NMR of textures in rotating ^{3}He-A

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(Received 5 January 1984; revised manuscript received 5 April 1984)

Four vortex textures in ^{3}He-A, one analytic and three singular, are proposed as possible candidates to explain recent transverse NMR results in the Helsinki rotating cryostat. Although the analytic texture, previously discussed by Seppälä and Volovik, has a transverse NMR shift and absorption closest to the observed values, the agreement with experiment is not wholly satisfactory. Longitudinal NMR shifts and absorption are calculated for the four textures; such measurements might determine which vortex is present.

I. INTRODUCTION

Recent transverse nuclear magnetic resonance (NMR) (Ref. 1) experiments at the rotating cryostat in Helsinki have indicated the existence of vortices in rotating ^{3}He-A. Seppälä and Volovik (SV) (Ref. 2) thereafter proposed two possible types of stable vortex structures for rotating ^{3}He-A in a magnetic field, and compared their NMR signatures with the experiments of Hakonen *et al.*[1] In the Ginzburg-Landau (GL) regime near T_c, SV found that 4π analytic vortices have a transverse NMR frequency and absorption compatible with the experiment, whereas singular vortices cannot explain the data. At the same time, the array of analytic 4π vortices has a higher free energy per unit circulation than the singular vortex array. SV explain this discrepancy by assuming that the analytic vortex lattice is metastable because analytic vortices form more easily when a stationary superfluid begins to rotate.

Hakonen *et al.*[1] find the same frequency of the NMR satellite line when a rotating normal fluid is cooled below T_c and when a stationary superfluid accelerates above ω_{c1} (the threshold for vortex creation). Evidently, the metastability arguments of SV cannot readily explain all the data. Furthermore, the free energy and NMR calculations of SV (Ref. 2) were limited to the GL region (except for their phenomenological parameter S), while the NMR measurements were made at temperatures below $0.85T_c$, where the GL approach is inaccurate.

This paper proposes two new singular textures as possible vortex configurations of slowly rotating ^{3}He-A in a high axial magnetic field. Using the hydrodynamic approach of Cross[3] and the hydrodynamic coefficients calculated by Williams,[4] we determine the free energies of these two vortices and the two SV vortices for all relevant temperatures and angular velocities. Our results show that the SV singular vortex has the lowest free energy (per unit circulation) of the four. For the vortex parameters found in the free-energy minimization, we calculate (variationally) the frequency and intensity of the low-frequency longitudinal spin-wave satellite as well as the transverse one. Comparing these results with the transverse NMR data of Hakonen *et al.*,[1] we find that all three singular vortices have transverse NMR shifts incompatible with the experimental data. The SV analytic vortex is compatible with the experimental data at lower temperatures, but at higher temperatures it also disagrees with the extrapolation of the NMR measurements.

Our results indicate that systematic measurements of the *longitudinal* NMR shift and intensity, as well as the temperature dependence of the transverse satellite intensity, could help decide whether the SV analytic texture is indeed present. If further experiments confirm the SV analytic vortex, it is important to study whether more accurate hydrodynamic parameters and more precise free-energy calculations can lower the energy of the SV analytic vortex relative to singular vortices. If not, is there some other explanation that precludes the singular vortices, which apparently have lower free energy? Should further experiments rule out the SV analytic vortex, then a new texture would be needed to explain the data.

II. FREE ENERGIES AND NMR OF VORTICES

The order parameter for superfluid ^{3}He-A is given by

$$A_{\mu i}=\frac{\Delta}{\sqrt{2}}d_\mu\psi_i \;, \tag{1}$$

where $\vec{\psi}=\hat{\Delta}^{(1)}+i\hat{\Delta}^{(2)}$, and $\hat{\Delta}^{(1)}$ and $\hat{\Delta}^{(2)}$ are real orthogonal unit vectors. The unit vector $\hat{d}$ is the local axis along which the spin of a Cooper pair is zero, while the unit vector $\hat{l}=\hat{\Delta}^{(1)}\times\hat{\Delta}^{(2)}$ is the axis of orbital angular momentum of a Cooper pair. In the hydrodynamic approach of Cross,[3] the free-energy density f is a functional of the vectors $\hat{d}$ and $\hat{l}$ and the normal and superfluid velocities $\vec{v}_n$ and $\vec{v}_s$ [with a suitable constraint on curl $\vec{v}_s$ (Ref. 3)]:

$$\begin{aligned}2f={}&\rho_s\vec{v}^{\,2}-\rho_0(\hat{l}\cdot\vec{v})^2+2C\vec{v}\cdot\mathrm{curl}\hat{l}-2C_0(\vec{v}\cdot\hat{l})(\hat{l}\cdot\mathrm{curl}\hat{l})+K_s(\mathrm{div}\hat{l})^2+K_t(\hat{l}\cdot\mathrm{curl}\hat{l})^2\\&+K_b(\hat{l}\times\mathrm{curl}\hat{l})^2+K_5(\hat{l}\cdot\vec{\partial}d_\alpha)^2+K_6(\hat{l}\times\vec{\partial}d_\alpha)^2+\frac{\rho_{\parallel}}{L_D^2}[1-(\hat{l}\cdot\hat{d})^2]+\frac{\rho_{\parallel}}{L_D^2}\frac{1}{H_D^2}(\vec{H}\cdot\hat{d})^2+K_4\mathrm{div}(\hat{l}\cdot\mathrm{div}\hat{l}+\hat{l}\times\mathrm{curl}\hat{l})\;,\end{aligned} \tag{2}$$

where $\vec{v}=\vec{v}_s-\vec{v}_n$, L_D is the dipolar length, and H_D is the characteristic magnetic field. The above formula for the free-energy density includes a pure divergence term (proportional to K_4) that is absent in the hydrodynamic approach, since it facilitates a smooth interpolation to the GL values as $T\to T_c$.

In the GL region, the hydrodynamic expression (2) represents the difference between the GL free-energy denisty of a texture given by $[\hat{l}(\vec{r}),\hat{d}(\vec{r}),\vec{v}(\vec{r})]$ and the GL free-energy density of the uniform, dipole-locked A phase with $\hat{d}\perp\vec{H}$. The values of the hydrodynamic coefficients expressed in units of the longitudinal mass density $\rho_{||}$ have been calculated by Williams[4] for all temperatures below T_c near the melting pressure. For the parameter K_4, we assume that the ratio $K_4/\rho_{||}$ remains equal to -0.5, as in the GL region.

In high magnetic fields ($H\gg H_D\approx 30$ G in the Helsinki experiments[1]), Eq. (2) shows that $\hat{d}$ lies perpendicular to the magnetic field to minimize the magnetic free energy. To reduce the gradient energy of $\hat{d}$, we assume[2] that the $\hat{d}$ vector remains along a given axis in a plane perpendicular to $\vec{H}$. We choose our axes so that the magnetic field and angular velocity of a container are along $\hat{z}$, while the $\hat{d}$ vector is along $\hat{x}$ (the vortex lattice and the axes $\hat{x}$, $\hat{y}$, and $\hat{z}$ are fixed with respect to the rotating container).

To introduce our new singular vortex textures, we approximate the true Wigner-Seitz (WS) cell by a cylinder of radius $L=(\hbar/2m_3\Omega)^{1/2}$, writing the orbital parameter near the cell boundary as

$$\vec{\psi}=e^{i\phi}(-\hat{z}+i\hat{y}) \tag{3}$$

up to a constant phase factor. Hereafter, all formulas for the order parameter are for the cell centered at the origin of the polar coordinate system (ρ,ϕ). The above order parameter has a circulation of 2π and a superfluid velocity $\vec{v}_s=\hat{\phi}/\rho$ (in units of $\hbar/2m_3$). Since the normal fluid velocity (in the same units) is given by $\vec{v}_n=2m_3\hbar^{-1}\Omega\rho\hat{\phi}$, the kinetic free energy would diverge if it continued unaltered to the origin. At distances of the order of the dipolar length L_D, the order parameter can deviate from the form (3) in order to reduce the kinetic energy, at the cost of raising the dipolar energy. At these distances the texture given by Eq. (3) can lower the total free-energy density by transforming into a radial disgyration with order parameter

$$\vec{\psi}=\hat{z}+i\hat{\phi} \tag{4}$$

corresponding to zero superfluid velocity.

In order to bend the order parameter from the form (3) where $\hat{l}=\hat{x}$ to the form (4) where $\hat{l}=-\hat{\rho}$, the $\hat{l}$ vector must rotate out of the xy plane at intermediate distances. One possibility is the z-in-x vortex, which takes the $\hat{l}$ vector from $\hat{l}=\hat{x}$ at the outer boundary to $\hat{l}=-\hat{z}$ at some intermediate distance r_1 and then to $\hat{l}=-\hat{\rho}$ near the center [as in the radial disgyration of Eq. (4)]. The full order parameter for the z-in-x vortex is defined by

$$\vec{\psi}=e^{i\phi}(-\hat{z}\sin\beta-\hat{x}\cos\beta+i\hat{y}) \tag{5}$$

at distances $r_1<\rho<L$, and

$$\vec{\psi}=\hat{z}\cos\eta-\hat{\rho}\sin\eta+i\hat{\phi} \tag{6}$$

at distances $r_0<\rho<r_1$. In our variational model, the radial angle β remains $\pi/2$ for $r_2\le\rho\le L$ and tends toward zero for $\rho=r_1$, while the radial angle η goes from $\pi/2$ at $\rho=r_1$ to zero at $\rho=r_0$. The radii r_2 and r_1 ($r_2>r_1$) are of the order of L_D, while r_0 is of the order of the coherence length $\xi\cong 93$ Å $(1-T/T_c)^{-1/2}$.

The free-energy density of the z-in-x vortex has a large contribution from the bending energy in the region $r_0\le\rho\le r_2$. One way to reduce this bending energy is to transform the outer-region order-parameter triad with $\hat{l}=\hat{x}$,

$$(\hat{\Delta}^{(1)},\hat{\Delta}^{(2)},\hat{l})=(-\hat{y}\cos\phi+\hat{z}\sin\phi\ ,-\hat{z}\cos\phi-\hat{y}\sin\phi,\hat{x})\ ,$$

into the triad $(\hat{z},\hat{\phi},-\hat{\rho})$ of the inner-region radial disgyration via the path of minimal rotation. Straightforward algebra shows that for a given polar angle ϕ, the required axis of rotation $\hat{n}(\phi)$ and the total angle of rotation $R(\phi)$ are given by

$$\hat{n}=[(\hat{x}+\hat{z})\sin(\phi/2)-\hat{y}\cos(\phi/2)][1+\sin^2(\phi/2)]^{-1/2}\ ,$$
$$\cos R=-\tfrac{1}{2}\sin^2(\phi/2)-\cos\phi\ . \tag{7}$$

The angle $R(\phi)$ lies between $R(0)=\pi$ and $R(\pi)=0$. At intermediate distances a variational interpolation for the trial $(\hat{\Delta}^{(1)},\hat{\Delta}^{(2)},\hat{l})$ gives

$$\hat{u}=(\hat{u}_0\cdot\hat{n})\hat{n}(1-\cos\beta)+\hat{u}_0\cos\beta+\hat{n}\times\hat{u}_0\sin\beta\ , \tag{8}$$

where $\hat{u}=(\hat{\Delta}^{(1)},\hat{\Delta}^{(2)},\hat{l})$ when $\hat{u}_0=(\hat{z},\hat{\phi},-\hat{\rho})$, and the angle $\beta(\rho,\phi)$ increases from $\beta(r_0,\phi)=0$ to $\beta(r_2,\phi)=R(\phi)$. We shall call the vortex in Eqs. (7) and (8) the asymmetric vortex (again $r_2\sim L_D$ and $r_0\sim\xi$).

To describe the z-in-x and the asymmetric vortexes completely, we need to describe the order parameter at distances less than or equal to r_0. According to Muzikar[5] and Fetter *et al.*,[6] the free energy of singular vortices is lowest for polar-phase core. An order parameter that interpolates between the radial disgyration at $\rho=r_0$ and the polar phase at $\rho=0$ is given by

$$\vec{\psi}=\sqrt{2}(\hat{z}\cos\theta+i\hat{\phi}\sin\theta)\ , \tag{9}$$

where the radial angle θ increases from $\theta(0)=0$ to $\theta(r_0)=\pi/4$. Since the hydrodynamic expression for the general free-energy density of a factorized order parameter is not known, the free energy of the polar core is calculated from the weak coupling GL expression (measured from the weak coupling GL free-energy density of the uniform dipole-locked A-phase with $\hat{d}\perp\vec{H}$). When adding the polar core free energy and the A-phase free energy of the region (r_0,L), we further assume that the dimensionless parameters used in the polar core calculation have their GL values at all temperatures below T_c. It turns out that the polar core energy is small (about 3%) compared to the total energy of a texture, which means that the above rough calculation of the polar core free energy will not lead to large errors in the total free energy of the texture.

To calculate the exact free energies of the z-in-x vortex

TABLE I. Free energy F of different vortices, per quantum of circulation and in units of $\pi\rho_{\parallel}$, for cell sizes $L/L_D=10$, 20, 50, and 100. Reduced temperature $T/T_c=0.99$, 0.90, 0.80, and 0.70 in the hydrodynamic approach, and $T/T_c=0.90$ when the BCS parameter values are used. The core radii r_2, r_1, and r_0 (in units of L_D or ξ), the angular velocity Ω, and the dipolar length L_D are also given.

	SV singular			SV analytic		z-in-x				Asymmetric			
L/L_D	F	r_2/L_D	r_0/ξ	F	r_2/L_D	F	r_2/L_D	r_1/L_D	r_0/ξ	F	r_2/L_D	r_0/ξ	Ω (rad/sec)
						$T/T_c=0.99$ ($L_D=5.96$ μm)							
100	10.74	1.73	2.80	15.11	3.46	13.97	1.56	0.49	2.42	11.57	1.89	2.44	0.03
50	9.70	1.72	2.80	13.03	3.46	12.93	1.56	0.49	2.42	10.53	1.88	2.44	0.12
20	8.33	1.72	2.80	10.29	3.45	11.55	1.56	0.49	2.42	9.18	1.85	2.44	0.74
10	7.29	1.71	2.80	8.23	3.43	10.51	1.56	0.49	2.42	8.20	1.74	2.43	2.96
						$T/T_c=0.90$ ($L_D=6.95$ μm)							
100	11.30	1.76	2.77	15.23	3.51	14.40	1.54	0.51	2.44	11.90	1.92	2.39	0.02
50	10.26	1.76	2.77	13.10	3.51	13.36	1.54	0.51	2.44	10.86	1.91	2.39	0.09
20	8.88	1.76	2.77	10.28	3.50	11.98	1.54	0.51	2.44	9.51	1.88	2.39	0.54
10	7.85	1.75	2.77	8.18	3.47	10.94	1.55	0.51	2.43	8.54	1.76	2.39	2.18
						$T/T_c=0.80$ ($L_D=8.10$ μm)							
100	11.34	1.80	2.67	15.40	3.56	14.40	1.56	0.55	2.26	11.81	1.97	2.22	0.02
50	10.30	1.80	2.67	13.20	3.56	13.36	1.56	0.55	2.26	10.77	1.96	2.22	0.06
20	8.92	1.80	2.67	10.31	3.55	11.99	1.56	0.56	2.26	9.42	1.93	2.22	0.40
10	7.89	1.79	2.67	8.14	3.52	10.94	1.57	0.56	2.26	8.45	1.80	2.22	1.60
						$T/T_c=0.70$ ($L_D=9.21$ μm)							
100	11.26	1.84	2.56	15.61	3.61	14.30	1.58	0.60	2.06	11.61	2.02	2.02	0.01
50	10.22	1.84	2.56	13.35	3.61	13.26	1.58	0.60	2.06	10.57	2.02	2.02	0.05
20	8.85	1.84	2.56	10.36	3.61	11.88	1.58	0.60	2.06	9.22	1.98	2.02	0.31
10	7.81	1.84	2.56	8.12	3.57	10.84	1.59	0.61	2.06	8.26	1.85	2.02	1.24
						BCS ($T/T_c=0.90$) ($L_D=6.00$ μm)							
20	9.02	1.71	2.87	10.29	3.45	12.10	1.52	0.47	2.58	9.72	1.82	2.55	0.73

[Eqs. (5), (6), and (9)] and the asymmetric vortex [Eqs. (7)–(9)], one would have to solve the Euler-Lagrange equations for the unknown functions $\beta(\rho)$, $\eta(\rho)$, and $\theta(\rho)$, and $\beta(\rho,\phi)$ and $\theta(\rho)$, respectively. However, a good approximation to the free energy can be found by assuming that each of the above functions increases linearly with the radial coordinate (the "stick approximation"), and then minimizing with respect to the variational parameters r_2, r_1, and r_0 for the z-in-x, and r_2 and r_0 for the asymmetric vortex. The integrals and the minimization procedure were done numerically for several reduced temperatures ($T/T_c=0.7$, 0.8, 0.9, and 0.99) and cell sizes ($L/L_D=10$, 20, 50, and 100), which correspond to low angular velocities. Hydrodynamic parameters for the above temperatures have been calculated by Williams[4] and have also been tabulated in Fetter *et al.*[6]

A free-energy calculation has also been done for the two SV vortices, and the results are summarized in Table I (the free energies are given per quantum of circulation). Comparing our variational free energies for the SV vortices at $T/T_c=0.90$ and $L/L_D=20$, calculated using BCS (weak coupling) values of the hydrodynamic parameters, with the exact values obtained by SV in the weak coupling GL region, we find that the stick approximation tends to overestimate the free energies by about 5%.

From Table I it follows that the SV singular vortex has the lowest effective free energy at low angular velocity and high axial magnetic fields. The asymmetric vortex has a slightly higher free energy due to the larger bending energy coming from the greater nonuniformity in the transition region between the $\hat{l}=\hat{x}$ order parameter and the corresponding disgyration: In the SV singular vortex one goes to a pure disgyration by rotating by $\pi/2$ about $\hat{\Delta}^{(2)}(\phi)$ for every angle ϕ, while in the asymmetric vortex one goes to the radial disgyration by rotating by $R(\phi)$ around in $\hat{n}(\phi)$, where $R(\phi)$ varies from π at $\phi=0$ to 0 at $\phi=\pi$. The SV analytic vortex lies even higher in energy than the asymmetric vortex except at lower temperatures and relatively high angular velocities. The z-in-x vortex is always more energetic than the asymmetric vortex.

Provided that there are no lower free-energy textures, the above considerations suggest that the SV singular vortex should be observed in the experiments. In order to check this prediction, we calculate frequencies and relative intensities of the satellite lines in the NMR experiments. So far only the transverse NMR has been measured, but we hope that our calculation of both NMR signatures will stimulate longitudinal NMR experiments as a possible means of deciding which texture is actually present.

The longitudinal $\omega_{\parallel}$ and the transverse $\omega_{\perp}$ NMR resonant frequencies are customarily parametrized in the form

$$\omega_{\parallel}^2=R_{\parallel}^2\omega_A^2\ ,\quad \omega_{\perp}^2=\omega_L^2+R_{\perp}^2\omega_A^2\ , \tag{10}$$

where $\omega_L=\gamma H$ is the Larmor frequency and $\omega_A=(\gamma/L_D)(\rho_{\parallel}/\chi_n)^{1/2}$ is the anisotropy or Leggett frequency. For uniform ^{3}He-A, the resonant frequencies occur at $R_{\perp}=1$ and $R_{\parallel}=1$. If there are regions with nonuniform order parameters, one also observes satellite lines (localized spin waves) with smaller intensities and with frequencies which lie below the main resonant lines.

The NMR equations can be obtained from the Leggett equations generalized to nonuniform situations.[7] Leggett's Hamiltonian density is

$$h=\tfrac{1}{2}\gamma^2\vec{S}\tilde{\chi}^{-1}\vec{S}-\gamma\vec{S}\cdot\vec{H}-\tfrac{1}{2}\rho_{\parallel}\frac{1}{L_D^2}(\hat{l}\cdot\hat{d})^2$$
$$+\tfrac{1}{2}K_5(\hat{l}\cdot\vec{\partial}d_\mu)^2+\tfrac{1}{2}K_6(\hat{l}\times\vec{\partial}d_\mu)^2\ , \tag{11}$$

where

$$\hat{\chi}^{-1}=\left[\tilde{1}+\frac{\lambda_m}{\chi_n-\lambda_m}\hat{d}\,\hat{d}\right]\Big/\chi_n$$

is the inverse magnetic susceptibility, $\lambda_m=\lambda_D/H_D^2=\rho_{\parallel}/(L_DH_D)^2$, χ_n is the susceptibility of normal ^{3}He, and $\gamma\vec{S}$ is the magnetization density. The equilibrium conditions are

$$0=\hat{d}\times\frac{\delta h}{\delta\hat{d}}=\hat{d}\times\left[\lambda_m(\hat{d}\cdot\vec{H})\vec{H}+\frac{\delta f}{\delta\hat{d}}\right]\ , \tag{12a}$$

$$0=\frac{\delta h}{\delta\vec{S}}=\gamma^2\tilde{\chi}^{-1}\vec{S}-\gamma\vec{H}\ , \tag{12b}$$

and (using the Poisson bracket relations between $\hat{d}$ and $\vec{S}$) the equations of motion are given by

$$\partial_t\vec{S}=\gamma\vec{S}\times\vec{H}-\hat{d}\times\frac{\delta f}{\delta\hat{d}}\ , \tag{13a}$$

$$\partial_t\hat{d}=\gamma\hat{d}\times[(\vec{H}-(\gamma/\chi_n)\vec{S}]\ , \tag{13b}$$

where $\delta f/\delta\vec{u}=\partial f/\partial\vec{u}-\partial_k[\partial f/\partial(\partial_k\vec{u})]$, and f is the sum of the last three terms in Eq. (11). Assuming that in the equilibrium the strong static magnetic field $\vec{H}_0$ lies along the z axis and the $\hat{d}$ vector along the x axis, and linearizing with respect to the small time-dependent components of the magnetic field $\vec{H}'e^{-i\omega t}$, the $\hat{d}$ vector $\vec{d}'e^{-i\omega t}$, and the magnetization density $\gamma\vec{S}'e^{-i\omega t}$, we find

$$\begin{bmatrix}\omega_A^2(l_x^2-l_y^2-P)-\omega^2 & \omega_A^2l_yl_z\\ \omega_A^2l_yl_z & \omega_L^2+\omega_A^2(l_x^2-l_z^2-P)-\omega^2\end{bmatrix}\begin{bmatrix}a'\\ b'\end{bmatrix}=\begin{bmatrix}i\omega\gamma H_z'\\ i\omega\gamma H_y'+\omega_L\gamma H_x'\end{bmatrix}\ , \tag{14}$$

where

$$P=L_D^2[\bar{K}_6\vec{\partial}^2+(\bar{K}_5-\bar{K}_6)(\vec{\partial}\cdot\hat{l})(\hat{l}\cdot\vec{\partial})]\ ,$$

$\bar{K}_5=K_5/\rho_{\parallel}$, $\bar{K}_6=K_6/\rho_{\parallel}$, and $d_z'=-b'$, $d_y'=a'$. To obtain Eq. (14), we eliminated the time-dependent component of the magnetization density in favor of the corresponding components of the d vector and the magnetic field:

$$\gamma S_x'=\chi_nH_0b'\ , \tag{15a}$$

$$\gamma S_y'=\chi_n[H_y'-i(\omega/\gamma)b']\ , \tag{15b}$$

$$\gamma S_z' = \chi_n [H_z' - i(\omega/\gamma) a'] \ . \tag{15c}$$

The last equation shows that oscillations of $\hat{d}$ in the xy plane correspond to the longitudinal resonance, while oscillations along the static magnetic field correspond to the transverse resonance. The NMR equations (14) and (15), also can be obtained using the Euler-angle approach of Maki and collaborators.[8]

From the NMR equation for $\vec{d}\,'$, we see that the longitudinal and the transverse response are coupled (unless $l_y l_z \approx 0$), so that strictly speaking there are no purely longitudinal and transverse eigenstates. It has been shown, however, that for $\omega_L \gg \omega_A$ (which is satisfied here), the correction terms for the longitudinal resonances of frequency approximately equal to ω_A and transverse resonances of frequency approximately equal to ω_L are of order $(\omega_A/\omega_L)^2$, and are therefore negligible.[9] In that case, the NMR matrix equation decouples, and one can solve for the longitudinal and transverse cases separately.

To find the lowest-lying resonance frequency we can use the variational principle[2] which gives

$$R_*^2 = \min \frac{\int d^2r [(l_x^2 - l_*^2) |g|^2 + \bar{K}_6 |(\vec{\partial} g)|^2 + (\bar{K}_5 - \bar{K}_6) |\vec{1} \cdot \vec{\partial} g|^2]}{\int d^2 r |g|^2} , \tag{16}$$

where $R_* = R_\perp$, $l_* = l_z$, and $g = b'$, and $R_* = R_{||}$, $l_* = l_y$, and $g = a'$ for the transverse and the longitudinal case, respectively, and (hereafter) we measure distances in units of the dipolar length. The form of the trial function g used to minimize R_* in Eq. (16) is

$$g = (1-w) g_0(x) + e^{i\theta} w \, g_c(x) \cos\phi \ , \tag{17a}$$

$$g_0 = (1 + a_0 x + b_0 x^2 + c_0 x^3) e^{-\sigma_0 x} \ , \tag{17b}$$

$$g_c = x^q (1 + a_1 x + b_1 x^2) e^{-\sigma_c x} \ , \tag{17c}$$

where w, $\sigma_0 = a_0$, σ_c, b_0, c_0, a_1, and b_1 are the variational parameters, and $x = \rho/L_D$. The weight w is constrained by $0 \le w \le 1$ and a_0 and σ_c must be positive. Examining the eigenvalue equation corresponding to Eq. (16) for distances much smaller than the dipolar length, but much larger than the core size, we find that $q = 1$ for the SV vortices and $q = \sqrt{2}$ for the z-in-x and asymmetric vortices. We have included the $\cos\phi$ term in Eq. (17) to check if the ϕ dependence of the $\hat{l}$-vector field is important (the $\sin\phi$ term is absent since we expect the ground state to be even).

The absorption can be calculated from

$$\frac{dU}{dt} = -\tfrac{1}{2} \int d^3 r \, \mathrm{Re}[\gamma \vec{S}^*(\vec{r},t) \cdot \partial_t \vec{H}(\vec{r},t)] \tag{18}$$

using Eq. (15). It follows that the relative absorption Q, defined as the intensity of a satellite line divided by the intensity of the corresponding main line, is given by

$$Q_{j\perp} = \frac{\left| \int d^3 r \, b_j'(r) \right|^2}{\left[\int d^3 r \right]} , \quad Q_{j_{||}} = \left| \frac{\omega_j}{\omega_A} \right|^3 \frac{\left| \int d^3 r \, a_j'(r) \right|^2}{\left[\int d^3 r \right]} \tag{19}$$

TABLE II. Transverse (tran) and longitudinal (long) values for R^2 and Q calculated for different reduced temperatures $T/T_c = 0.99$, 0.90, 0.80, and 0.70 in the hydrodynamic approach, and for $T/T_c = 0.90$ when the BCS values are used. Relative intensities Q are for the angular velocity Ω corresponding to $L = 20 L_D$. The experimental values for R_1^2 were taken from Hakonen *et al.* (Ref. 1).

	SV singular		SV analytic		z-in-x		Asymmetric		Experiments
	R^2	Q (%)	R^2	Q (%)	R^2	Q (%)	R^2	Q (%)	R^2
$T/T_c = 0.99$									
Tran	0.97	29	0.42	2.4	0.87	12	0.95	23	0.72
Long	0.97	28	0.42	0.67	0.99	82	0.96	21	
$T/T_c = 0.90$									
Tran	0.95	17	0.36	2.2	0.81	8.1	0.92	14	0.56
Long	0.95	16	0.36	0.49	0.98	36	0.93	13	
$T/T_c = 0.80$									
Tran	0.91	11	0.30	2.1	0.73	5.3	0.86	8.6	0.41
Long	0.91	9.1	0.30	0.35	0.96	19	0.87	6.5	
$T/T_c = 0.70$									
Tran	0.85	6.7	0.24	2.0	0.62	3.6	0.79	5.3	0.28
Long	0.85	5.2	0.24	0.23	0.92	9.9	0.81	4.1	
BCS									
Tran	0.96	29	0.42	2.4	0.88	12	0.96	23	
Long	0.96	28	0.42	0.67	0.99	88	0.96	23	

for the transverse and longitudinal case, respectively. To obtain Eq. (19) we have used the fact that for transverse resonances $\omega_j \cong \omega_L \gg \omega_A$, normalized all wave functions by $\int d^3r\,|a'|^2 = \int d^3r\,|b'|^2 = 1$, and noted that a' and b' for the main lines are approximately uniform in space.

Using the parameters r_2, r_1, and r_0, which are given in Table I for the different textures, we can find the parameter R^2 and the relative intensity Q of the lowest-lying NMR satellite from Eqs. (16), (17), and (19). Since the NMR equations for the general case of the factorized phase are not known, the radial integrals in Eqs. (16) and (19) do not include the polar core region of the singular vortices. We expect that the exact treatment of the polar core regions would give corrections of order $(\xi/L_D)^2$.

From Table I it follows that the radii for r_0, r_1, and r_2 have very weak Ω dependences. The form of the satellite wave function and the parameter R will then be roughly Ω independent, while the relative absorption Q (being proportional to the number of localized spin waves) will scale linearly with Ω. Our results for R and Q for the two SV vortices, the z-in-x, and the asymmetric vortex are listed in Table II. The experimental values for $R_\perp$ were obtained using the formula $R_\perp = 0.86 - 1.1(1 - T/T_c)$ which summarizes the NMR results of Hakonen *et al.*[1]

According to Hakonen *et al.*,[1] the relative absorption is given by $Q_\perp = 0.058\Omega$ (in rad/sec) $= 1.5/L_D^2$ [in $(\mu\text{m})^{-2}$] using $L/L_D = 20$ and the relation between the angular velocity and the WS cell size $\Omega = \hbar/2m_3L^2$. Hakonen *et al.* do not specify at which temperature this value for $Q_\perp/\Omega$ holds, but since precise measurements are very difficult for $T/T_c > 0.8$, we assume that the above formula for $Q_\perp$ holds for $T/T_c < 0.8$. Then, using $L_D = 8.5$ μm, we obtain $Q_\perp = 2.1\%$, which can be compared with theoretical values for different vortices listed in Table II.

From Table II it seems that none of the investigated vortices explains the data, although values of $R_\perp^2$ and $Q_\perp$ for the SV analytic vortex are somewhat closer to the experimental values than those for the singular vortices. This may mean that some other texture with lower free energy can explain the observed NMR, or, assuming that the SV analytic vortex is present, that the free-energy calculations are off by about 10%. More accurate determination of the polar core free energy and small changes in the values of the hydrodynamic parameters could lower the free energy of the SV analytic vortex relative to singular vortices.

We propose measurements of the longitudinal NMR response, $R_{\parallel}^2$ and $Q_{\parallel}$, and the temperature dependence of $Q_\perp$ as a way to decide which of these vortices (if any) is seen in the experiment. Table II shows that SV vortices have $R_\perp^2$ very nearly equal to $R_{\parallel}^2$, while the z-in-x and the asymmetric vortex has $R_\perp^2$ (considerably) smaller than $R_{\parallel}^2$. Also, the SV analytic vortex has small and nearly temperature-independent $Q_\perp$, in contrast to singular vortices. It is also interesting to notice that all vortices except the asymmetric vortex have an optimum trial wave function with the parameter $w = 0$, which means negligible ϕ dependence. For the asymmetric vortex, it is essential to have $w \neq 0$ in order to obtain the lowest R^2; the typical value of the ratio between $\int d^2x\,|wg_c\cos\phi|^2$ and $\int d^2x\,|(1-w)g_0|^2$ is 0.1.

We conclude by noting that 4π vortices seem necessary to obtain the large frequency shifts measured by Hakonen *et al.*[1] These shifts come only from wave functions that are localized well within the core, where the effective potential $(l_x^2 - l_*^2 - 1)$ in Eq. (16) is attractive. If the core is small, however, then the gradient terms in Eq. (16) hinder the localization of a spin wave. The 4π vortices have larger cores than the 2π vortices (since their cell size is $\sqrt{2}$ larger than the cell size of a 2π vortex), and the wave function can be localized in the core with relatively less kinetic energy. We therefore expect a frequency shift for a typical 4π vortex to be larger than that for a typical 2π vortex.

ACKNOWLEDGMENTS

We thank J. A. Sauls and P. Kumar for many helpful discussions. This work was partially supported by National Science Foundation Grant Nos. DMR 802063 (V.Z.V. and D.L.S.) and DMR 81-18386 (A.L.F.).

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