Percolation theory for nonlinear conductors

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Under broad conditions, a network of nonlinear conductors has an I-V characteristic uniquely determined by Kirchhoff's rules. By means of a renormalization calculation, we show that near the percolation threshold the details of the microscopic I-V characteristic are averaged out, so that the bulk material approaches power-law conductor behavior ($V=I^{\alpha}$). The threshold exponents $t(\alpha)$ and $s(\alpha)$ are discussed in the limiting cases of two dimensions (where they are related by duality) and high dimensionality (by solving the Cayley-tree model).

I. INTRODUCTION

In a previous publication[1] we discussed the behavior of a random network of nonlinear conducting elements having an arbitrary I-V characteristic. We showed that this system can be discussed in terms of the exponent theory of the percolation conductivity problem, but with the additional complication that the I-V curve for a network is not the same as that of its component parts; indeed, it is not clear how it can be calculated in general. There are two special cases in which the problem simplifies.

(1) Power-law conductors, for which

$$V=r\,|\,I\,|^{\alpha}\mathrm{sgn}I\ , \tag{1.1}$$

give a simple generalization of the percolation problem. The I-V characteristic for a random network of such elements, all having the same value of α (but arbitrarily chosen coefficients r), is also a power law in the exponent α.

(2) Near the percolation threshold the I-V characteristic of any nonlinear network composed of monotonically increasing $V(I)$ reduces to a power-law form:

$$V/L\sim\rho_{\mathrm{eff}}\,|\,I/A\,|^{\alpha}\mathrm{sgn}I\ , \tag{1.2}$$

where L is the length of the sample, A is the cross-sectional area, and ρ_{eff} is the appropriately generalized resistivity, and in this limit

$$\rho_{\mathrm{eff}}\sim\,|\,p-p_c\,|^{-\tau}\ , \tag{1.3}$$

where the exponent τ depends on dimensionality and on α but is otherwise universal. In addition to arguments in support of these conclusions, we gave some discussion of dependence of τ on α and dimensionality.

In the present paper we will use renormalization methods to discuss the functional form of the I-V characteristic near threshold, giving support to the second assertion above; we will find the high-dimensionality limit form of $\tau(\alpha)$ by considering the Cayley-tree model; we will note the implications of duality in two dimensions; we will discuss the generalization of the hyperscaling relation[2] to the present problem.

It is useful to define the conductivity

$$J=I/A=\sigma_{\mathrm{eff}}\,|\,V/L\,|^{1/\alpha}\mathrm{sgn}V \tag{1.4}$$

and its exponent

$$\sigma_{\mathrm{eff}}\sim(p-p_c)^{t}\ , \tag{1.5}$$

where obviously $\rho_{\mathrm{eff}}=\sigma_{\mathrm{eff}}^{-\alpha}$ and $\tau=\alpha t$. This is particularly important because the two limits $\alpha\to 0$ and $\alpha\to\infty$ can be identified, respectively, with the varistor (or bipolar Zener diode) and the saturating conductor (it also describes the critical currents of superfluid[3] or superconducting networks); thus the values of $\tau(0)$ and $t(\infty)$ have already been discussed in those special contexts.[4–6]

II. KIRCHHOFF'S RULES AND UNIQUENESS

The current distribution in a random network with fixed voltage boundary conditions is determined by the generalization of Kirchhoff's rules: We seek an assignment of potentials to the nodes of the network such that the currents in each link (determined by the potential difference across it and by its I-V characteristic) satisfy the condition that the net current into each node is zero. With the condition that the dependence of I on V is monotonically increasing, this assignment is unique, as is readily shown: define

$$G_{ij}(V)=\int_0^V I_{ij}(v)\,dv \tag{2.1}$$

for each link and construct the function

$$S=\sum_{i,j}G_{ij}(V_i-V_j)\ . \tag{2.2}$$

Since each of the functions G_{ij} is bounded below, so is S, and S has a minimum. The existence of this minimum is equivalent to the existence of a solution to Kirchhoff's equations, as can be seen by explicitly calculating the partial derivative of S with respect to a site voltage: The

mathematical statement that the derivative vanishes becomes translated into the physical statement that the currents balance into the node.

To demonstrate uniqueness we suppose that there were two minima of S for voltage assignments $\{V_i^{(1)}\}$ and $\{V_i^{(2)}\}$. Then S would also have a saddle point at $\{V_i^{(s)}\}$, since along any path in $\{V\}$ space connecting $\{V_i^{(1)}\}$ and $\{V_i^{(2)}\}$ the value of S would have a maximum [allowing V_i to depend on a parameter λ by the rule $V_i=V_i^{(1)}+\lambda(V_i^{(2)}-V_i^{(1)})$ generates such a path]. This saddle point would also be a solution to Kirchhoff's equations (again $\partial S/\partial V_i=0$). In the case that $I(V)$ is differentiable with a positive definite derivative, it is readily shown that no such saddle point can exist, since S can be expanded in a series

$$S=S^{(s)}+\sum_{i,j}\partial G_{ij}/\partial(V_i-V_j)\,|_s(V_i-V_j^{(s)}-V_j+V_j^{(s)}) + \sum_{i,j}\partial^2 G_{ij}/\partial(V_i-V_j)^2\,|_s(V_i-V_j^{(s)}-V_j+V_j^{(s)})^2+\cdots, \tag{2.3}$$

where the linear terms must cancel because $\partial S/\partial V_i$ does, and quadratic terms are all positive since $\partial^2 G(\Delta V)/\partial\Delta V^2=\partial I(\Delta V)/\partial\Delta V$ is positive. Since more general $I(V)$ can be approximated arbitrarily closely by differentiable functions, the proof of uniqueness is readily generalized.

In the limiting cases where $\partial I(\Delta V)/\partial\Delta V=0$ for a range of its argument, one readily produces examples wherein the solution set $\{V_i\}$ is not unique, but the current distribution is then the same for all solutions; likewise discontinuous $I(V)$ (so that $\partial V/\partial I=0$) can lead to a multiplicity of current distributions with a unique $\{V_i\}$.

The fact that solving Kirchhoff's equations is equivalent to minimizing S underlies the applicability of the relaxation method to these problems: For any distribution $\{V_i\}$ that is not the solution set, there must be a V_i for which $\partial S/\partial V_i\neq 0$; then it is possible to change the V_i so that S decreases. Thus the algorithm must converge. Redner and Mueller[7] recently claimed that the relaxation method failed for diode networks, but this claim is based on a faulty algorithm which failed to check whether the replacement V_i is better (in the sense that S is smaller) than the original value. A minor change in the algorithm solves the problem.

III. DUALITY IN TWO DIMENSIONS

Two-dimensional networks, by virtue of their topology, have a special property ("duality") which constrains their behavior. The underlying geometric principle is as follows: Consider an arbitrary planar network of sites connected by bonds; the bonds subdivide the plane into polygons. Now choose the center of each polygon as the sites of the dual network, and join the dual sites by dual bonds if the corresponding polygons share an edge. The dual of the dual network is again the original network, and every bond intersects exactly one dual bond.

The implications of duality for linear networks have already been noted,[8-11] and the basic construction also applies to the present problem: Given a potential $\{V_i\}$ defined on the nodes of the original network which satisfies Kirchhoff's equations, one can construct another function $\{W_i\}$ on the dual network such that the potential difference W_i-W_j between neighboring dual sites is the current flowing on the bond (of the original network) that separates the dual sites, and a corresponding dual current distribution which is equal to the potential difference V_i-V_j between the corresponding network sites. The dual current is related to dual potential difference by the same function that relates potential difference to current in the original problem:

$$I_{\text{dual}}=f(\Delta W)\leftrightarrow\Delta V=f(I)\leftrightarrow I=f^{-1}(\Delta V)\,. \tag{3.1}$$

The implications of this result are best appreciated by considering a specific problem: a lattice of power-law conductors [described by Eq. (1.1)], a fraction x of which have been replaced by nonconducting links, but with $p=1-x$ greater than the percolation threshold so that the conductivity is nonzero. The problem dual to this is another lattice of power-law conductors, but now described by

$$V=r^{-1}\,|\,I\,|^{1/\alpha}\text{sgn}I \tag{3.2}$$

[which is the same form as Eq. (1.1) but with a different power law], a fraction x of which have been replaced by links of infinite conductivity, but with x less than the percolation threshold so that the conductivity is finite. As x approaches the percolation threshold, the conductivity of this latter problem will diverge, so that we might define $\sigma\sim(x_c-x)^{-s}$ to describe it; then it is implied by (3.1) that

$$\sigma_{\text{dual}}=\rho_{\text{original}} \tag{3.3}$$

or

$$s(1/\alpha)=\tau(\alpha)=\alpha t(\alpha)\,. \tag{3.4}$$

IV. RENORMALIZATION OF THE I-V CHARACTERISTIC: THE POWER-LAW CASE

We consider random networks of power-law conductors in which all elements are described by the same exponent α, but with randomly varying coefficients r_i. Since any two-terminal network of this sort is electrically equivalent to a single power-law conductor,[1] the renormalization scheme is relatively simple and very similar to what has been done in the linear case. Indeed, our principal reason for discussing this problem is to introduce the concepts which will be needed to treat the general problem in the next section.

We then proceed as usual:[12-15] a small network ("cell") is chosen, and also a slightly arbitrary rule for determining its conductivity. For example, consider the cell shown in Fig. 1 (inset). Let the four elements of the cell that we have chosen have characteristics of the form

$$I=g_i\,|\,V\,|^{1/\alpha}\text{sgn}V\,, \tag{4.1}$$

where the coefficients g_i are chosen independently from a distribution $P(g)$. The V characteristic of the cell also has this form, with

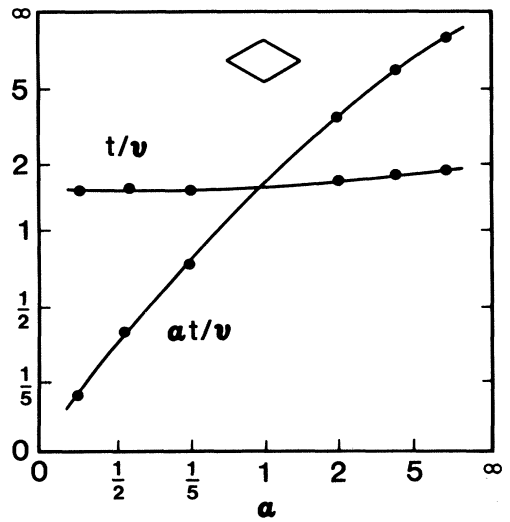

FIG. 1. Conduction exponents $t(\alpha)$ and $\alpha\tau(\alpha)$ given by a simple two-dimensional renormalization transformation. Inset: the renormalization cell used.

$$g_{\text{cell}}=(g_1^{-\alpha}+g_2^{-\alpha})^{-1/\alpha}+(g_3^{-\alpha}+g_4^{-\alpha})^{-1/\alpha}\ , \tag{4.2}$$

the distribution of values received defines a new distribution P' (g_{cell}); the process as a whole defines a transformation T which generates P' from P.

One of the effects of this transformation is to change the fraction of elements p for which g is nonzero. This aspect is described by conventional percolation theory with the result[12]

$$p'-p_c=b^{1/\nu}(p-p_c)\ , \tag{4.3}$$

where b is the size of the cell ($b=\sqrt{2}$ in our case) and ν and p_c are approximations to the correlation exponent and percolation threshold. For the cell chosen, $p_c=0.618$ and $\nu=0.818$, which are to be compared to $p_c=0.5$ and $\nu=1.33$ for the two-dimensional bond problem. Better results are obtained with larger cells.

At p_c there is a distribution $P^*(g)$ for which the only effect of T is a shift of scale of g,

$$T(P^*(g))=\lambda P^*(g\lambda)\ . \tag{4.4}$$

This is also the g distribution of a large network near the percolation threshold.

The value of the conductivity exponent $t(\alpha)$ can be extracted from λ by the rule

$$\lambda=b^{t/\nu}b^{-(d-1)+1/\alpha}\ . \tag{4.5}$$

The second term arises because the g_{cell} is the conductance of the cell (rather than its conductivity), which has an explicit dependence ($g\sim AL^{-1/\alpha}$) on cell size in addition to the dependence implied by scaling.

The transformation T was implemented by computer as follows: The distribution P was represented by an array of 10000 values (of which only 6180 were nonzero), and values were chosen from it at random to construct cells. The conductances of these cells [as given by Eq. (4.2)] were stored in an array P'; new cells were constructed until P' also had 6180 nonzero elements. Repeating this transformation ten times gives a sequence of distributions with corresponding averages $\langle g\rangle_P$, and the ratio of successive averages

$$\langle g\rangle_{P^{(n)}}/\langle g\rangle_{P^{(n-1)}}=\lambda \tag{4.6}$$

gives a series which converges to λ.

Figure 1 shows the results of performing this calculation for various values of α. In our previous paper[1] we gave another example based on a different cell. The results look different because in the present case $\tau(0)$ is exactly zero [so that $t(0)$ is finite], whereas in the previous example $\tau(0)$ was small but nonzero.

The limit $\alpha\to 0$ is best represented by the bipolar Zener diode ("spark gaps" in Lobb's terminology;[16] the item of commerce is called a varistor), which switches from nonconduction to conduction at an onset voltage V_0. A random network of such devices would have a critical voltage whose concentration dependence near the percolation threshold is of the form

$$V_0(p)\simeq(p-p_c)^{\tau(0)}V_0\ . \tag{4.7}$$

Physically, the onset voltage is determined by the shortest conducting path, and so our result is that the ratio of arc length to end-to-end distance of the shortest conducting path is only weakly dependent on $p-p_c$.

The exponent $t(0)$ is almost meaningless: in general, $t(\alpha)$ describes the concentration dependence of the current at fixed voltage, but in the limit $\alpha\to 0$, the current is either zero or extremely large, depending on the voltage level chosen. Thus the discrepancy between $t(0)\simeq 1$ (here) and $t(0)\simeq\infty$ (previously) is artificial. Furthermore, in the present case $\tau(0)=0$, because all paths through the renormalization cell are of the same length—clearly a special feature of the case chosen. Thus this result should not be taken as support for Lobb's conjecture[16] that $\tau(0)=0$; the small-cell renormalization study presented previously does not contradict it either (Lobb's conjecture was based on results derived from *large* renormalization cells).

V. RENORMALIZATION OF *I-V* CHARACTERISTIC: THE GENERAL CASE

In the case described above, the functional form of the *I-V* characteristic could be described by just two parameters (g and α), only one of which was affected by the renormalization, and thus the problem reduced to the determination of a one-dimensional distribution $P(g)$. More general functional forms can be treated by the renormalization method by considering a larger parameter set. For example, it would be very interesting to study the behavior under renormalization of functions of the form

$$I=g_1|V|^{\beta_1}+g_2|V|^{\beta_2} \tag{5.1}$$

for chosen values of β_1 and β_2. Functions of this sort can be represented as a vector (g_1,g_2), and the analysis would focus on the evolution under the renormalization transformation of distributions in this parameter space. This particular example is unworkable, however, because the functions generated by the renormalization transformation

will, in general, fail to have a form that can be represented by two parameters (g_1,g_2): this subspace of the set of all functions is not closed under renormalization. It seems quite likely that the only closed parametrizations are infinite, however, it is a reasonable approximation to use a large finite set and project the functions produced by the renormalization back in this set. We chose to represent an arbitrary I-V curve as a vector of values of I for the 30 values

$$V_n=2^{-n/3},\quad n=0,1,\ldots,29 \tag{5.2}$$

(which spans approximately three decades of voltage with logarithmically spaced intervals), and determined the current for other values of the voltage by interpolation or extrapolation. The function P was represented by 1000 vectors, which initially was either the function

$$I=\begin{cases} V, & |V|<1 \\ \mathrm{sgn}V, & |V|\geq 1 \end{cases} \tag{5.3}$$

(with probability p) or the zero function. The renormalization transformation chosen was the parallel combination of series elements described above. Unlike the previous case, however, the fraction of conducting elements was above (rather than at) the percolation threshold. Under these circumstances, iteration of the renormalization transformation eventually produces a distribution for which every element has the same I-V characteristic, which can be taken to be the I-V characteristic for a bulk sample whose microscopic structure is described by the initial distribution. These functions for various p are shown in Fig. 2.

Figure 3 shows another example. The I-V characteristic for the conducting elements (the top curve) was chosen to be irregular in form. Dilution with nonconducting elements causes the irregularities to be smoothed out.

These results show that far from p_c the bulk I-V characteristic is very similar to that of the conducting elements, but as p_c is approached, it becomes more featureless. Power laws, which are straight lines in the logarithmic representation, are unchanged by the renormalization

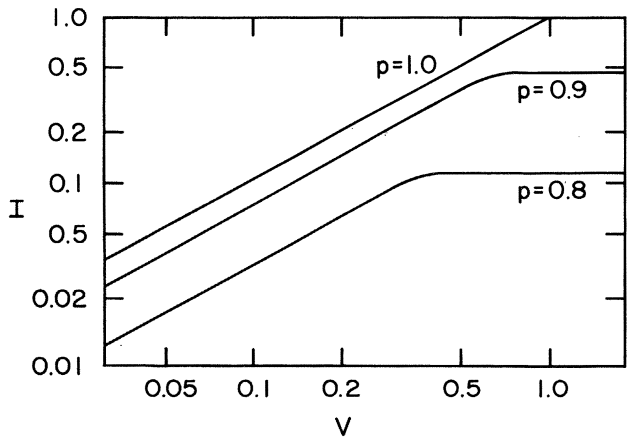

FIG. 2. Concentration dependence of the I-V characteristic for an inhomogeneous conductor, as given by a renormalization transformation.

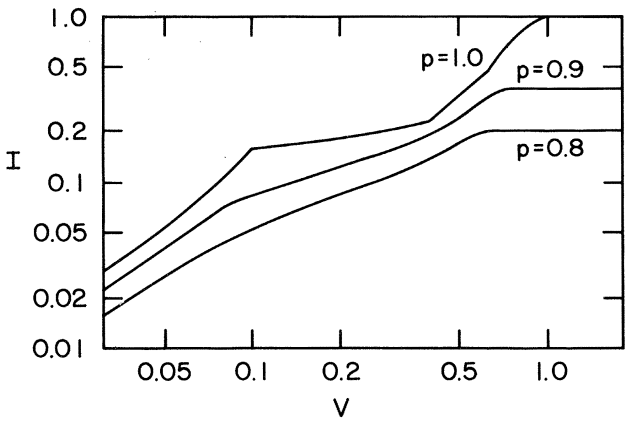

FIG. 3. Same as Fig. 2, but with a different initial I-V characteristic. This suggests that near p_c arbitrarily complicated functions give rise to power laws plus a smooth crossover.

(except for a vertical shift), and close to p_c the I-V characteristic has reduced to a high-field power law, a low-field power law, and a smooth transition between them.

VI. CAYLEY-TREE MODEL

The Cayley tree is an infinite branching network with no closed loops. Locally it resembles a regular lattice in that each site is connected to z others, but globally it does not, since the number of nth neighbors grows exponentially [as $z(z-1)^{n-1}$] rather than as a power law ($\simeq n^{d-1}$); the network cannot be embedded in a lattice of finite dimensionality.

It has often proved useful to study network problems on the Cayley tree. Owing to the fact that the tree can be defined recursively, it is possible to find the conductivity exponents for the linear percolation problem exactly.[17–20] The exponent values received are those appropriate to an infinite system and therefore also to a system above the critical dimensionality ($d^*=6$). The discussion of conduction problems on the Cayley tree is conveniently divided into three steps.

The first step determines the geometric properties of a random tree. This does not depend on the nature of the conducting bonds, and was solved by Essam and Fisher.[17] The principal results are that the percolation threshold for a tree of coordination number z is given by

$$p_c=\frac{1}{z-1}\ , \tag{6.1}$$

and that for $p>p_c$ the probability that a given site is connected by a particular bond to an infinite cluster is

$$\epsilon=p-p_c\ . \tag{6.2}$$

The probability that the site is not connected by any of its outgoing bonds to an infinite cluster is $(1-\epsilon)^z$, since the outgoing paths are independent.

For the second step we assume that the tree is grounded at infinity, apply a unit potential at an arbitrary site, and define the "conductivity along a bond" as the current that flows out along one of the outgoing bonds. This quantity will depend on the environment of the site chosen, so that we must discuss its probability distribution $P(g)$. In view of the results of the geometry problem, we may assume this has the form

$$P(g)=1-\epsilon+\epsilon H(g)\ . \tag{6.3}$$

Since an outgoing branch can be decomposed into the first bond in series with the parallel combination of $(z-1)$ branches, a recursion relation for P can be written as

$$P(g)=\int \delta(g-[g_0(g_1+\cdots+g_{z-1})]/[g_0^{\alpha}+(g_1+\cdots+g_{z-1})^{\alpha}]^{1/\alpha})Q(g_0)P(g_1)\cdots P(g_{z-1})\ , \tag{6.4}$$

where the integration is over all the g_i, and

$$Q(g)=(1-p)\delta(g)+p\delta(1-g) \tag{6.5}$$

is the distribution of bond conductivities. From this recursion we can derive a recursion for the average of g over H,

$$\langle g\rangle_H=(z-1)p\epsilon(1-\epsilon)^{z-2}\langle g/(1+g^{\alpha})^{1/\alpha}\rangle_H+p(z-1)(z-2)\epsilon^2(1-\epsilon)^{z-3}\langle\langle (g_1+g_2)/[1+(g_1+g_2)^{\alpha}]^{1/\alpha}\rangle_H\rangle_H\ , \tag{6.6}$$

where terms higher than second order in ϵ have been omitted, and the notation

$$\langle f(g)\rangle_H=\int f(g)H(g)dg \tag{6.7}$$

has been introduced. Near p_c, the range of g will be small, and Eq. (6.6) can be further simplified as

$$\langle g\rangle_H=(z-1)p\epsilon[1-(z-2)\epsilon]\langle g-\alpha^{-1}g^{\alpha+1}\rangle_H +p(z-1)(z-2)\epsilon^2\langle g\rangle_H \tag{6.8}$$

or

$$p\langle g^{\alpha+1}\rangle_H=\alpha(p-p_c)\langle g\rangle_H\ . \tag{6.9}$$

If we now assume that near p_c the p dependence of H can be represented by

$$H(g)=(p-p_c)^{-x}f(g(p-p_c)^{-x})\ , \tag{6.10}$$

where f is independent of p, Eq. (6.9) determines x to be $1/\alpha$, so that

$$\langle g\rangle_H\simeq(p-p_c)^{1/\alpha} \tag{6.11}$$

and

$$\langle g\rangle_p\simeq(p-p_c)^{1+1/\alpha}\ . \tag{6.12}$$

This result agrees with that of Stinchcombe[18] in the case $\alpha=1$ that he treated; the method is essentially the same as that of Heinrichs and Kumar[20] except that the use of Laplace transforms has been avoided.

The results (6.11) can be achieved by a simpler argument. Near the percolation threshold, the current-carrying paths are long simple chains, and sites at which the current divides are relatively infrequent. A simple agrument shows that the chain length is of the order of $\epsilon^{-1}\simeq(p-p_c)^{-1}$. If we ignore the existence of sites where four finite paths join (which occurs with probability ϵ^4), then one-half the applied voltage drop is between the origin and the first branch site, and the current in the chain is

$$(V/2\mathscr{L})^{1/\alpha}\approx V^{1/\alpha}(p-p_c)^{1/\alpha}\ , \tag{6.13}$$

which is then an estimate for $\langle g\rangle_H$.

This is not the same as the effective conductivity g_{eff}, however, which was defined above [Eq. (1.4)] as the average current density per unit applied field, and which also determines the average power $\mathscr{P}$ dissipated per unit volume,

$$\mathscr{P}=(I/A)(V/L)=g_{\text{eff}}(V/L)^{1+1/\alpha}\ . \tag{6.14}$$

In either case the important point is that the geometry of the applied field and the boundary conditions at infinity are completely different from the case just considered.

We can estimate the form of g_{eff} by an argument similar to that just given for the conductance along a bond. The potential difference across a chain will be determined by the geometrical distance ξ between the ends of the chain, which is less than the length $\mathscr{L}$ of the chain because the chain is twisted. In the high-dimensionality limit, the chain performs a random walk in space. Then $\xi^2\simeq\mathscr{L}$ and the current carried by a chain is

$$I_{\text{chain}}=(\xi V/L\mathscr{L})^{1/\alpha}=(p-p_c)^{1/2\alpha}(V/L)^{1/\alpha}\ , \tag{6.15}$$

except that the chain can carry no current at all unless both ends are connected to the infinite cluster, which occurs with probability $\epsilon^2\simeq(p-p_c)^2$. Then the average power dissipated in a link is

$$\mathscr{P}=\epsilon^2[(p-p_c)^{1/2\alpha}(V/L)^{1/\alpha}]^{1+\alpha} =(p-p_c)^{(5+1/\alpha)/2}(V/L)^{1+1/\alpha}\ . \tag{6.16}$$

Comparison with (6.14) then gives

$$\sigma_{\text{eff}}\simeq(p-p_c)^{(5+1/\alpha)/2}\ . \tag{6.17}$$

To use the definition (6.14) to calculate g_{eff}, we must appreciate that the current density J is different from the current I_{chain} carried by the chain, because a given link of the chain is almost as likely to be carrying current in the backward direction as forward (as defined by the direction of the applied field), and because the chain will carry no current unless both ends are connected to the infinite cluster. The fraction of bonds pointing forward exceeds that of the backward bonds by[21] $\xi/\mathscr{L}$, and the probability that the chain carries any current is ϵ^2, so

$$J=(\xi/\mathscr{L})I_{\text{chain}}\epsilon^2=(p-p_c)^{(5+1/\alpha)/2}(V/L)^{1/\alpha}\ , \tag{6.18}$$

which agrees with (6.17).

In the linear case it was possible to define and solve a recursion relation for J; the result was $g_{\text{eff}}\simeq(p-p_c)^3$ which agrees with (6.18), since $\alpha=1$ in this case. The discussion given here is implicitly present in the former discussion when one looks at which terms were kept and which were deleted, except that in that case it was possible to calculate the required averages in a more explicit manner. We were not able to treat the nonlinear problem this way.

A similar discussion can be given for the exponent s, which characterizes the divergence of the conductivity of a mixture of unit and infinite conductors as the threshold for percolation of the "super" elements is approached from below. Now the relevant geometrical fact is that highly conducting clusters of average diameter $\xi\simeq(p_c-p)^{1/2}$ are separated by only a few links of finite conductivity. Thus the potential drop across these boundary links will be proportional to ξ and the applied field V/L, giving a current

$$I_{\text{boundary}}\simeq(\xi V/L)^{1/\alpha}\ . \tag{6.19}$$

This current is almost as likely to be antiparallel as parallel to the field; the same argument introduced above implies that the average current in a finite conductivity circuit element is

$$J\simeq(\xi/\mathscr{L})(\xi V/L)^{1/\alpha}\simeq(p_c-p)^{(\alpha-1)/2\alpha}(V/L)^{1/\alpha}\ , \tag{6.20}$$

giving

$$s(\alpha)=(1-\alpha)/2\alpha\ . \tag{6.21}$$

VII. DIMENSIONALITY DEPENDENCE

Skal and Shklovskii[22] and de Gennes[23] have given a model which describes percolating systems. They envision that the current-carrying parts of the cluster can be idealized as chains of conductors—possibly multiply stranded, but essentially one dimensional in nature—of resistance $\mathscr{L}\simeq(p-p_c)^{-\zeta}$, which join at nodes with average spacing $\xi\simeq(p-p_c)^{-\nu}$. The voltage difference across the chain is approximately $\xi V/L$, and the current carried by a chain can then be estimated from the generalized Ohm's law

$$I=(\xi V/L)^{1/\alpha}\ . \tag{7.1}$$

The lateral chain spacing is also ξ, and so the current density is $I/A=(\xi V/L\mathscr{L})^{1/\alpha}\xi^{1-d}$, giving

$$t(\alpha)=(d-1)\nu+(\zeta-\nu)/\alpha\ . \tag{7.2}$$

This model has been criticized[24] on the grounds that the topology is overly simple. However, it does have the advantages that (1) Eq. (7.2) fits all estimated values for $t(d,\alpha)$ with $\zeta(d,\alpha)$ showing very little dependence on d or α, and (2) the equation correctly predicts the high-dimensional behavior: it becomes exact at $d^*=6$, with $\nu^*=\frac{1}{2}$ and $\zeta^*=1$. Alternate models that have been proposed[24] are much less satisfactory in this regard.

A corresponding relation for $s(\alpha)$ can be given by assuming that Eq. (6.20), which was derived for the Cayley tree, can be meaningfully used at finite dimensionality by defining $\mathscr{L}\simeq(p_c-p)^{-\zeta}$ and $\xi\simeq(p_c-p)^{-\nu}$ with the same exponents as before. The assumption with regard to ξ is standard; the other assumption is more suspicious since the property used in Eq. (6.20) is the arc length, rather than the resistance (which are the same thing only for perfectly one-dimensional chains of the Cayley model). Setting aside the objections, we come to the relation

$$s(\alpha)=\nu+\nu/\alpha-\zeta\ , \tag{7.3}$$

which can be combined with (7.2) to give

$$\alpha t(\alpha)+s(1/\alpha)=d\nu\ , \tag{7.4}$$

which is a generalization of the relationship $s+t=d\nu$ proposed previously for the linear case.[2] It is the only generalization which is consistent with the high-dimensionality results (6.17) and (6.21).

Equation (7.2), combined with Lobb's conjecture[16] that $\tau(0)=0$ in two dimensions, would imply $\zeta=\nu=t(\alpha)$; this is also forced by the combination of (7.4) and the dual relation (3.4).

In some respects (7.4) strengthens the case for the hyperscaling relationship, since it includes the functional forms for $s(\alpha)$ and $t(\alpha)$; however, in other respects it weakens the case; in particular, the arguments originally advanced for the relationship would give a different combination of $s(\alpha)$ and $t(\alpha)$.

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