrt{\pi/2}-2f(\Delta_{20},\Delta_{30}-\Delta_{40})$$
$$+2\int_0^\infty d\rho\frac{e^{-\rho^2}}{\rho}\left[-f_2(\rho,\Delta_{20},\Delta_{30})\left[\frac{1}{M(\rho,\Delta_{20},\Delta_{30})-\Delta_{40}+\Delta_{20}}\right]-f_3(\rho,\Delta_{20},\Delta_{30})\left[\frac{1}{N(\rho,\Delta_{20},\Delta_{30})-\Delta_{40}+\Delta_{20}}\right]\right.$$
$$\left.+f_2(\rho,\Delta_{30},\Delta_{40})\left[\frac{1}{M(\rho,\Delta_{30},\Delta_{40})+\Delta_{40}}\right]+f_3(\rho,\Delta_{30},\Delta_{40})\left[\frac{1}{N(\rho,\Delta_{30},\Delta_{40})+\Delta_{40})}\right]\right], \quad \text{(A4)}$$

where $\Delta_i=\epsilon_{h_i}/(e^2/7\epsilon l)$ or $\epsilon_{p_i}/(e^2/7\epsilon l)$ and $\Delta_{ij}=\Delta_i-\Delta_j$; $f_1(a,b)$, $f_2(\rho,a,b)$, $f_3(\rho,a,b)$, $M(\rho,a,b)$, and $N(\rho,a,b)$ are in Eqs. (3.27) and (3.28).

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