Formalism for the quantum Hall effect: Hilbert space of analytic functions

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We develop a general formulation of quantum mechanics within the lowest Landau level in two dimensions. Making use of Bargmann's Hilbert space of analytic functions we obtain a simple algorithm for the projection of any quantum operator onto the subspace of the lowest Landau level. With this scheme we obtain the Schrödinger equation in both real-space and coherent-state representations. A Gaussian interaction among the particles leads to a particularly simple form in which the eigenvalue condition reduces to a purely algebraic property of the polynomial wave function. Finally, we formulate path integration within the lowest Landau level using the coherent-state representation. The techniques developed here should prove to be convenient for the study of the anomalous quantum Hall effect and other phenomena involving electron-electron interactions.

I. INTRODUCTION

Recent experiments which have studied the quantum Hall effect[1–3] and the anomalous quantum Hall effect[4–6] have pointed up the need for a concise theoretical formulation of the two-dimensional Landau-level problem including electron-electron interactions. The remarkable phenomena associated with the anomalous quantum Hall effect are possible because of two strong constraints on the system. The very high magnetic field dominates the physics, and, as has been discussed previously,[7] severely restricts the dynamics of the particle motion. Low temperatures limit the electronic inversion layer to occupancy of the lowest spin state of the lowest Landau level. Under these conditions a commensuration energy exists which lowers the ground-state energy and introduces an excitation gap whenever the Landau-level filling is a simple rational fraction of the form p/q with q odd.[6] The origin of this commensuration effect is currently the object of intense experimental and theoretical interest.[7–15]

We wish to take advantage of the constraints on the dynamics mentioned above to present a general formalism for doing quantum mechanics within the lowest Landau level in two dimensions. The formalism we have developed is quite simple, easy to implement, and provides a useful language with which to discuss the physics of Landau levels in two dimensions. This formalism should find application in both analytic and numerical studies of the interacting electron problem.

The paper is organized as follows. Section II introduces a Hilbert space of analytic functions which spans the states of interest and with which the Schrödinger equation is very conveniently projected onto the lowest Landau level. Section III discusses the formal evaluation of the partition function. Section IV introduces coherent states and a formulation of path integrals for the lowest Landau level. A summary is presented in Sec. V. The formalism developed here is illustrated with simple analytical examples. Applications of these techniques will be presented in a sequel to the present paper.

II. HILBERT SPACE OF ANALYTIC FUNCTIONS

Consider a two-dimensional electron gas lying in the x-y plane and subject to a perpendicular magnetic field $\vec{B}=B\hat{z}$. The Hamiltonian is taken to be

$$H=H_0+V\,, \tag{1}$$

$$H_0=\sum_i \frac{1}{2m}\left[\vec{p}_i+\frac{e}{c}\vec{A}_i\right]^2\,, \tag{2}$$

$$V=\sum_{i<j} v(\vec{r}_i-\vec{r}_j)\,, \tag{3}$$

where V is the Coulomb interaction or some other model interaction among the particles. The eigenvalues of the kinetic energy lie in discrete, highly degenerate Landau levels uniformly spaced in energy by $\hbar\omega_c$ where ω_c is the classical cyclotron frequency. We will assume B is sufficiently large that the magnetic energy greatly exceeds characteristic thermal and potential energies, so that mixing of Landau levels can be neglected. This (by now standard) assumption is not necessarily strictly valid for real systems, but will presumably lead only to quantitative errors not qualitative changes in the physics. Restriction of the electrons to the lowest spin state of the lowest Landau level yields a considerable advantage since the wave functions for these states have a simple analytic form. Our central purpose in this section is to develop a systematic formalism for doing quantum mechanics in the lowest Landau level.

We begin with the kinetic energy. In the symmetric gauge with vector potential $\vec{A}_i=-\frac{1}{2}\vec{r}_i\times\vec{B}$ the lowest Landau level eigenfunctions of H_0 have the form[8] (in units where $l^{-2}=eB/\hbar c=1$)

$$\psi[z]=f[z]\exp\left[-\tfrac{1}{4}\sum_i |z_i|^2\right]\,, \tag{4}$$

where $[z]$ means $(z_1,z_2,\ldots,z_N)$ and f is a polynomial in the N variables $z_k\equiv x_k-iy_k$. The exponential factor in

(4) will be common to all wave functions, and the manipulations we wish to carry out are simplified considerably if this common factor can be removed. We do this now by formally defining a Hilbert space of analytic functions following Bargmann.[16]

Consider the set of entire functions of N complex variables,

$$\Theta \equiv \{f\} . \tag{5}$$

These functions are analytic in each of their arguments everywhere in the complex plane. Thus for $N=1$, for example, the function defined by

$$f(z)=z^3$$

is an element of Θ, but the function defined by

$$f(z)=z^*$$

is *not* analytic (since z^* cannot be expressed as a power series in z) and is thus excluded from Θ. This is a crucial point to which we shall return later.

Define an inner product on Θ via

$$(f,g)=\int d\mu[z] f^*[z] g[z] , \tag{6}$$

where the measure is

$$d\mu[z]=\prod_{i=1}^{N} \frac{1}{2\pi} e^{-|z_i|^2/2} dx_i dy_i . \tag{7}$$

We restrict Θ to include only those functions with finite norm $(f,f)<\infty$.

The Hilbert space thus defined is realized by the wave functions of the lowest Landau level since these may always be written as in Eq. (4) with f being a member of Θ. Furthermore, the inner product on Θ has been defined in such a way that wave function overlaps are given by

$$\langle \psi' | \psi \rangle = (f',f) . \tag{8}$$

The primary advantage of defining the Hilbert space so that we can work with f instead of ψ in (4) is that f is analytic while ψ is not.

We now investigate what linear operators can be defined on Θ, focusing on the case $N=1$ to obtain some useful results which are easily generalized to arbitrary N. Consider

$$g=\hat{O} f , \tag{9}$$

where $\hat{O}$ is a linear operator and f and g are in Θ. The requirement of analyticity severely restricts the form of $\hat{O}$. There are only three fundamental operations allowed: (1) multiplication by a complex constant, (2) multiplication by a power of z (but not z^*), and (3) differentiation with respect to z. Any linear operator on Θ can be expressed in terms of these fundamental operations.

In order to study these fundamental operators it is useful to define orthonormal basis functions by

$$f_n(z)=\frac{z^n}{(2^n n!)^{1/2}} . \tag{10}$$

These have the property

$$z f_m = \sqrt{2}\sqrt{m+1} f_{m+1} , \tag{11}$$

$$\frac{d}{dz} f_m = \sqrt{m/2} f_{m-1} . \tag{12}$$

Hence,

$$a^\dagger \equiv z/\sqrt{2} , \tag{13}$$

$$a \equiv \sqrt{2}\frac{d}{dz} \tag{14}$$

are boson ladder operators[7] and are easily seen to be mutually adjoint with respect to the inner product defined on Θ. Our present discussion of these operators in terms of the Bargmann space Θ makes formal Laughlin's[14] procedure of having d/dz not apply to the exponential part of the wave function in Eq. (4).

Having obtained these fundamental operators we now consider how to project the Hamiltonian onto the lowest Landau level by expressing it in terms of these operators. Because the lowest-Landau-level eigenfunctions are all degenerate, the kinetic energy commutes with any operator that has been projected on to that level. The kinetic energy is thus a constant which can be ignored. The Schrödinger equation becomes simply

$$\hat{V}\psi[z]=E\psi[z] , \tag{15}$$

where $\hat{V}$ is the projection of the potential operator onto the lowest Landau level. Returning to the case of N particles let us assume a central two-body potential which may be expanded in the form

$$V=\sum_{i<j}^{N}\sum_{n=0}^{\infty} \gamma_n (\vec{r}_{ij}\cdot\vec{r}_{ij})^n ,$$

which may be rewritten

$$V[z^*,z]=\sum_{i<j}^{N}\sum_{n=0}^{\infty} \gamma_n (z_i^*-z_j^*)^n (z_i-z_j)^n . \tag{16}$$

We are now faced with the problem that z^* takes states out of the Hilbert space (it mixes Landau levels). We need to project z^* onto our fundamental operators. Consider the matrix element

$$\alpha_{nmk}=(f_n, z_k^* f_m) . \tag{17}$$

Although $z_k^* f_m$ is outside the Hilbert space, α_{nmk} is perfectly well defined. Indeed from the definition of the inner product it is clear that

$$(f_n, z_k^* f_m)=(z_k f_n, f_m) \tag{18}$$

since z_k^* is the Hermitian conjugate of z_k. However, using (13) and (14) we have that the adjoint of z_k is

$$z_k^\dagger = 2\frac{\partial}{\partial z_k} , \tag{19}$$

so that (18) becomes

$$(f_n, z_k^* f_m)=\left[f_n, 2\frac{\partial}{\partial z_k} f_m \right] . \tag{20}$$

Thus z_k^* and $2\partial/\partial z_k$ have the same matrix elements within the space Θ. Despite this, the two operators are

not completely equivalent since z_k^* commutes with z_k and $\partial/\partial z_k$ does not. For example, we have

$$(f,z_k z_k^* g)=(f,z_k^* z_k g)\ , \tag{21}$$

but

$$\left[f,z_k 2\frac{\partial}{\partial z_k}g\right]\neq\left[f,2\frac{\partial}{\partial z_k}z_k g\right]\ . \tag{22}$$

Only the right-hand side of (22) is in agreement with (21), showing that occasionally it is important even for physicists to distinguish between the Hermitian conjugate and the adjoint.

The message in (21) and (22) is that z^* makes sense only when operating to the left. Hence, in order to project the potential energy onto the lowest Landau level one simply uses

$$V[z^*,z]\rightarrow\hat{N}V\left[2\frac{\partial}{\partial z},z\right]\ , \tag{23}$$

where $\hat{N}$ is a normal ordering operator that keeps all the derivatives on the left. Note that if one has the product of two operators each of which is separately projected onto the lowest Landau level one has

$$A[z^*,z]B[z^*,z]\rightarrow\left[\hat{N}A\left[2\frac{\partial}{\partial z},z\right]\right]\left[\hat{N}B\left[2\frac{\partial}{\partial z},z\right]\right]\ , \tag{24}$$

whereas if only the product is to be projected one has

$$A[z^*,z]B[z^*,z]\rightarrow\hat{N}\left[A\left[2\frac{\partial}{\partial z},z\right]B\left[2\frac{\partial}{\partial z},z\right]\right]\ , \tag{25}$$

Using these rules we obtain our central result: the projection of the Schrödinger equation onto the lowest Landau level,

$$\left[\hat{N}V\left[2\frac{\partial}{\partial z},z\right]\right]\psi[z]=E\psi[z]\ , \tag{26}$$

where $V[z^*,z]$ is the classical potential.

We now turn to some illustrative examples using (26). Consider the case of harmonic interaction among the particles,

$$V[z^*,z]=\tfrac{1}{2}\lambda^2\sum_{i<j}^{N}(z_i^*-z_j^*)(z_i-z_j)\ . \tag{27}$$

The Schrödinger equation becomes

$$\lambda^2\sum_{i<j}^{N}\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right](z_i-z_j)\psi[z]=E\psi[z]\ . \tag{28}$$

Let us choose for ψ (the polynomial part of) Laughlin's wave function[9]

$$\psi_m[z]=\prod_{k<l}^{N}(z_k-z_l)^m\ , \tag{29}$$

where m is an odd integer. Separating terms involving only z_i or z_j individually yields

$$V\psi_m=\lambda^2(m+1)N(N-1)\psi_m + \lambda^2\sum_{i<j}^{N}\sum_{k\neq i,j}^{N}m\left[\frac{z_i-z_j}{z_i-z_k}-\frac{z_i-z_j}{z_j-z_k}\right]\psi_m\ . \tag{30}$$

The summation in (30) may be evaluated by choosing any three electrons (e.g., 1,2,3), and noting

$$\sum_{i<j}^{3}\sum_{k\neq i,j}^{3}\left[\frac{z_i-z_j}{z_i-z_k}-\frac{z_i-z_j}{z_j-z_k}\right]=3\ . \tag{31}$$

There are

$$\begin{bmatrix}N\\3\end{bmatrix}$$

distinct ways of choosing these three electrons so that (30) becomes

$$V\psi_m=N(N-1)(1+mN/2)\psi_m\ , \tag{32}$$

and as previously found by other means,[7] Laughlin's wave function is an exact eigenfunction of the harmonic interaction.

As a second example, we consider how one can represent the $1/r^2$ potential,

$$V[z^*,z]=\sum_{i<j}^{N}\frac{1}{(z_i^*-z_j^*)(z_i-z_j)}\ . \tag{33}$$

We can rewrite this by means of the following integral representation:

$$V[z^*,z]=\sum_{i<j}^{N}\int_0^\infty d\lambda\, e^{-\lambda(z_i^*-z_j^*)(z_i-z_j)}\ . \tag{34}$$

Projection onto the lowest Landau level yields

$$V\left[2\frac{\partial}{\partial z},z\right]=\sum_{i<j}^{N}\hat{N}\int_0^\infty d\lambda\exp\left[-2\lambda\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right](z_i-z_j)\right]\ . \tag{35}$$

Similarly, the Coulomb potential may be expressed as a Gaussian integral,

$$V\left[2\frac{\partial}{\partial z},z\right]=\sum_{i<j}^{N}\frac{1}{\sqrt{\pi}}\hat{N}\int_{-\infty}^{\infty}d\lambda\exp\left[-2\lambda^2\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right](z_i-z_j)\right]\ . \tag{36}$$

These results are not very useful as they stand because of the presence of the normal ordering operator. Fortunately, this difficulty can be bypassed as we now demonstrate. Consider the operator (which contains no normal ordering)

$$V=\sum_{i<j}^{N} v_{ij} , \tag{37}$$

$$v_{ij}=\int_0^\infty d\lambda\, g(\lambda)\exp\left[-\tfrac{1}{2}\lambda(z_i-z_j)\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right]\right] , \tag{38}$$

where g is an arbitrary, real-valued function. To see the effect of V we note that any wave function may always be written (for a given i and j)

$$\phi[z]=\sum_{n,m=0}^{\infty} a_{nm}(z_i+z_j)^n(z_i-z_j)^m , \tag{39}$$

where a_{nm} depends only on coordinates other than z_i and z_j and where m is odd. Applying (38) yields

$$v_{ij}\phi[z]=\sum_{n,m=0}^{\infty}\int_0^\infty d\lambda\, g(\lambda)e^{-\lambda m}a_{nm}(z_i+z_j)^n(z_i-z_j)^m . \tag{40}$$

Suppose the actual potential obeys

$$v_{ij}(z_i+z_j)^n(z_i-z_j)^m=\epsilon(m)(z_i+z_j)^n(z_i-z_j)^m . \tag{41}$$

If we regard ϵ as a function of a continuous variable m then we can always achieve (41) by choosing g in (40) to be the inverse Laplace transform of ϵ. Thus it is possible to avoid the normal ordering problem. For example, the choice $g(\lambda)=\frac{1}{4}$ reproduces the matrix elements of $1/r_{ij}^2$ exactly. The choice $g(\lambda)=\delta(\lambda-\lambda_0)$ corresponds to a Gaussian interaction. With the use of (37) and (38) the Schrödinger equation becomes

$$\int_0^\infty d\lambda\, g(\lambda)\sum_{i<j}^{N}\exp\left[-\tfrac{1}{2}\lambda(z_i-z_j)\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right]\right]\psi[z]=E\psi[z] . \tag{42}$$

For the remainder of the discussion we shall specialize to the case of a Gaussian interaction which is obtained by simply dropping the coupling constant integration in (42).

We see from (42) that the Schrödinger equation contains an infinite number of derivatives, but fortunately we can take advantage of the fact that $\exp(d/dz)$ is a displacement operator and $\exp(zd/dz)$ is a dilation operator. Defining

$$z_i'=z_i+Q(z_i-z_j) , \tag{43}$$

$$z_j'=z_j-Q(z_i-z_j) , \tag{44}$$

where

$$Q=(e^{-\lambda}-1)/2 , \tag{45}$$

we see that Eq. (40) may be rewritten (dropping the coupling constant integration)

$$v_{ij}\phi[z]=\sum_{n,m=0}^{\infty} a_{nm}(z_i'+z_j')^n(z_i'-z_j')^m . \tag{46}$$

Hence, the Schrödinger equation becomes

$$\sum_{i<j}^{N}\phi(z_1,z_2,\ldots,z_i',\ldots,z_j',\ldots,z_N)=E\phi[z] . \tag{47}$$

We have thus reduced the Schrödinger equation to a purely algebraic statement about the polynomial ϕ.

We can gain further insight into the meaning of (47) if we think of it as a statement about the dilation symmetry of ϕ. Note that the effect of the potential v_{ij} on the left-hand side of (47) is to compress the distance between particles i and j by a factor $1+2Q=\exp(-\lambda)$. The multiplication by E on the right-hand side of (47) can also be related to a dilation in the following way. The wave function ϕ may always be taken to be an eigenfunction of the total angular momentum. However, ϕ has angular momentum L if and only if ϕ is homogeneous of degree L. Hence,

$$E\phi[z]=\phi[Kz] , \tag{48}$$

where $K=E^{1/L}$. Thus, the right-hand side of (47) corresponds to a global dilation (or contraction). The Schrödinger eigenvalue condition is simply that the sum of local contractions of individual bonds generated by the potential is equivalent to a single global contraction produced by the eigenvalue.

It is useful to consider the Gaussian interaction for various limiting values of the coupling constant λ. For infinitesimal λ we obtain the harmonic interaction discussed earlier. In the opposite limit $\lambda\to\infty$ we have $Q=-\frac{1}{2}$ so that [as can be seen from (43) and (44)] the effect of the potential is to bring the particles very close together. This is consistent with the short-range nature of the potential in this limit. If we allow for the possibility of complex coupling constants we see that for $\lambda=i\pi$, $Q=-1$ and $z_i'=z_j$, and $z_j'=z_i$. Hence,

$$\Xi_{ij}=\exp\left[(i\pi/2)(z_i-z_j)\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right]\right] \tag{49}$$

is an explicit representation of the particle exchange operator. Since ϕ must be totally antisymmetric we may deduce that the energy eigenvalue obeys

$$E(\lambda\pm i\pi)=-E(\lambda)\ . \tag{50}$$

Hence, E must have the form

$$E(\lambda)=\sum_{m\ \text{odd}}^{\infty}\alpha_m e^{-\lambda m}\ , \tag{51}$$

a fact which could also be deduced from (40).

In summary we have, by making use of Bargmann's Hilbert space of analytic functions, derived a simple formalism for projecting the Schrödinger equation onto the lowest Landau level. We have shown how to do this for arbitrary forms of central interactions among the particles. We found that the case of a Gaussian interaction leads to a particularly simple form in which the Schrödinger eigenvalue condition reduces to a purely algebraic property of the eigenfunction polynomial. Finally, we note that this last result suggests that it may be possible to attack the problem of solving the many-body wave equation by abstract group-theoretic methods. Of particular interest in this regard is the connection made by Bargmann[16] between polynomials in two variables, SU(2) and the rotation group. It may be possible to extend these ideas to polynomials in many variables.

III. PARTITION FUNCTION

Having established how to project the Hamiltonian onto the lowest Landau level we are in a position to discuss the formal evaluation of the partition function

$$Z\equiv\mathrm{Tr}\,e^{-\beta(H_0+V)}\ . \tag{52}$$

We begin by considering the recent work of Tosatti and Parrinello[15] (TP) which we believe treats the projection onto the lowest Landau level incorrectly. TP approximate (52) by

$$Z=\mathrm{Tr}\,e^{-\beta H_0}e^{-\beta V}\ , \tag{53}$$

and then restrict the trace to the lowest Landau level. The use of (53) ignores the fact that H_0 and V do not commute. We have been able to show that the use of this approximation destroys any possibility of observing a commensuration energy. We will demonstrate this by reviewing the derivation of TP and then extending it to obtain some new results. Following this we will present what we believe is the correct formulation of the partition function using the projection technique developed in the preceding section.

Assuming (53) to be valid we may neglect the kinetic energy since it is a constant for the lowest Landau level. Taking the trace in (53) in a coordinate representation yields

$$Z=\prod_{j=1}^{N}\int d^2z_j e^{-\beta V[z^*,z]}P[z]\ , \tag{54}$$

where $d^2z_j\equiv dx_j dy_j$ and

$$P[z]\equiv\sum_S|\psi_S[z]|^2\ , \tag{55}$$

where the sum is over all Slater determinants ψ_S of N electrons in the lowest Landau level [note the ψ_S used here is the actual wave function including the exponential factor as in (4)]. Equation (54) resembles the classical partition function modified by a quantum correction P. TP argue that P contains the commensuration energy. We will now extend the analysis of TP by explicitly performing the summation in (55) to show that P does not produce a significant commensuration energy.

The Sth wave function is

$$\psi_S[z]=\frac{1}{\sqrt{N!}}\mathrm{Det}M_S\ , \tag{56}$$

where M_S is an $N\times N$ matrix with

$$(M_S)_{ij}=\phi_{Si}(z_j)\ , \tag{57}$$

and ϕ_{Si} is the ith orbital in the Sth configuration. Thus (55) becomes

$$P[z]=\frac{1}{N!}\sum_S(\mathrm{Det}M_S^\dagger)(\mathrm{Det}M_S)\ . \tag{58}$$

However, from the Cauchy-Binet theorem[17] we have

$$P[z]=\frac{1}{N!}\mathrm{Det}(L^\dagger L)\ , \tag{59}$$

where L is a rectangular matrix given by

$$L_{kj}=\phi_k(z_j)\ , \tag{60}$$

with j running from 1 to N, and k running over *all* orbitals (occupied or empty). Setting (60) into (59) gives

$$P[z]=\frac{1}{N!}\mathrm{Det}G[z]\ , \tag{61}$$

where G is the matrix defined by

$$G_{jj'}=\sum_k\phi_k^*(z_j)\phi_k(z_{j'})\ . \tag{62}$$

We see that $G_{jj'}$ is simply the one-body Green's function for propagation from z_j to $z_{j'}$. In the symmetric gauge,

$$G_{jj'}=\frac{1}{2\pi}e^{-|z_j-z_{j'}|^2/4}e^{-(i/2)(x_jy_{j'}-y_jx_{j'})}\ , \tag{63}$$

so that G is a Gaussian with a magnetic flux phase factor. The partition function may now be written

$$Z=\frac{1}{N!}\prod_{j=1}^{N}\int d^2z_j e^{-\beta V[z^*,z]}\mathrm{Det}G[z]\ . \tag{64}$$

Because of the strong Gaussian falloff of $G_{jj'}$, it is useful to separate G into a diagonal and an off-diagonal part,

$$G_{jj'}=\frac{1}{2\pi}(\delta_{jj'}+F_{jj'})\ , \tag{65}$$

where $F_{ii}=0$. We now use $\mathrm{Det}(G)=\exp(\mathrm{Tr}\ln G)$ and expand

$$Z=\frac{1}{N!}\prod_{j=1}^{N}\int\frac{d^2z_j}{2\pi}e^{-\beta V[z^*,z]}e^{\mathrm{Tr}[-(1/2)F^2+(1/3)F^3-\cdots]}. \tag{66}$$

The first term in the expansion may be viewed as a correction to the classical potential which introduces an effective interaction

$$\beta V_2=\sum_{i<j}^{N}e^{-|z_i-z_j|^2/2}. \tag{67}$$

This short-range repulsion represents the lowest-order exchange interaction. The next terms in the expansion represent cyclic three-particle exchange, etc.

We seek in this expansion a source of the commensuration energy which is observed experimentally to lower the free energy whenever the Landau-level filling is a simple rational fraction.[6] The first such term in (66) is the cyclic three-particle exchange. The three particles i, j, and k form a triangle of area A_{ijk} containing a certain magnetic flux Φ which controls the phase of the three-particle exchange term

$$\tfrac{1}{3}\sum_{i,j,k}e^{-(|z_i-z_k|^2+|z_k-z_j|^2+|z_j-z_i|^2)/4}\tfrac{1}{2}\cos(A_{ijk}). \tag{68}$$

The oscillating area term does produce a commensuration energy, but unfortunately the Gaussian falloff is quite severe. For a filling factor ν the quantity in (68) is of order $\exp(-\sqrt{3}\pi/\nu)$ which is approximately 10^{-7} for $\nu=\frac{1}{3}$. The higher-order contributions are even smaller. Thus, this is much too small to explain the experimentally observed commensuration energy.[4–6]

The origin of this difficulty is the approximation made by TP in using Eq. (53). We will now remedy this by going back to (52) and first projecting the Hamiltonian onto the lowest Landau level and likewise restricting the trace. One then has

$$Z=\mathrm{Tr}\,e^{-\beta(\tilde{H}_0+\tilde{V})}. \tag{69}$$

The kinetic energy $\tilde{H}_0$ is a constant for the lowest Landau level and, because the projection has already been performed, now commutes with the potential energy $\tilde{V}$.

The corrected version of (54) becomes [using (23)]

$$Z=\frac{1}{N!}\int d\mu[z]\sum_{S}\Phi_S^*[z]\exp\left[-\beta V\left[2\frac{\partial}{\partial z},z\right]\right]\Phi_S[z], \tag{70}$$

where Φ_S is the polynomial part of ψ_S defined in (56). It is assumed that V has been expressed in the manner described previously so that the normal ordering problem has been eliminated.

We would like to perform the summation in (70) to obtain the corrected version of Eq. (64). This may be done as follows. We rewrite (70) as

$$Z=\frac{1}{N!}\prod_{j=1}^{N}\int d^2z_j\sum_{S}\left[\Phi_S^*\exp\left[-\tfrac{1}{4}\sum_i|z_i|^2\right]\right]\exp\left[-\tfrac{1}{4}\sum_i|z_i|^2\right]\exp\left[-\beta V\left[2\frac{\partial}{\partial z},z\right]\right]\Phi_S. \tag{71}$$

We may move the second Gaussian weight factor to the right provided we make the transformation

$$2\frac{\partial}{\partial z}\to 2\frac{\partial}{\partial z}+\tfrac{1}{2}z^*, \tag{72}$$

and make the rule

$$\frac{\partial}{\partial z}z^*=0. \tag{73}$$

This leaves

$$Z=\frac{1}{N!}\sum_{j=1}^{N}\int d^2z_j\sum_{S}\psi_S^*[z]\exp\left[-\beta V\left[2\frac{\partial}{\partial z}+\tfrac{1}{2}z^*,z\right]\right]\psi_S[z], \tag{74}$$

$$Z=\lim_{[z']\to[z]}\frac{1}{N!}\prod_{j=1}^{N}\int d^2z_j\exp\left[-\beta V\left[2\frac{\partial}{\partial z}+\tfrac{1}{2}z^*,z\right]\right]\mathrm{Det}G[z',z], \tag{75}$$

where $G[z',z]$ is an $N\times N$ matrix elements given by the one-body propagator,

$$G_{kj}[z',z]=\frac{1}{2\pi}e^{-|z_k'-z_i|^2/4}e^{(z_k'^*z_i-z_k'z_i^*)/4}. \tag{76}$$

This formal expression for the partition function should prove to be a useful starting point for evaluation by a variety of numerical and analytic techniques, both perturbative and nonperturbative. The expression we have obtained is necessarily more complicated than that used by TP, but we have demonstrated that the latter does not yield a commensuration energy. In the next section we

will discuss path-integral techniques for the evaluation of this more complicated expression for the partition function.

IV. COHERENT STATE REPRESENTATION AND PATH INTEGRATION

Consider the analytic function defined by

$$\phi_\xi[z]=\exp\left[\tfrac{1}{2}\sum_i \xi_i^* z_i\right] . \tag{77}$$

Recalling Eq. (13), which shows that z_i is effectively a harmonic-oscillator raising operator, we see that ϕ_ξ is nothing more than a coherent state of the oscillator.[18] This is an eigenfunction of the lowering operator,

$$a_i\phi_\xi=\sqrt{2}\frac{\partial}{\partial z_i}\phi_\xi=\frac{1}{\sqrt{2}}\xi_i^*\phi_\xi . \tag{78}$$

One can verify, in a straightforward manner, that ϕ_ξ has the norm

$$(\phi_\xi,\phi_\xi)=\exp\left[\tfrac{1}{2}\sum_i |\xi_i|^2\right] , \tag{79}$$

and corresponds to a Gaussian wave packet centered at point $[\xi]$ (when the exponential factor is restored to the wave function). These coherent states have several possible uses which we now explore.

There is currently a very active search underway for analytic wave functions to describe the ground state of interacting Landau-level electrons and to explain the energy excitation gap required by the existence of the anomalous quantum Hall effect.[9,11,12,14] One potentially useful approach to this problem is to take advantage of an integral representation of the wave function through the following identity:[16]

$$\psi[z]=\int d\mu[\xi]\psi[\xi]\phi_\xi[z] , \tag{80}$$

where the measure is defined in Eq. (7). As Bargmann points out, the meaning of this identity is that ϕ_ξ is analogous to the Dirac δ function. Consistent with this is the fact that ϕ_ξ is the most hightly localized wave packet that can be constructed within the lowest Landau level.

As an example of the use of Eq. (80), Laughlin's wave function given by Eq. (29) may be written

$$\psi_m[z]=\int d\mu[\xi]\prod_{i<j}(\xi_i-\xi_j)^m\exp\left[\tfrac{1}{2}\sum_k \xi_k^* z_k\right] . \tag{81}$$

With this representation it is quite easy to see how to include additional correlations in Laughlin's wave function by simply writing

$$\Psi[z]=\int d\mu[\xi]\prod_{i<j}(\xi_i-\xi_j)^m f(|\xi_i-\xi_j|)\exp\left[\tfrac{1}{2}\sum_k \xi_k^* z_k\right] . \tag{82}$$

The form of f determines the additional correlations. For example,

$$f=|\xi_i-\xi_j|^p$$

or

$$f=1-e^{-p|\xi_i-\xi_j|^2}$$

both discourage close encounters of the particles. By using the first form of f, the integral in (82) may be explicitly carried out to yield

$$\Psi[z]=\prod_{i<j}^{N}\left[\frac{\partial}{\partial z_i}-\frac{\partial}{\partial z_j}\right]^p\prod_{k<l}^{N}(z_k-z_l)^{m+p} .$$

This seems likely to have a more favorable ground-state energy than Laughlin's wave function.

One of the attractive features of this representation is that since f is real, the total angular momentum, and hence the particle density, is independent of the form of f. This is thus a very convenient way in which to include variational freedom in the wave function while keeping the density automatically constrained. We note that Eq. (82) could also be easily generalized to include additional three-body and higher correlations as variational degrees of freedom.

A further advantage of the representation given in (80) is the complete factorization of the z dependence [as can be seen in the specific examples (81) and (82)]. We can take advantage of this factorization to cast the Schrödinger equation for the Gaussian interaction into a new form. Setting (80) into (47) yields

$$\int d\mu[\xi]\psi[\xi]\exp\left[\tfrac{1}{2}\sum_i \xi_i^* z_i\right]\sum_{k<l} e^{(Q/2)(\xi_k^*-\xi_l^*)(z_k-z_l)} =E\psi[z] , \tag{83}$$

or, equivalently,

$$\int d\mu[\xi]\psi[\xi]\exp\left[\tfrac{1}{2}\sum_i \xi_i^* z_i\right]\left[E-\sum_{k<l} e^{(Q/2)(\xi_k^*-\xi_l^*)(z_k-z_l)}\right] =0 . \tag{84}$$

This is simply the integral form of the Schrödinger equation conjugate to the differential form obtained previously.

Another important use of the coherent state representation is in the development of a path-integration scheme projected onto the lowest Landau level. To verify that the resolution of the identity within the coherent representation is[18]

$$I=\int d\mu[\xi]\,|\phi_\xi\rangle\langle\phi_\xi| ,$$

is a straightforward procedure. Specializing to the case $N=1$, the single-particle propagator may be rewritten using

$$G(z_f,t;z_i)=A^{-1}(\phi_{zf},e^{-iHt}\phi_{zi}) \tag{85a}$$

$$=\prod_{j=1}^{n+1}\int d\mu(\xi_j)(\phi_{\xi j},e^{-iH\epsilon}\phi_{\xi j-1})\ , \tag{85b}$$

where

$$A\equiv(\phi_{zi},\phi_{zi})^{1/2}(\phi_{zf},\phi_{zf})^{1/2}\ ,$$

and $\epsilon\equiv t/(n+1)$, $\xi_0\equiv z_i$, and $\xi_{n+1}\equiv z_f$. Following Schulman[18] a series of standard manipulations leads to the path integral

$$G=\int D\xi\, e^{-iS(t)}\ , \tag{86}$$

where $\int D\xi$ means

$$\lim_{n\to\infty}\prod_{i=1}^{n}\int\frac{dx_i dy_i}{2\pi}\ ,$$

and the action S is given by

$$S(t)\equiv\int_0^t d\tau\left[\frac{1}{4i}\left[\xi^*\frac{d\xi}{d\tau}-\xi\frac{d\xi^*}{d\tau}\right]-H(\xi^*,\xi)\right]\ , \tag{87}$$

with

$$H(\alpha^*,\beta)\equiv\frac{(\phi_\alpha,H\phi_\beta)}{(\phi_\alpha,\phi_\beta)}\ . \tag{88}$$

Since the coherent states are all degenerate eigenstates of the kinetic energy (which is therefore being neglected), Eq. (88) is readily evaluated

$$H(\xi^*,\xi)=\int\frac{d^2z}{2\pi}V(z^*,z)e^{-|z-\xi|^2/2}\ , \tag{89}$$

where V is the classical potential. This is simply the expectation value of the potential energy for a Gaussian wave packet centered at $z=\xi$.

From the action in (87) we may find the variational equations for the extremal path

$$\frac{d\xi}{d\tau}=2i\frac{\partial H(\xi^*,\xi)}{\partial\xi^*}\ , \tag{90}$$

$$\frac{d\xi^*}{d\tau}=-2i\frac{\partial H(\xi^*,\xi)}{\partial\xi}\ . \tag{91}$$

Since H is real these are consistent. Returning to the original coordinates x and y via $\xi=x-iy$ yields

$$\dot{x}=\frac{\partial H}{\partial y}\ , \tag{92}$$

$$\dot{y}=-\frac{\partial H}{\partial x}\ . \tag{93}$$

These are simply the classical $\vec{E}\times\vec{B}$ drift equations.

We notice that the action in (87) is peculiar in that it is linear in the time derivatives. This violation of time reversal symmetry is, of course, due to the presence of the magnetic field. The form of the action suggests that we do not have an ordinary path integral, but rather something similar to a phase-space path integral.[18] Indeed if we treat ξ as a canonical coordinate and identify

$$\Pi\equiv\frac{-i}{2}\xi^* \tag{94}$$

as the canonically conjugate momentum, the path integral (86) assumes the phase-space form

$$G=\int D\xi\, D\Pi\, e^{iS}\ , \tag{95}$$

where the action is now

$$S=\int_0^t d\tau\,\tfrac{1}{2}(\Pi\dot{\xi}-\dot{\xi}\Pi)-H(\Pi,\xi)\ . \tag{96}$$

The variational equations for the extremal path now become

$$\dot{\xi}=\frac{\partial H}{\partial\Pi}\ , \tag{97}$$

$$\dot{\Pi}=-\frac{\partial H}{\partial\xi}\ , \tag{98}$$

which are nothing more than the usual Hamilton's equations.

There is a very nice connection between this and our previous results for the projection of the Schrödinger equation onto the lowest Landau level. To see this we consider the canonical quantization procedure for the classical theory represented by Eqs. (97) and (98). In a coordinate representation we quantize the classical theory by replacing the momentum by the operator

$$\Pi_{\rm op}=-i\frac{\partial}{\partial\xi}\ . \tag{99}$$

From Eq. (94) we see that this requires the substitution

$$\xi^*\to 2\frac{\partial}{\partial\xi}\ . \tag{100}$$

This is, however, precisely the rule given previously in Eq. (23) for projecting the quantum Hamiltonian onto the lowest Landau level. The fact that the path-integration scheme we have found is analogous to a phase-space path integral suggests that caution is required in making use of this scheme. Schulman[18] points out that phase-space path integrals are notoriously ill behaved. The problem here is that when we project the free propagator onto the lowest Landau level it becomes time independent. Thus, even for infinitesimal times the particle can propagate a finite length (on the order of the magnetic length). Formal manipulations of the path-integral expression must therefore be treated with care. Klauder has recently considered these questions in connection with the quantum Hall effect.[19]

One standard manipulation is to expand the path integral in fluctuations about the extremal path obtained in Eqs. (90) and (91). This procedure fails because of the pathological nature of the paths.[18] It is possible, however, to avoid this difficulty, and we plan to present a rigorous derivation of the propagator in the semiclassical limit elsewhere.[20]

V. SUMMARY

We have presented a general formulation of quantum mechanics within the lowest Landau level in two dimensions. This scheme involves study of the Hilbert space of functions analytic in the complex coordinate $z=x-iy$. The quantity z^* is related to the conjugate momentum and the simple replacement $z^* \rightarrow 2\,d/dz$ converts any classical quantity $f(z^*,z)$ into the associated quantum operator f_{op} properly projected onto the lowest Landau level. Within this formalism we have obtained expressions for the Schrödinger equation in both the real-space and coherent state representations. We have shown that a Gaussian interaction between the particles leads to a particularly simple form for the Schrödinger equation with the eigenvalue condition being reduced to a purely algebraic property of the polynomial wave function. We have also formulated path integration within the lowest Landau level by making use of the coherent state representation. Numerical application of the formalism presented here is currently in progress.

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