Some properties of the spectrum of the Sierpinski gasket in a magnetic field

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The spectrum of the Sierpinski gasket in a magnetic field is discussed using a synthetic Green's-function technique. This directly relates the spectrum of an $(n+1)$-stage gasket to that of its n-stage components and allows effective use of the implicit symmetry. It is found that the $(n+1)$-stage spectrum is nested with three eigenvalues belonging to the three different representations between any two consecutive stage-n eigenvalues. For the special points where the eigenvalues for stage n and $n+1$ coincide we provide proofs for the two Rammal-Toulouse [Phys. Rev. Lett. 49, 1194 (1982)] nesting properties, derive explicit expressions for the evolution of the degeneracies, and construct the eigenfunctions. Some of the implications and remaining problems are also discussed.

I. INTRODUCTION

A considerable number of recent investigations deal with the solutions and spectrum of linear difference equations on the Sierpinski gasket.[1–8] The motivation for this interest is largely the belief that it will lead to insight into the properties of random fractals such as percolation clusters.[1–3,5] There is also, however, an intrinsic interest in studying a problem which has no translational symmetry and is nevertheless exactly solvable, at least in principle, because of its dilation symmetry. In Ref. 3 we derived recursion relations for the linearized Landau-Ginzburg equations in a magnetic field, which are also applicable to the free-particle Schrödinger equations, by using an exact decimation technique. When there is no external field the equations simplify considerably. The spectrum and eigenfunctions were studied in detail by Domany *et al.*[6] and, using somewhat different decimation techniques, by Rammal[7] and by Tremblay and Southern.[8] The spectrum is only defined on a Cantor set with very high degeneracies (of the order of volume). The eigenfunctions are also localized as can be seen explicitly[6] or from the fact that a finite fraction of the eigenstates ($\sim\frac{1}{3}$) disappears at each stage of the scale transformation and cannot be associated with eigenstates on the dilated gasket.

Rammal and Toulouse[4] have studied the spectrum in a magnetic field using the recursion relations derived in Ref. 3. They found some remarkable symmetries and nesting properties of the spectrum. The decimation technique is extremely cumbersome, however, in this case, because both the eigenvalues and the distribution of magnetic fluxes are renormalized. One does not obtain any simple relationship between the spectra of gaskets of different size. As a result, Rammal and Toulouse[4] were not even able to prove all the properties they observed and missed others.

The hierarchical structure of the gasket was used in a different way in determining the scaling behavior of the conductance[1] and in solving the London equations for superconductivity.[2,5] Instead of using a successive decimation technique on a large gasket, these authors use the reverse procedure and consider the formation of a large gasket by combining smaller ones. This allows one to use the properties of the smaller gaskets explicitly. In this paper we apply the same technique to a study of the spectrum of the Schrödinger equation in a magnetic field. We use the Green's functions and eigenfunctions of the n-stage problem to construct solutions on the larger $(n+1)$-stage gasket. Since this only requires matching at three common boundary points, the resulting algorithm is relatively simple. It also allows one to utilize the implicit symmetries. The gasket (in a field) only has an explicit threefold rotational symmetry. It has additional symmetry properties because each of its component gaskets had the same symmetry originally.

Our main result is a new general nesting property. We show that between any two consecutive eigenvalues of the n-stage gasket there are three eigenvalues on the $(n+1)$-stage gasket belonging to the three irreducible representations.

We also obtain simple proofs for the two nesting properties of Rammal and Toulouse[4] concerning eigenvalues of an n-stage gasket which remain in the spectrum for all larger gaskets. We construct the eigenfunctions and, when the eigenvalues are degenerate, determine the evolution of the degeneracy.

II. ITERATION PROCEDURE

A. Formulation of the problem

As shown in Ref. 3, the solutions of the free-particle Schrödinger equation

$$(i\vec{\nabla}-\vec{A}/\Phi_0)^2\phi=q^2\phi \tag{1}$$

on a net composed of thin wires of equal length a lead to the difference equations

$$\sum_j (t\phi_i-\eta_{ij}\phi_j)=0 \tag{2}$$

for the amplitudes of ϕ at the vertices of the net (ϕ_i,ϕ_j). The summation in Eq. (2) is over the connected neighbors of vertex i, and the eigenvalue of the difference equations (2) is related to that of the Schrödinger equation through

$$t=\cos qa \tag{3}$$

and

$$\eta^{ij}=\exp(-i\gamma_{ij}),\quad \gamma_{ij}=\left[\frac{2\pi}{\Phi_0}\right]\int_i^j \vec{A}\cdot d\vec{l}\ , \tag{4}$$

where $\Phi_0=(\hbar c/e)$ is the flux quantum. The γ_{ij} must obey the (gauge-invariant) loop condition

$$\sum_{\text{around loops } l}\phi_{ij}=2\pi\Phi_l/\Phi_0=\gamma_l\ , \tag{5a}$$

where Φ_l is the flux through the loop l encircled by the segments ij. It will be convenient to supplement this by the network analog of a London gauge:

$$\sum_j \gamma_{ij}=0\ . \tag{5b}$$

This is formally equivalent to a Kirchoff law.[2,5] Equations (5) determine the γ_{ij} uniquely.

We want to solve Eqs. (2) on a (two-dimensional) Sierpinski gasket in constant external magnetic field such that

$$\gamma_0=\frac{\pi\sqrt{3}a^2H}{2\Phi_0} \tag{6}$$

for an elementary triangle.

Assume we know the solutions of Eqs. (2) and (5) on an n-stage gasket, i.e., the solution of:

$$(t\hat{Z}^n-\hat{A}^n)\,|\,\alpha\rangle_n=0\ , \tag{7}$$

where $\hat{Z}_n$ is a diagonal matrix whose elements Z_i^n are the numbers of connected neighbors of site i [see Eq. (1)]. $\hat{A}^n$ is a Hermitian matrix with elements η_{ij}^n [Eq. (4)] determined by solving Eqs. (5) on the n-stage gasket. Equations (7) have

$$P_n=3\frac{3^n+1}{2} \tag{8}$$

independent solutions $|\,\alpha\rangle_n$ with eigenvalues $t=\alpha_n$. Because of the somewhat unusual form of Eqs. (7) (the Z_i^n are not all equal) we replace the standard orthonormality conditions by

$$\langle\alpha\,|\,\hat{Z}^n\,|\,\beta\rangle=\delta(\alpha\beta)\ . \tag{9}$$

The gasket as a whole only has a point symmetry (C_3). Thus the eigenfunctions can be classified according to the irreducible representations of this group. We write

$$\zeta^\nu=e^{2\pi\nu i/3},\quad \nu=0,\ \pm1 \tag{10}$$

and use ν as an index for the representations. For special values of the field

$$\gamma_0=\pi m \tag{11}$$

(m being an integer), two of the representations always become degenerate.

We want to combine three n-stage gaskets to form an

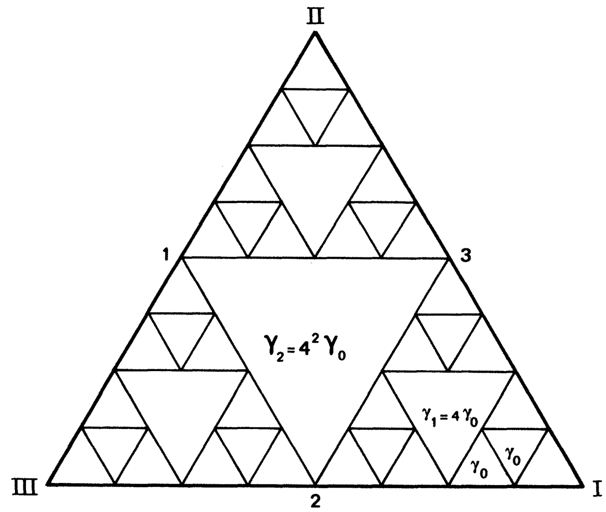

FIG. 1. The $(n=3)$ Sierpinski gasket. The notation for the matching vertices (1,2,3) and the external vertices (I, II, III) is shown.

$(n+1)$-stage gasket and use the known (n-stage) solutions to construct the $(n+1)$-stage solutions. There are three effects.

(a) The phase factors (γ_{ij}) are modified. Equations (5) for each of the three separate (n-stage) gaskets must be supplemented by the loop condition [Eq. (5a)] for the new central hole for which (Fig. 1)

$$\gamma_n=4^n\gamma_0\ . \tag{12}$$

(b) Equations (2) for the three common vertices (1,2,3; see Fig. 1) are modified because each of these vertices now has connected neighbors on two adjacent n-stage gaskets.

(c) The amplitudes at these vertices ($|\,\alpha 1\rangle$, $|\,\alpha 2\rangle$) are unique and each shows up in the equations originating in two separate (n-stage) gaskets.

B. The phase factors

Consider first the modification of the phase factors. We write

$$\gamma_{ij}^{n+1}=\gamma_{ij}^n+\delta_{ij}^{n+1}\ , \tag{13}$$

where the γ_{ij}^n are the solutions of Eqs. (5) on the separated n-stage gasket. Thus the δ_{ij}^{n+1} obey

$$\sum_{\text{around loop}}\delta_{ij}^{n+1}=0 \tag{14a}$$

for all loops belonging to the separate n-stage gaskets. Also[6]

$$\sum_{\text{around loop}}\delta_{ij}^{n+1}=2\gamma_n=2.4^n\gamma_0 \tag{14b}$$

for paths enclosing the new central loop generated by the iteration. Finally one has the Kirchoff equations (5b) for all vertices. One notes that, because of Eqs. (14a) the δ_{ij} are irrotational on each of the n-stage gaskets. They are in fact equivalent to the currents induced by imposing a voltage[2]

$$v = 2.4^n \frac{\gamma_0}{3} \quad (15)$$

between the two connected external vertices (e.g., 2 and 3 for gasket I). Thus on each separate (n-stage) gasket (M) one can define site voltages so that

$$v_i^M - v_j^M = \delta_{ij}^{n+1}, \quad M = \mathrm{I},\mathrm{II},\mathrm{III} . \quad (16)$$

Clearly

$$v_i^{\mathrm{III}} - v_2^{\mathrm{III}} = v_2^{\mathrm{I}} - v_3^{\mathrm{I}} = v_3^{\mathrm{II}} - v_1^{\mathrm{II}} = v \quad (17)$$

but in general $v_i^M \neq v_i^{M'}$.

C. Formal solution

We can now write Eqs. (2) for the $(n+1)$-stage gasket as three equations:

$$(t\hat{Z}^n - e^{-iV^M}\hat{A}^n e^{iV^M}) \mid \alpha^M \rangle_{n+1} = \mid X^M \rangle , \quad (18)$$

where V^M is a diagonal matrix with elements v_i^M [Eq. (16)] and $\hat{Z}^n$ and $\hat{A}^n$ are defined in Eq. (7). The inhomogeneous term $\mid X^M \rangle$ on the right-hand side of (18) results from the fact that [matching condition (b) above] the equations for the common vertices ($m = 1,2,3$) are modified. We write

$$X_1^{\mathrm{III}} = -X_1^{\mathrm{II}} = X_1,$$
$$X_2^{\mathrm{I}} = -X_2^{\mathrm{III}} = X_2, \quad (19)$$
$$X_3^{\mathrm{II}} = -X_3^{\mathrm{I}} = X_3 .$$

All other X_i^M vanish. The explicit expression for the X_i in terms of the site amplitudes ($\mid \alpha i \rangle$ is obvious from Eq. (2). Solution of Eqs. (18) with the subsidiary condition (c) that the amplitudes at the common vertices, each of which shows up twice, must be unique—is equivalent to a solution on the $(n+1)$-stage gasket.

We thus want to invert the matrice on the right-hand side of Eq. (18). Consider first the inversion of the n-stage matrix [Eq. (7)],

$$G_{ij}^n = (t\hat{Z}^n - \hat{A}^n)_{ij}^{-1} = \sum_\alpha \frac{\mid \alpha i \rangle_n \langle \alpha j \mid_n}{t - \alpha} \quad (20)$$

and, including the δ_{ij} phase factor [Eqs. (16) and (18)]:

$$(\tilde{G}_{ij}^n)^M = (t\hat{Z}^n - e^{-iV^M}\hat{A}^n e^{iV^M})^{-1} = G_{ij}^n e^{-i\delta_{ij}} . \quad (21)$$

To solve Eqs. (18) we only need the elements of G for the external vertices

$$G_{MM}^n = \sum_\mu \Psi_\mu^n, \quad G_{MM+1}^n = \sum_\mu \Psi_\mu^n , \quad (22)$$

where

$$\Psi_\mu^n = \sum_{\alpha \in \mu} \frac{\mid \alpha, M, \mu \rangle_n \mid^2}{t - \alpha_n} \quad (23)$$

is the part of G_{MM}^n associated with eigenfunctions α belonging to the representation μ. Solving Eqs. (18) for the common vertices ($i = 1,2,3$) gives

$$\phi_i = G_{MM}^n X_i - (\lambda^+)^2 G_{MM+1}^n X_{i+1}$$
$$= -G_{MM}^n X_i + \lambda^2 (G_{MM+1}^n)^+ X_{i-1}, \quad i = 1,2,3 \quad (24)$$

where

$$\lambda = \exp\left[-i4^n \frac{\gamma_0}{3}\right] . \quad (25)$$

Finally, eliminating the ϕ_i from Eqs. (24), one finds

$$\hat{\Gamma} \mid X \rangle = 0 , \quad (26)$$

where

$$\hat{\Gamma} = \left\| \begin{matrix} 2G_{MM}^n & -(\lambda^+)^2 G_{MM+1}^n & -\lambda^2 G_{MM+}^{n+} \\ -\lambda^2 G_{MM+1}^n & 2G_{MM}^n & -(\lambda^+)^2 G_{MM+1}^{n+} \\ -(\lambda^+)^2 G_{MM+1}^n & -\lambda^2 G_{MM+1}^{n+} & 2G_{MM}^n \end{matrix} \right\| . \quad (27)$$

This has two types of solutions: (a),

$$X_i \equiv 0, \quad i = 1,2,3 . \quad (28)$$

The eigenvalues, and eigenfunctions, on the $(n+1)$-stage gasket are then automatically also solutions on the n-stage gaskets separately [Eq. (18)]. The second type of solution is (b),

$$\sum_{\mu = 0 \pm 1} \sin^2\beta_n(\nu + \mu)\Psi_\mu^n = 0, \quad \nu = 0, \pm 1 \quad (29)$$

where

$$\beta_n(l) = (4^n\gamma_0 - \pi l)/3 , \quad (30)$$

and ν is the representation to which the solution belongs. One notes that the three equations (29) only differ in a permutation of the (non-negative) coefficients of the Ψ_μ^n.

For completeness we also give the recursion relations for the Ψ_μ^n,

$$\Psi_\nu^{n+1} = \frac{\sum_{\mu < \mu'} \cos^2\beta_n \left[\nu + \frac{\mu + \mu'}{2}\right] \Psi_\mu^n \Psi_{\mu'}^n}{\sum_\mu \sin^2\beta_n(\nu + \mu)\Psi_\mu^n} . \quad (31)$$

The derivation of this expression is straightforward. For special values of the field, when solutions of Eq. (28) appear, Eq. (31) does not describe Ψ_ν^{n+1} completely.

III. PROPERTIES OF THE SPECTRUM

A. The general case

For most values of the external field (i.e., γ_0) there are P_n distinct eigenvalues on the n-stage gasket and, moreover, none of the three coefficients $\sin^2\beta(l)$ vanishes. Thus each of the three Eqs. (29) has poles at all the n-stage eigenvalues (α_n), and therefore one solution α_{n+1} between any two consecutive α_n. This leads to an interesting new nesting property for the spectrum.

Nesting property III: There are three eigenvalues α_{n+1} on the $(n+1)$-stage gasket, one for each irreducible representation ν, between any two consecutive eigenvalues of the n-stage gasket.

We have retained the notation of Rammal and Toulouse[4] for the nesting properties. We note that nesting property I of this reference (not proven there) is in fact a corollary and follows from continuity (see Sec. III D below).

Rammal and Toulouse[4] also found that all eigenvalues changed sign under the transformation

$$\gamma_0 \to \pi - \gamma_0 \to \alpha_n \to -\alpha_n \ . \tag{32}$$

It is easy to see that this holds on the zero-stage gasket (a triangle). The coefficients $\sin^2\beta_n(l)$ in Eq. (29) are invariant under this transformation except for a permutation of indices. Thus this symmetry follows trivially.

In this general case each of Eqs. (29) has $P_n - 1$ distinct solutions, all different from the α_n. Thus one obtains all $(P_{n+1} = 3P_n - 3)$ solutions on the $n+1$ stage from Eqs. (29).

One also notes that the only degeneracies which can occur in the spectrum are accidental. Two α_ν belonging to different representations (ν) can coincide. [A threefold degeneracy can only occur if $\Psi_\nu^n(t;\gamma) \equiv 0$.]

Exceptions to this general situation occur when, for some reason, Eqs. (29) have less than $3(P_n - 1)$ solutions. This can only happen if there are less than $3P_n$ distinct poles in these equations because of degeneracies (in the n-stage spectrum) or because the relevant residues vanish.

It follows from our analysis in Sec. II that all eigenvalues obey Eq. (26). Thus if one misses solutions in Eqs. (29) they must show up as solutions of Eq. (28). There are then eigenvalues on the $(n+1)$-stage gasket which also belong to the n-stage spectrum for the same field $[\alpha_{n+1}(\gamma) = \alpha_n(\gamma)]$. The two nesting properties of Ref. 4 describe situations of this type. We consider the different possibilities separately.

B. Special values of the field

Consider fields such that

$$\gamma_n = 4^n \gamma_0 = m\pi \ . \tag{33}$$

One of the three coefficients $\sin^2\beta_n(l)$ in Eqs. (29) will then vanish. Thus one misses P_n solutions. They can be constructed from the solutions on the n-stage gasket.

Let α_n be a nondegenerate eigenvalue belonging to representation μ. On the $(n+1)$-stage gasket one thus has

$$|\alpha,i\rangle = \zeta^\mu(\lambda^+)^2 |\alpha,i+1\rangle, \quad i=1,2,3 \tag{34a}$$

[from Eq. (18)], for the amplitudes at the common vertices. To form a consistent solution at stage $n+1$, one also requires

$$|\alpha,i\rangle = \zeta^{-\nu} |\alpha,i+1\rangle \ . \tag{34b}$$

When γ_0 obeys Eq. (33) this will determine the new representation ν. We have thus constructed an eigenfunction for the $(n+1)$-stage gasket belonging to the eigenvalue α_n.

It is also straightforward to see that

$$\alpha_{n+1}(\gamma) = \alpha_n(\gamma) + (\gamma - \gamma_c)^2 F[(\gamma-\gamma_c)^2] \ , \tag{35}$$

where γ_c is a special value obeying Eq. (32). One way of seeing this is to expand the $\delta_{ij}^{(n+1)}$, but not the γ_{ij}^n, in $\gamma - \gamma_c$ and then consider the resulting perturbation expansion for the $(n+1)$-stage gasket. To zero order in $\gamma - \gamma_c$ one obtains $\alpha_{n+1}(\gamma) = \alpha_n(\gamma)$. Since all odd powers of $\gamma - \gamma_c$ in the perturbation are pure imaginary, the form (35) follows. Alternatively one could note that the residue at $\alpha_n(\gamma)$ in Eq. (29) is proportional to $(\gamma-\gamma_c)^2$, which leads to the same result.

We have thus proven *nesting property II:* the lines $\alpha_n(\gamma)$ and $\alpha_{n+1}(\gamma)$ are tangential at γ_c.

This property of the spectrum was discovered by Rammal and Toulouse.[4] They state that they were able to prove it by using the decimation technique but the proof is complicated. The present derivation seems straightforward. We have also constructed the eigenfunctions.

C. Vanishing residues

Consider situations where, for some eigenvalue α_n the amplitude at the external vertices ($|\alpha,M\rangle_n$) vanishes:

$$|\alpha(\bar\gamma),M\rangle_n = 0 \ . \tag{36}$$

The solutions on the three n-stage gaskets are then independent. If one had a p-fold degeneracy at stage n the $(n+1)$-stage degeneracy for the same α is $3p$.

One also notes that the residue at the pole near $\bar\gamma$ is $\mathcal{O}((\gamma-\bar\gamma)^2)$ so that $\alpha_{n+1}(\gamma)$ and $\alpha_n(\gamma)$ are tangential for the solutions we have considered. Since Eqs. (29) have no pole at $\alpha_n(\bar\gamma)$ one may, however, accidentally have additional solutions (of those equations) for which $\alpha_{n+1}(\bar\gamma) = \alpha_n(\bar\gamma)$.

An example for this type of behavior is evident in the spectra calculated by Rammal and Toulouse.[4] One of the three eigenvalues for $\gamma_0 = 3\pi/4$ on the $n=1$ gasket (Fig. 3 of Ref. 4) has the property (36). As a result it becomes threefold degenerate in the $n=2$ spectrum (Fig. 4 of Ref. 4). Eigenfunctions of this type also show up in zero field,[(6)] where they can be related to the B representation of the symmetry group (D_3). An example is the eigenfunction for $\gamma = \pi$; $t = +1$ in the same spectrum. We know of no reason why the field condition (33) and Eq. (36) should be connected in general.

D. Degenerate eigenvalues

Let $\alpha_n(\gamma)$ be a p-fold degenerate eigenvalue ($p \geq 2$) on the n-stage gasket. This means that p distinct lines $\alpha_n(\gamma)$ in the α,γ plane intersect (or meet) at the point (α,γ). From nesting property III (Sec. III A above) one therefore has $3p-3$ lines $\alpha_{n+1}(\gamma)$ nested between the $\alpha_n(\gamma)$ which must go through (α,γ). In most cases this will determine the new degeneracy

$$p_{n+1} = 3p_n - 3 \ . \tag{37}$$

We note, however, that when (36) holds for all degenerate eigenfunctions one can have additional solutions [of Eqs. (29)] with the same eigenvalue.

It is also straightforward to construct the new eigenfunctions. For the given eigenvalue one has $3p$ linearly independent (but not orthogonal) eigenfunctions on the n-

stage gaskets. One has three matching conditions for the amplitudes at the vertices 1,2,3 which must be obeyed by linear combinations of these functions. In general, this will lead to $3p-3$ independent solutions.

One notes that for $p \geq 3$ one can always choose the basis functions on the n-stage gasket so that only two of them have nonvanishing amplitudes at the external vertices of the n-stage gasket. One can therefore choose an essentially localized basis set for these functions similar to the situation for zero field.[6]

This property of the spectrum was noticed by Rammal and Toulouse[4] (nesting property I) without proof. Our derivation provides a proof and also allows us to determine the evolution of the degeneracy.

An interesting case occurs when there are degeneracies for fields which obey Eq. (33). This certainly happens for $\gamma_0=0,\pi$ (e.g., the point Q_2 in Fig. 3 of Ref. 4). Each of the p intersecting branches (at stage n) then generates a tangential branch at stage $n+1$ by the construction of Sec. III B. In addition there are $2p-3$ branches which can not be related to a unique representation (μ) a stage n.

IV. DISCUSSION

The synthetic technique we have developed seems to have advantages. The role of symmetry and the relationship between the spectra of gaskets of different size, in the same field, come out much more clearly this way. Our main new result is nesting property III (Sec. III A). When combined with the results of Sec. III B and III D (nesting properties II and I of Ref. 4), this amounts to a nocrossing property. Three lines $\alpha_{n+1}(\gamma)$ belonging to the three irreducible representations ν are trapped between any two consecutive lines $\alpha_n(\gamma)$ and can only cross them when they intersect.

We were also able to assign representations to the lines and to relate the representations at stages n and $n+1$ through Eqs. (34). This amounts to serious constraints on the spectrum and has interesting implications. We only discuss the bounds of the spectrum.

The ground-state eigenvalue of the spectrum at stage n is described by a series of intersecting arcs[4] each belonging to some representation μ. They are also a lower bound on the spectrum at stage $n+1$. The successive ground states (of stages n and $n+1$) coincide at the cusps (of stage n) and when $\gamma_k=k\pi/4n$. In most cases there will be more than one γ_k on a given (n-stage) arc. The $\alpha_{n+1}(\gamma)$ touching at successive k must belong to different representations [ν; Eq. (34)]. This generates new cusps. One also notes that old cusps develop rapidly increasing degeneracies. This is closely related to the decoupling of loops, which dominates the magnetic susceptibility.[5] One also notes that the low-field expansion of the ground-state energy[2–4] is only relevant at very low fields ($\gamma<\pi/4^n$).

As emphasized in Ref. 4 a proper understanding of the spectrum in a field would be of considerable interest. One would like to understand the evolution of the gaps in the spectrum and the density of states for large n. This seems surprisingly elusive. We note that Eqs. (29) and (31) have a form which does not easily lend itself to numerical iteration to large n. In this respect they are not superior to Eqs. (6.16) and (6.20) of Ref. 3 [Eqs. (5) and (6) of Ref. 4] which are also very inconvenient numerically. It would certainly be of considerable interest to develop a suitable algorithm possibly along the lines of Refs. 7 and 8.

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