PHYSICAL REVIEW B VOLUME 29, NUMBER 11 1 JUNE 1984

Many-body properties of a one-dimensional Bose liquid

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(Received 2 May 1983)

The elementary excitations of one-dimensional Bose liquids interacting via a soft potential with a Lennard-Jones-type attractive tail are evaluated in the ring-diagram approximation. Starting with this potential, we have evaluated the structure factor, thermodynamic functions, pair-distribution function, ground-state energy, and sound velocity explicitly. The energy spectrum and structure factor resemble the bulk cases, but the minimum energy is much lower and the first peak of the structure factor is higher. The pair-distribution function decreases as r^{-2} at large distances, which is characteristic of phonons in one dimension. The ground-state energy is expressed analytically in powers of a dimensionless parameter. The expression agrees with the one obtained from a δ-function-type potential.

I. INTRODUCTION

In many-body theory there are a few one-dimensional models of Bose or Fermi gases or liquids with two-body interactions among the particles. For instance, in the pioneering work of Bijl,[1] the ground-state energy of a one-dimensional system was estimated. Later, Rubin, Girardeau, and others developed more reliable approaches.[2] Lieb and Liniger[3] considered a one-dimensional Bose gas with a two-body repulsive δ-function potential and obtained the exact eigenfunction, ground-state energy, and other quantities. In dealing with a many-body system, it is worth evaluating the pair-distribution function because it yields not only the thermodynamic functions, which can be derived from the partition function, but also the structure factor. Therefore, we have recently treated the pair-distribution function of two- and three-dimensional liquid helium[4] in the chain-diagram approximation.[5]

It is the purpose of the present paper to derive the pair-distribution function of a one-dimensional Bose liquid, and then its excitation spectrum, and thermodynamic functions. The interaction potential is assumed to be given by a soft potential with a Lennard-Jones attractive tail. In Sec. II, we shall treat the pair-distribution function and present some basic formulas, which will be used in the following sections. The excitation spectrum and the structure factor will be evaluated in Sec. III. The thermodynamic functions and fluid density will be given in Sec. IV. We shall evaluate explicitly the pair-distribution function and ground-state energy in Sec. V. Finally, in Sec. VI, we shall discuss our present results in comparison with related works.

Throughout this paper, we shall take the units such that $\hbar=1$ and $2m=1$, where m is the particle mass, except for the case in which explicit restoration of $\hbar$ or $2m$ is desirable.

II. BASIC FORMULAS

In this section, we present the formula for the pair-distribution function in the ring-diagram approximation and related formulas, and then derive the excitation spectrum.

In the ring-diagram approximation, the pair-distribution function of a one-dimensional Bose liquid is given by

$$\rho_2(x)=n^2+I_2(x)-\frac{1}{2\pi\beta}\sum_{j=-\infty}^{\infty}\int dq\,\frac{u(q)\lambda_j^2(q)}{1+u(q)\lambda_j(q)}e^{iqx}\,, \tag{2.1}$$

where n is the number density, $\beta=1/k_BT$, $u(q)$ is the Fourier transform of the interaction potential, λ_j is the jth eigenvalue of the effective boson propagator representing the unit of a chain, and $I_2(x)$ is the ideal-gas contribution

$$I_2(x)=\left[\frac{1}{2\pi}\int dp\, f(p)e^{ipx}\right]^2 . \tag{2.2}$$

The eigenvalues of the effective propagator are obtained from

$$\begin{aligned}\lambda_j(q)=\frac{1}{2\pi}\int dp\, f^{(0)}(p)[1+f^{(0)}(p+q)]\\ \times\int_0^\beta d\alpha\exp\{\alpha[p^2-(p+q)^2]\}\\ \times\exp\left[\frac{2\pi i}{\beta}j\alpha\right],\end{aligned} \tag{2.3}$$

where $f^{(0)}(p)$ is the Bose distribution function

$$f^{(0)}(p)=\frac{ze^{-\beta p^2}}{1-ze^{-\beta p^2}} . \tag{2.4}$$

Performing the x and α integrations and changing the integration variable from $p+q$ to $-p$, we obtain the eigenvalues as follows:

$$\begin{aligned}\lambda_j(q)&=\frac{1}{2\pi}\int dp\left[\frac{f^{(0)}(p)}{[p^2-(p+q)^2]+(2\pi i/\beta)j}\right.\\&\qquad\left.-\frac{f^{(0)}(p)}{[(p+q)^2-p^2]+(2\pi i/\beta)j}\right]\\&=\frac{1}{\pi}\int dp\, f^{(0)}(p)\frac{q^2+2pq}{(q^2+2pq)^2+[(2\pi i/\beta)j]^2} .\end{aligned} \tag{2.5}$$

Neglecting higher-order terms, we arrive at

$$\lambda_j(q)=\frac{2nq^2}{q^4+[(2\pi/\beta)j]^2} . \tag{2.6}$$

The integrand of Eq. (2.1) can be rewritten as

$$\frac{u(q)\lambda_j^2(q)}{1+u(q)\lambda_j(q)}=\lambda_j(q)-\frac{\lambda_j(q)}{1+u(q)\lambda_j(q)} .$$

The first term on the right-hand side yields

$$-\frac{1}{2\pi\beta}\sum_j\lambda_j e^{iqx}=-I_2(x)-n\delta(x) .$$

Hence,

$$\rho_2(x)=n^2-n\delta(x)+\frac{1}{2\pi\beta}\sum_j\int dq\frac{e^{iqx}}{1+u(q)\lambda_j(q)} . \tag{2.7}$$

The structure factor is given by

$$S(q)=\frac{1}{\beta n}\sum_j\frac{\lambda_j(q)}{1+u(q)\lambda_j(q)} . \tag{2.8}$$

This result is correct within the chain-diagram approximation. When the pair-distribution function in this approximation is used for the evaluation of the internal energy, the result corresponds to the ring-diagram approximation because the graphical bonding between the two representative particles in the pair-distribution function is completed.

In terms of the pair-distribution function, the internal energy is given by

$$U(T)=U_0(T)+\frac{L}{2}\int_0^1 d\xi\int dx\,\phi(x)\frac{\partial}{\partial\beta}[\beta\rho_2(x,\beta,\xi)] , \tag{2.9}$$

where ξ is a coupling constant, U_0 is the ideal-gas energy, L is the one-dimensional volume, and $\phi(x)$ is the interaction potential. Using the Eq. (2.1) for the pair-distribution function, we obtain the internal energy as follows:

$$\begin{aligned}U(T)&=U_0(T)+\tfrac{1}{2}\pi^2L\int dx\,\phi(x)-\tfrac{1}{2}nL\int\frac{dq}{2\pi}u(q)\\&\quad+\frac{L}{2}\frac{\partial}{\partial\beta}\left[\sum_{j=-\infty}^{\infty}\int_0^1 d\xi\int\frac{dq}{2\pi}\frac{u(q)\lambda_j(q)}{1+\xi u(q)\lambda_j(q)}\right] ,\end{aligned} \tag{2.10}$$

where $u(q)$ is the Fourier transform of the interaction potential. In the j sum, we may use the formula

$$\sum_{\eta=-\infty}^{\infty}\frac{1}{x^2+\eta^2}=\frac{\pi}{x}\coth(\pi x) \tag{2.11}$$

to obtain

$$\begin{aligned}&\sum_{j=-\infty}^{\infty}\frac{u(q)\lambda_j(q)}{1+\xi u(q)\lambda_j(q)}\\&\qquad=n\beta u(q)q^2\frac{\coth\{\frac{1}{2}\beta[q^4+2nu(q)q^2\xi]^{1/2}\}}{[q^4+2nu(q)q^2\xi]^{1/2}} .\end{aligned} \tag{2.12}$$

The final expression for the internal energy is

$$\begin{aligned}U(T)&=\tfrac{1}{2}n^2L\int dx\,\phi(x)+\frac{L}{2}\int\frac{dq}{2\pi}[\epsilon(q)-q^2-nu(q)]\\&\quad+L\int\frac{dq}{2\pi}\epsilon(q)f(\epsilon(q)) ,\end{aligned} \tag{2.13}$$

where $f(\epsilon)=1/e^{\beta\epsilon}-1$ is the Bose distribution function and

$$\epsilon(q)=[q^4+2nu(q)q^2]^{1/2} \tag{2.14}$$

is the excitation energy which agrees with the Bogoliubov-Zubarev form.

In the expression of the internal energy given by Eq. (2.13), the first and second terms correspond, respectively, to the ground-state energy and the quasiparticle excitation energy. The latter demonstrates that a Landau-type quasiparticle excitation is possible even for a one-dimensional Bose system.

If the quasiparticles move with an average drift velocity with respect to the rest frame, the total momentum of the quasiparticles is related to the density given by

$$\rho_N(T)=\frac{1}{2\pi}\int q^2dq\left[-\frac{\partial f(\epsilon(q))}{\partial\epsilon(q)}\right] . \tag{2.15}$$

In three dimensions, a similarly defined quantity represents the normal-fluid density. In our case, there is no such normal-fluid density, but since this density is related to the net momentum of the excitations, it characterizes certain kinetic properties.

III. EXCITATION SPECTRUM

The excitation energy given by Eq. (2.14) is a one-dimensional version of the Bogoliubov-Zubarev[6] expression and consists of phononlike and rotonlike parts. The energy spectrum is meaningful for $u(0)\neq\infty$ and is reduced to the free-particle type if $u(0)=0$. For small momentum, Eq. (2.14) becomes phononlike,

$$\epsilon(q)\simeq q[2nu(0)]^{1/2} . \tag{3.1}$$

Note that $u(0)$, assumed to be finite and positive, determines the sound velocity $[2nu(0)]^{1/2}$.

For large momentum, Eq. (2.14) depends strongly on the potential $u(q)$ and can be expanded to

$$\epsilon(q)=q^2+nu(q)-\frac{1}{8}\left[\frac{2nu(q)}{q}\right]^2+\cdots . \tag{3.2}$$

To investigate the energy explicitly, we introduce a soft potential with a Lennard-Jones tail,

$$\phi(x)=\begin{cases} V_0 , & |x| \leq a \\ \epsilon_0[(a/x)^{12}-(a/x)^6] , & |x| \geq a . \end{cases} \tag{3.3}$$

We find the Fourier transform of Eq. (3.3) in one dimension as follows:

$$\begin{aligned} u(q)={}&2aV_0\frac{\sin(qa)}{qa}-2a\epsilon_0\left[\frac{(qa)^{11}}{11!}+\frac{(qa)^5}{5!}\right]\sin(qa) \\ &+2a\epsilon_0\left[\sum_{k=0}^{4}(-)^{k+1}\frac{(9-2k)!}{11!}(qa)^{2k+1}-\sum_{k=0}^{1}(-)^{k+1}\frac{(3-2k)!}{5!}(qa)^{2k+1}\right]\sin(qa) \\ &+2a\epsilon_0\left[\sum_{k=0}^{5}(-)^{k}\frac{(10-2k)!}{11!}(qa)^{2k}-\sum_{k=0}^{2}(-)^{k}\frac{(4-2k)!}{5!}(qa)^{2k}\right]\cos(qa) . \end{aligned} \tag{3.4}$$

Expanding the right-hand side of Eq. (3.4), we obtain a series

$$2nu(q)=A_0^2+A_1^2q^2+A_2^2q^4+A_3^2q^5+A_4^2q^6+\cdots , \tag{3.5}$$

where

$$A_0^2=4na(V_0-\tfrac{6}{55}\epsilon_0) , \quad A_1^2=4na^3\left[-\frac{V_0}{3!}+\frac{\epsilon_0}{9}\right] ,$$

$$A_2^2=4na^5\left[\frac{V_0}{5!}-\frac{\epsilon_0}{28}\right] , \quad A_3^2=2na^6\frac{\pi}{5!}\epsilon_0 , \quad \cdots .$$

For small q, we obtain the excitation energy as follows:

$$\epsilon(q)\simeq A_0q+Bq^3+Cq^5+Dq^6+\cdots , \tag{3.6}$$

where the coefficients are given by

$$B=\frac{1+A_1^2}{2A_0} , \quad C=\frac{A_2^2}{2A_0}-\frac{3(1+A_1^2)}{8A_0^3} , \quad D=\frac{A_3^2}{2A_0} , \quad \cdots . \tag{3.7}$$

Note that Eq. (3.6) represents an anomalous dispersion because B is positive.

For large momentum, we can take only the first term in $u(q)$, which is dominant, and express the excitation energy as follows:

$$\epsilon(q)=q\left[q^2+4naV_0\frac{\sin(qa)}{qa}\right]^{1/2} . \tag{3.8}$$

The energy $\epsilon(q)$ is oscillatory. Let the first minimum occur at q_0. Around this q_0, Eq. (3.8) may be replaced approximately in a Landau form,

$$\epsilon(q)=\Delta+\frac{\hbar^2}{2m^*}(q-q_0)^2 , \tag{3.9}$$

where Δ and m^* are the energy gap and effective mass, respectively.

According to the theory of Feynman *et al.* for liquid helium,[7] the excitation energy is related to the structure factor $S(q)$ as follows:

$$\epsilon(q)=\frac{\hbar^2q^2}{2mS(q)} . \tag{3.10}$$

In our case, a reverse relation is obtained from Eq. (2.8),

$$S(q)=\frac{\hbar^2q^2}{2m\epsilon(q)}\coth[\tfrac{1}{2}\beta\epsilon(q)] . \tag{3.11}$$

Hence, for absolute zero,

$$S(q)=\frac{q}{[q^2+2nu(q)]^{1/2}} . \tag{3.12}$$

Figure 1 illustrates the Fourier transform $u(q)$ of the interaction potential. The potential parameters have been chosen in correspondence with the bulk values which we have adopted earlier. They are $a=2.55$ Å, $V_0=5$ K, and $\epsilon_0=6$ K.

Figure 2 represents the excitation spectrum. The solid curve corresponds to $V_0=5$ K and the double-dashed curve to $V_0=6$ K. These curves are similar to the familiar three-dimensional (3D) excitation spectrum. However, the roton energy is much lower.

The structure factor $S(q)$ is shown in Fig. 3 as a function of dimensionless variable qa. In comparison with the 3D case, the first peak is higher. This corresponds to the reduction in the roton energy.

IV. THERMODYNAMIC FUNCTIONS

We follow Landau's theory to obtain the phonon and roton energies. For small q, we obtain

$$E_{\mathrm{ph}}(T)=\frac{1}{2\pi}\left[\frac{\zeta(2)}{A_0}(k_BT)^2-\frac{3\times 3!\zeta(4)}{A_0^4}B(k_BT)^4+\frac{5!\zeta(6)}{A_0^6}C(k_BT)^6+\cdots\right] . \tag{4.1}$$

For large q, we take Eq. (3.9) to obtain

$$E_{\mathrm{rot}}(T)=\left[\frac{m^*k_BT}{2\pi\hbar^2}\right]^{1/2}(\Delta+\tfrac{1}{2}k_BT)e^{-\Delta/k_BT} . \tag{4.2}$$

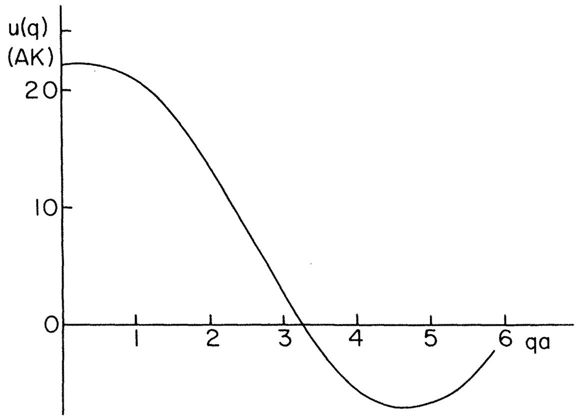

FIG. 1. Potential $u(q)$ as a function of dimensionless parameter aq with the numerical parameters: $a=2.55$ Å, $V_0=5$ K, $\epsilon_0=6$ K, and $n=0.167/\text{Å}$ (This corresponds to $n=2.18\times10^{-2}/\text{Å}^3$ in three dimensions.)

Using Eqs. (4.1) and (4.2), we obtain the corresponding specific heats

$$C_{v,\text{ph}}(T)=\frac{k_B}{2\pi}\left[\frac{2\zeta(2)}{A_0}(k_BT)-\frac{3\times4!\zeta(4)}{A_0^4}B(k_BT)^3+\frac{6!\zeta(6)}{A_0^6}C(k_BT)^5+\cdots\right], \tag{4.3}$$

$$C_{v,\text{rot}}(T)=\left[\frac{m^*k_BT}{2\pi\hbar^2}\right]^{1/2}\times k_B\left[\frac{3}{4}+\frac{\Delta}{k_BT}+\left[\frac{\Delta}{k_BT}\right]^2\right]e^{-\Delta/k_BT}. \tag{4.4}$$

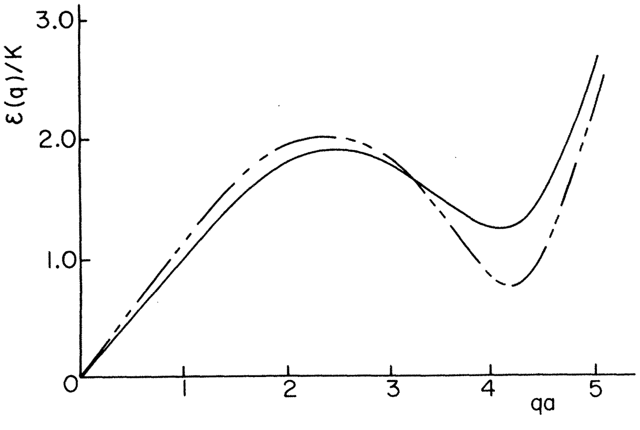

FIG. 2. Excitation spectrum vs dimensionless parameter qa. The solid and double-dashed lines are the spectrum for $V_0=5$ and 6 K, respectively, with the other parameters fixed.

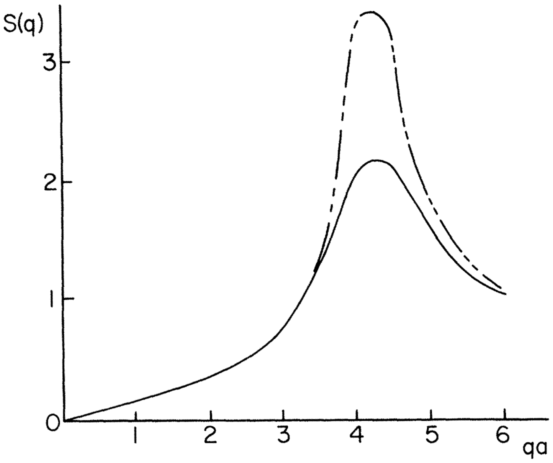

FIG. 3. Structure factor against dimensionless parameter qa. Solid line, $V_0=5$ K; double-dashed line, $V_0=6$ K.

We note that the phonon specific heat is an odd function of temperature with the first linear term which is characteristic of one dimension. The roton energy gap Δ and the roton minimum characterize the roton specific heat. The exponential factor makes its contribution very small for low temperatures.

Our theoretical expressions for various thermodynamic functions are given by

$$F_{\text{ph}}(T)=-\frac{1}{2\pi}\left[\frac{\zeta(2)}{A_0}(k_BT)^2-\frac{3!\zeta(4)}{A_0^4}B(k_BT)^4+\frac{4!\zeta(6)}{A_0^6}C(k_BT)^6+\cdots\right], \tag{4.5}$$

$$F_{\text{rot}}(T)=-\left[\frac{m^*(k_BT)^2}{2\pi\hbar^2}\right]^{1/2}e^{-\Delta/k_BT}, \tag{4.6}$$

$$S_{\text{ph}}(T)=\frac{k_B}{2\pi}\left[\frac{2\zeta(2)}{A_0}(k_BT)-\frac{4!\zeta(4)}{A_0^4}B(k_BT)^3+\frac{6!\zeta(6)}{A_0^6}C(k_BT)^5+\cdots\right], \tag{4.7}$$

$$S_{\text{rot}}(T)=\left[\frac{m^*k_BT}{2\pi\hbar^2}\right]^{1/2}k_B\left[\frac{3}{2}+\frac{\Delta}{k_BT}\right]e^{-\Delta/k_BT}, \tag{4.8}$$

$$P_{\text{ph}}(T)=\frac{1}{2\pi}\left[\frac{3}{2}\frac{\zeta(2)}{A_0}(k_BT)^2-\frac{3!\zeta(4)}{A_0^4}B_1(k_BT)^4+\cdots\right], \tag{4.9}$$

$$P_{\text{rot}}(T)=\frac{1}{2}\left[\frac{m^*(k_BT)^3}{2\pi\hbar^2}\right]^{1/2}e^{-\Delta/k_BT}, \tag{4.10}$$

where P represents pressure and

$$B_1=\frac{5A_1^2+7}{2A_0} .$$

The phonon and roton parts of the normal-fluid density are

$$\rho_{N,\mathrm{ph}}(T)=\frac{1}{2\pi}\left[\frac{2!\zeta(2)}{A_0^3}(k_BT)^2-\frac{5!\zeta(4)}{A_0^6}B(k_BT)^4+\frac{7!\zeta(6)}{A_0^8}C(k_BT)^6+\cdots\right], \tag{4.11}$$

$$\rho_{N,\mathrm{rot}}(T)=\left[\frac{m^*}{2\pi k_BT}\right]^{1/2}qe^{-\Delta/k_BT} . \tag{4.12}$$

V. PAIR-DISTRIBUTION FUNCTION AND GROUND-STATE ENERGY

In this section, we evaluate the pair-distribution function explicitly and show the characteristics of one-dimensional phonons in the Bose liquid. Introducing Eq. (2.6) into (2.1), we have

$$\Delta\rho_2(x)=n^2-\frac{n}{\pi^2}\int_0^\infty y^2dy\cos(xy)\int_{-\infty}^{\infty}dz\left[\frac{1}{y^4+z^2}-\frac{1}{[y^4+2nu(y)y^2+z^2]}\right],\quad \Delta\rho_2(x)=\rho_2(x)-I_2(x) , \tag{5.1}$$

where we changed the variables

$$z=2\pi j/\beta ,\quad y=q .$$

Performing the integration, we obtain

$$\Delta\rho_2(x)=n^2-\frac{n}{\pi}\int_0^\infty dy\left[1-\frac{1}{[y^2+2nu(y)]^{1/2}}\right]\cos(xy) . \tag{5.2}$$

The exact evaluation of the pair-distribution function depends strongly on the potential $u(q)$, which is given by Eq. (3.4). It is very difficult to evaluate Eq. (5.2) analytically for a soft potential with a Lennard-Jones tail. Therefore, we retain only the constant term of $u(q)$ to obtain the asymptotic expression of the pair-distribution function. Replacing $u(q)$ by $u(0)$ and integrating by parts, we obtain

$$\frac{\Delta\rho_2(x)}{n^2}=1-\frac{2u(0)}{\pi\alpha}\frac{1}{(\alpha x)^2}\left[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!(2n+1)!!}{(\alpha x)^{2n}}\right], \tag{5.3}$$

where α is $[2nu(0)]^{1/2}$. The x^{-2} proportionality of the pair-distribution function at large distances is a characteristic property of one-dimensional phonons.

Taking only the ground-state energy in Eq. (2.13), we have

$$\begin{aligned}E_g&=\tfrac{1}{2}n^2L\int dx\,\phi(x)+\tfrac{1}{2}L\left[\int_{-q_0}^{q_0}+\int_{-\infty}^{-q_0}+\int_{q_0}^{\infty}\right]\\&\quad\times\frac{dq}{2\pi}[\epsilon(q)-q^2-nu(q)]\\&=I_1+I_2+I_3 .\end{aligned} \tag{5.4}$$

To evaluate these integrals, we use Eqs. (3.6) and (3.2) in the integral of Eq. (5.4) for small and large momenta, respectively. The first integral I_1 of Eq. (5.4) is obtained easily,

$$I_1=\tfrac{1}{2}n^2L\int\phi(x)dx=Nna(V_0-\tfrac{6}{55}\epsilon_0)=NnaV^* . \tag{5.5}$$

We insert Eq. (3.6) into (5.4) and take up to order q^2 for the second integral. We then obtain

$$\begin{aligned}I_2&=\frac{L}{4\pi}\int_{-q_0}^{q_0}dq[\epsilon(q)-q^2-nu(q)]\\&=-\frac{LA_0^2}{4\pi a}=-\frac{nLV^*}{\pi} ,\end{aligned} \tag{5.6}$$

where

$$V^*=V_0-\tfrac{6}{55}\epsilon_0 ,\quad A_0^2=4naV^* .$$

To evaluate the remaining integral for large momenta, we use Eq. (3.2) and choose only the dominant term to arrive at

$$\begin{aligned}I_3&=\frac{L}{4\pi}\left[\int_{-\infty}^{-q_0}+\int_{q_0}^{\infty}\right]\\&=-\frac{L}{\pi}n^2a^3V_0^2\int_{\alpha_0}^{\infty}\frac{\sin^2x}{x^4}dx ,\quad \alpha_0=q_0a\\I_3&=-\frac{n^2a^3}{\pi}cLV_0^2 ,\end{aligned} \tag{5.7}$$

where $c=1.3290$.

Adding these results, we finally obtain

$$E_0=\frac{E_g}{N}=naV^*\left[1-\frac{1}{\pi na}-\frac{ca^2}{\pi V^*}V_0^2\right] . \tag{5.8}$$

We note that the ground-state energy obtained by Lieb and Liniger is given by

$$E_0=\frac{E_g}{N}=nc\left[1-\frac{4}{3\pi}\left[\frac{c}{n}\right]^{1/2}\right], \tag{5.9}$$

where $2c$ is the strength of the δ function.

Equation (5.8) gives the ground-state energy per particle for the soft potential. The first term agrees essentially with the first term of Eq. (5.9). Our second term includes the contribution from the attractive potential so that it differs from Eq. (5.9). The third term is mainly determined by the hard-sphere diameter and the soft potential. A similar calculation for the 3D case yields the same ground-state energy as Lee, Huang, and Yang.[8]

VI. RESULTS AND DISCUSSIONS

It is well known that Bose-Einstein condensation does not exist in two and one-dimensions because of the forms for the average number of particles. The same eigenvalue expression can be used for two and three dimensions[9] in the neglect of higher-order terms, but the number density appearing in this expression for reduced dimensions does not include the condensate.

By adopting a soft potential, we have obtained the excitation spectrum as a function of dimensionless parameter qa in Fig. 2 with parameters n, a, ϵ_0, and V_0. The behavior of this spectrum is analogous to Landau's case: for small momentum, it is phononlike, while it is rotonlike at around q_0 for large momenta. Passing the roton minimum the slope of the dispersion is slightly steeper than that of phonons. We note that as the numerical values of three parameters n, a, and V_0 increase, the phonon part becomes steeper, followed by the concave part which is depleted progressively. Finally, the roton minimum disappears and eventually becomes an inflection point. Thus, the excitation spectrum becomes that of the pseudopotential.

It is interesting to note that, if the attractive potential is neglected in Eq. (3.4), the one-dimensional excitation reduces to the 3D excitation spectrum of Brueckner and Sawada.[10] In this case, the energy gap and the effective mass are given by

$$\Delta=q_0\left[q_0^2+4naV_0\frac{\sin(q_0a)}{q_0^a}\right]^{1/2}, \tag{6.1}$$

$$m^*=\frac{(q_0^4+4nV_0q_0\sin q_0a)^{1/2}}{6q_0+2naV_0(2\cos q_0a-q_0a\sin q_0a)}, \tag{6.2}$$

and q_0 can be determined as follows:

$$2q_0^3+nV_0[\sin(q_0a)+(q_0a)\cos(q_0a)]=0\,. \tag{6.3}$$

The detailed analysis of Eqs. (6.1)–(6.3) for the Brueckner-Sawada spectrum was given by Pathria and Singh.

Lieb[11] showed that the excitation spectrum of the one-dimensional Bose gas interacting via a repulsive δ-function potential could be analyzed by a double spectrum: type-I excitation (so called Bogoliubov spectrum), and type-II excitation ("hole" states). Our excitation spectrum Eq. (2.14) is just Bogoliubov type and thus type-II excitation does not appear.

The structure factor corresponding to the excitation spectrum of Fig. 2 increases linearly as $q/[2nu(0)]^{1/2}$ for small momenta, and reaches the peak which corresponds to the roton minimum. The slope is more or less the same as those of two and three dimensions.[9]

The thermodynamic functions [Eqs. (4.1)–(4.11)] based on the phonon spectrum Eq. (3.6) agree with what was obtained by Padmore,[12] who used the simple two- and one-dimensional models of restricted dimensionality: fluid flow in two parallel plates and also in the straight rectangular tube. If we take only the first term, the specific-heat and the normal-fluid densities in two and three dimensions are obtained as follows: For three dimensions,

$$C_{v,\mathrm{ph}}=\frac{k_B}{2\pi}\frac{4!\zeta(4)}{A_0^3}(k_BT)^3\,,\quad \rho_{N,\mathrm{ph}}=\frac{1}{6\pi^2}\frac{4!\zeta(4)}{A_0^5}(k_BT)^4\,,$$

and for two-dimensions,

$$C_{v,\mathrm{ph}}=\frac{k_B}{2\pi}\frac{3!\zeta(3)}{A_0^2}(k_BT)^2\,,\quad \rho_{N,\mathrm{ph}}=\frac{1}{4\pi}\frac{3!\zeta(3)}{A_0^4}(k_BT)^3\,.$$

Let N be the dimensionality. Then the specific heat and the normal-fluid density can be written as T^N and T^{N+1}, respectively. In the case of the roton part, both are not different from those of other dimensions except for small modifications. We have shown elsewhere that the second and third terms play important roles in two-dimensional fluid density.[13] We expect a similar role played by these terms in the one-dimensional case.

We have evaluated the pair-distribution function of the one-dimensional Bose liquid with the soft potential in the small-momentum region [Eq. (5.3)]. This pair-distribution function is subject to the condition $x \gg \sqrt{3}/\alpha$. It decreases as r^{-2} at large distances and rapidly approaches unity as $x\rightarrow\infty$. We note that the pair-distribution functions[14] of two- and three-dimensional hard-sphere Bose systems decrease as r^{-3} and r^{-4}, respectively, at large distances. The x^{-2} proportionality is the main long-distance behavior of one-dimensional phonons in a Bose liquid. We expect that the pair-distribution function at short distances will oscillate and tends to unity as $x\rightarrow\infty$. This will be mainly due to the contribution from the core part of the soft potential.

ACKNOWLEDGMENTS

This work was supported by the U. S. Navy Office of Naval Research under Contract No. N00014-79-C0451. The support from the Korea Science and Engineering Foundation to the Korea University is also acknowledged.

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