PHYSICAL REVIEW B VOLUME 29, NUMBER 9 1 MAY 1984

Temperature variation of sound velocity in liquid He II

Chung-In Um, Woo-Hyung Kahng, and Kyu-Hwang Yeon
Department of Physics, College of Sciences, Korea University, Seoul, Korea

Soon-Tahk Choh
Department of Physics, College of Natural Science, Hangyang University, Seoul, Korea

A. Isihara
Statistical Physics Laboratory, Department of Physics, State University of New York at Buffalo, Buffalo, New York 14260
(Received 2 May 1983)

The temperature dependences of the sound velocity in liquid helium II in bulk and in film are evaluated explicitly for low temperatures. In the bulk case, our theory yields results which agree with experiment better than the previous theories of Andreev and Khalatnikov and of Singh and Prakash. The film case is shown to be somewhat different from the bulk case.

I. INTRODUCTION

Some years ago, Andreev and Khalatnikov[1] investigated the temperature variation of the sound propagation in liquid helium using a collisionless kinetic equation and normal energy dispersion. More recently, Singh and Prakash[2] investigated the behavior of the retarded single-particle Green's function for a weakly interacting Bose gas and showed that a new temperature-dependent term arises in addition to what Khalatnikov and Andreev obtained. However, through theoretical considerations,[3] heat capacity,[4] scattering,[5] thermal expansion,[6] ultrasonic propagation,[7] and other measurements, evidence has been accumulated that the dispersion is actually "anomalous." While there are some different forms proposed for anomalous dispersion,[3] Kim, Um, Choh, and Isihara[3] derived the following form based on a microscopic theory:[3]

$$\epsilon = cq(1+\delta_1 q^2 - \delta_2 q^3 + \cdots) \ , \tag{1.1}$$

where the δ *are positive.* On the other hand, the normal dispersion was assumed to be of the following form,

$$\epsilon = cq(1-\gamma q^2) \ , \tag{1.2}$$

where the units are such that $\hbar = 1$, $2m = 2$, and c is the phonon velocity and γ *is positive.* These two dispersion relations represent different phonon decay mechanisms. In the case of the normal dispersion, it has been discussed[8] that four-phonon processes are the lowest, while in the anomalous case, three-phonon processes become most important at low momenta.

It has been shown by Maris[3] that anomalous dispersion gives sound velocity in better agreement with experiment than normal dispersion. It is the purpose of the present article to show that the anomalous dispersion relation given by Eq. (1.1) results in a sound velocity which agrees with experiment better than previous results. On the other hand, we have shown elsewhere that ρ_n/T^3 (the normal fluid density divided by temperature cubed) shows interesting temperature dependence if Eq. (1.1) is adopted.[9] Thus, the temperature dependences of these two different quantities are described in a consistent way.

We shall report also that the case of helium films is different from the bulk case. Since experimental data on helium films seem unavailable, our theoretical results can be useful in the future.

II. BASIC EQUATIONS

The velocity of sound can be obtained by solving the equations of motion of the liquid. We consider sound propagation near absolute zero following Andreev and Khalatnikov. Thus, we deal with the kinetic equation for the phonon distribution function $n(\vec{q}, \vec{r}, t)$:

$$\frac{\partial n}{\partial t} + [n, H] = 0 \ , \tag{2.1}$$

$$H = \epsilon(q) + \vec{q}\cdot\vec{v}_s \ , \tag{2.2}$$

together with the equation of continuity and the equation for the superfluid velocity $\vec{v}_s$:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \ , \tag{2.3}$$

$$\frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla}(\mu + \tfrac{1}{2}v_s^2) = 0 \ , \tag{2.4}$$

where j is the momentum of a unit volume of the liquid and μ is the chemical potential, which are given by

$$\vec{j} = \rho\vec{v}_s + \int \vec{q} n \frac{d\vec{q}}{(2\pi)^3} \ , \tag{2.5}$$

$$\mu = \mu_0 + \frac{\partial}{\partial \rho}\int \epsilon n \frac{d\vec{q}}{(2\pi)^3} \ , \tag{2.6}$$

where μ_0 is the chemical potential at absolute zero.

As usual, we assume that deviations from equilibrium are small and are proportional to $\exp[i(\vec{q}\cdot\vec{r} - \omega t)]$. After a straightforward calculation for small deviations from equilibrium and making use of Eq. (1.1), we arrive at a sound velocity expression:

$$\omega/q = c + \delta c \ ,$$

$$\delta c = 3c\left\{\frac{1}{4}(u+1)^2\ln\left[\frac{2}{27\delta_1}\left(\frac{c}{kT}\right)^2\right] - (u+1)^2 - \frac{1}{3} - \frac{3}{8}\frac{\rho_0^2}{c}\left(\frac{\partial^2 c}{\partial\rho_0^2}\right)\right\}\frac{\rho_n}{\rho_0} \quad (2.7)$$

[in three dimension (3D)], where we have neglected the effect of δ_2 and where ρ_0 is the equilibrium density of the liquid, c is given in the Appendix, and

$$u = \frac{\rho_0}{c}\frac{\partial c}{\partial \rho_0} \quad (2.8)$$

is a dimensionless parameter introduced by Khalatnikov.[8] The normal fluid mass density in Eq. (2.7) can be evaluated from a consideration of the total momentum carried by quasiparticles as follows:

$$\rho_n = \frac{1}{6\pi^2}\int dq\, q^4\left(-\frac{\partial n_0(\epsilon)}{\partial \epsilon}\right) ,$$

when n_0 is the equilibrium distribution. The resulting low-temperature expression is

$$\rho_n(T) = \frac{\Gamma(5)\zeta(4)}{6\pi^2 c}\left(\frac{kT}{c}\right)^4 - \frac{\Gamma(8)\zeta(6)}{6\pi^2 c}\delta_1\left(\frac{kT}{c}\right)^6 + \frac{\Gamma(9)\zeta(7)}{6\pi^2 c}\delta_2\left(\frac{kT}{c}\right)^7 + \frac{1}{3\pi}q_0^4\left(\frac{m^*}{2\pi kT}\right)^{1/2} e^{-\Delta/T} \quad (2.9)$$

(in 3D), where m^*, Δ, and q_0 are, respectively, the effective mass, roton energy, and momentum.

Note that for very low temperatures, δc is proportional to $T^4\ln(1/T)$ because in Eq. (2.7) the first term is dominant, in agreement with Andreev and Khalatnikov. At slightly higher temperatures, the constant terms in the curly brackets of Eq. (2.7) will contribute, leading to a T^4 proportionality. Since these terms are minus, they will reduce δc when it is plotted against temperature.

III. RESULTS AND DISCUSSIONS

In order to evaluate the temperature variation of sound velocity explicitly, it is necessary to determine the theoretical parameters. The coefficients δ_1 and δ_2 in the energy dispersion relation (1.2) have been determined elsewhere.[9,10] We list these parameters for two and three dimensions, respectively, as follows:

$$\delta_1 = 1.47\ \text{Å}^2,\ \ \delta_2 = -40.24\ \text{Å}^3\ \ (\text{for 2D})\ ;$$

$$\delta_1 = 1.51\ \text{Å}^2,\ \ \delta_2 = 3.25\ \text{Å}^3\ \ (\text{for 3D})\ .$$

These values are close to what Aldrich, Pethick, and Pines[11] found independently and reproduce the experimental dispersion curve, the cutoff momentum, and the ratio of the group velocity to the sound velocity very well. We have listed other parameters in Table I including the case for helium films. Because of the lack of motion perpendicular to the surface, the sound velocity in helium films differs from that in bulk. In this table, the parameter u defined by Eq. (2.8) for two dimensions is assumed to be 1.8 which is the value used by the previous workers[1,2] for the bulk case.

Figure 1 compares our theoretical results (solid curve) with the experimental data of Whitney and Chase.[12] Curves I and II represent, respectively, the results of Singh and Prakash and of Andreev and Khalatnikov. In comparison with these previous results, our agreement with the data is much better.

The two-dimensional case can be treated as in the bulk case. The sound velocity expression (2.7) is replaced by

$$\delta c = -c\left[(u+1)^2 + \frac{1}{2} - \frac{1}{3}\frac{\rho_0^2}{c}\frac{\partial^2 c}{\partial \rho_0^2}\right]\frac{\rho_n(T)}{\rho_0} , \quad (3.1)$$

where the normal fluid density is

$$\rho_n(T) = \frac{\Gamma(4)\zeta(3)}{4\pi c}\left(\frac{kT}{c}\right)^3 - \frac{\Gamma(7)\zeta(6)}{8\pi c^3}[1 + \tfrac{1}{4}(\pi n a^4)(\tfrac{3}{2}\epsilon^* - V_0)]\left(\frac{kT}{c}\right)^5 + \frac{1}{2}q_0^3\left(\frac{m^*}{2\pi kT}\right)^{1/2} e^{-\Delta/T} \quad (3.2)$$

(for 2D).

Using these results, we have illustrated the theoretical sound velocity in Fig. 2 as a function of temperature. As we see, the temperature dependence is very different from the three-dimensional case.

The main reason for the above difference is due to the difference in the angle integrals. In the Appendix, we have given the relevant angle integrals for both dimensions. An angle integral produces a logarithmic term for three dimensions. As we see, the two-dimensional integral does not include a logarithmic term, resulting in the difference. It will be very interesting to have an experimental test of such a difference.

The basic physics behind the present work is in investigating the energy dispersion and in finding the role played by three-phonon processes. In contrast, four-phonon processes

TABLE I. Theoretical parameters.

	m^*/m	q_0	Δ (K)	n	c (cm/sec)	u
2D	0.75	1.02	4.12	2.79×10^{-2} (Å^{-2})	164.4	1.8
3D	1.71	1.92	8	2.18×10^{-2} (Å^{-3})	238	1.8

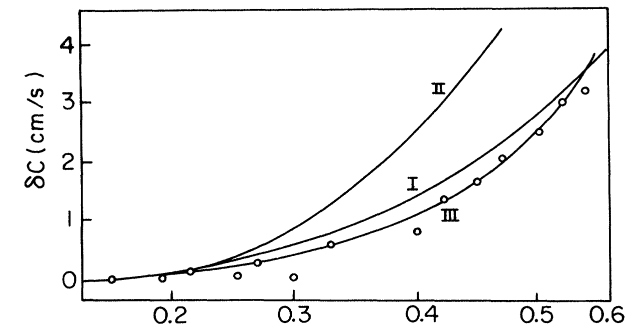

FIG. 1. Temperature dependence of the sound velocity of bulk liquid helium. Curve I: Singh and Prakash (Ref. 2); Curve II: Andreev and Khalatnikov (Ref. 1); Curve III: present theory. The ordinate represents the sound velcoity deviation $\delta c = c - \omega/q$ (cm/sec).

play a major role in Andreev-Khalatnikov and Singh-Prakash theories. From the analysis of the present work, together with our other studies,[9,10] it seems clear that three-phonon processes are important.

Concerning the energy dispersion itself, we remark that several different forms have been proposed. For instance, Greywall[4] and Dynes and Narayanamurti[7] have used an even-powered polynomial expression, while Maris[3] has proposed his own. In comparison, our dispersion relation has been derived from a microscopic theory of liquid helium. Different from some other microscopic theories, such as those based on variational principles, our theory treats density fluctuations due to collective couplings of the helium particles and derives the internal energy. It is very interesting that this internal energy expression is of the Landau form. That is, the theory justifies the Landau theory itself rather than deriving only the excitation energy for absolute zero, and thereby massless quasibosons appear in a natural way although the theory deals with actual helium particles.

Because of different dispersion relations, it is by no means a simple matter to compare various values of the coefficients. We must remember that higher-order terms in the dispersion relations affect the lower-order coefficients such as δ_1. In addition, as pointed out by Greywall,[4] experimental determinations may be based on some questionable assumptions. Nevertheless, we remark that our values 1.51 and 3.25 for δ_1 and δ_2 are in excellent agreement with the values of Aldrich *et al.*[11] which are 1.5 and 3.7, respectively. The errors in our determination are ± 0.15 for δ_1 and ± 0.20 for δ_2. On the other hand, δ_1 is around 0.9, 0.45, and 0.13 according to the analyses of Maris and co-workers, Philips, Waterfield, and Hoffer, and Bhatt and McMillan, respectively. Therefore, together with higher-order coefficients, further studies of the dispersion relation are definitely needed.

At temperatures of order 0.4 K, the δ_1 and δ_2 terms contribute approximately 15% and 1.2% of the first phonon term in Eq. (2.9). Hence, we expect a reduction in δc of order 14%. This explains why our result is lower than the previous results. As temperature increases the logarithmic term in Eq. (2.7) becomes smaller. Finally, around 0.65 K, the constant terms, which are negative, become larger. Hence, δc is expected to decrease, in agreement with Whitney and Chase and also with Kebukawa.[13] A further increase in temperature will make the roton contribution more and more important. Hence, a precise measurement of the temperature variation of the sound velocity is desirable.

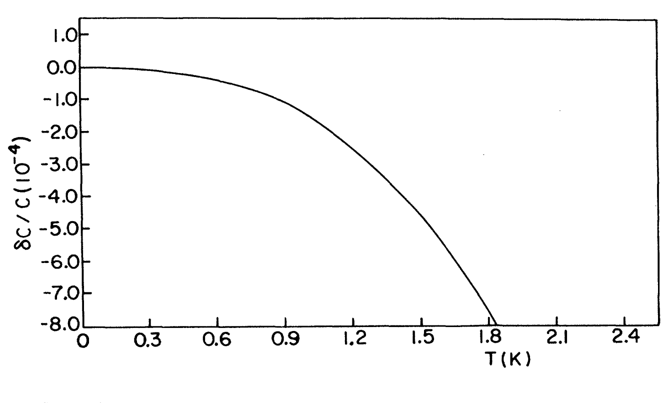

FIG. 2. Sound velocity in helium films. The curve represents Eq. (3.1).

ACKNOWLEDGMENTS

This work was supported by the U.S. Office of Naval Research (ONR) under Contract No. N000-14-79-C0451. One of the authors (C.-I.U) is grateful for the support from the Korea Science and Engineering Foundation. The work of another (A.I.) is supported by ONR under contract N-00014-79-0451.

APPENDIX

Our microscopic theory[9,10] adopts a soft potential of the following form:

$$\phi(r) = \begin{cases} V_0, & r \leq a \\ \epsilon^*[(a/r)^{12} - (a/r)^6], & r > a \ . \end{cases} \tag{A1}$$

The second velocity $c = \omega/q$ in the limit of small q is given by

$$c = [nu(0)/m^*]^{1/2} \ , \tag{A2}$$

where

$$u(0) = \frac{4\pi}{3} a^2 (V_0 - \tfrac{2}{3} a \epsilon^*) \quad \text{(for 3D)} \ ,$$
$$u(0) = \pi a^2 (V_0 = \tfrac{3}{10} \epsilon^*) \quad \text{(for 2D)} \ . \tag{A3}$$

The kinetic and hydrodynamical equations (2.1)–(2.4) are reduced in small n' and ρ' by expressing $n = n_0 + n'$ and $\rho = \rho_0 + \rho'$ as follows:

$$n' + q \frac{\partial n_0}{\partial \epsilon} \frac{v \cos\theta}{\omega/q - v\cos\theta} \left[\rho' \frac{\partial c}{\partial \rho_0} + v_s \cos\theta \right] = 0 \ , \tag{A4}$$

$$-\frac{\omega}{q} \rho' + \rho_0 v_s + \int q n' \cos\theta \, d\tau = 0 \ , \tag{A5}$$

$$-\frac{\omega}{q} v_s + \left[\frac{c^2}{\rho_0} + \int \frac{\partial^2 \epsilon}{\partial \rho_0^2} n_0 d\tau \right] \rho' + \int n' \frac{\partial \epsilon}{\partial \rho_0} d\tau = 0 \ , \tag{A6}$$

where $d\tau$ is the volume element in momentum space and $v = \partial\epsilon/\partial q$.

For three dimensions, the angle integration of Eq. (A4) leads to

$$\int dq\, q^4 \frac{\partial n_0}{\partial \epsilon} \ln\left|\frac{\omega/q+v}{\omega/q-v}\right| \approx \ln(2/3\delta q^2), \quad (\delta_1 q^2 << 1) \ . \quad (A7)$$

q^2 on the right-hand side is then replaced by its mean value.

On the other hand, for two dimensions, such a logarithmic term does not appear:

$$\int_0^{2\pi} \frac{\cos^2 d\theta}{\omega/q - v\cos\theta} = -2\pi\omega/qv^2 \ . \quad (A8)$$

Accordingly, in δc we expect a two-dimensional peculiarity.

[1]A. Andreev and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 44, 2053 (1963) [Sov. Phys. JETP 17, 1384 (1963)].

[2]K. K. Singh and J. Prakash, Phys. Rev. B 17, 1253 (1978).

[3]H. J. Maris and W. E. Massey, Phys. Rev. Lett. 25, 220 (1970); H. J. Maris, *ibid.* 28, 277 (1972); Phys. Rev. A 8, 2629 (1973); Yuh-Ren Lin-Liu and Chia-Wei Woo, J. Low Temp. Phys. 14, 317 (1974); R. N. Bhatt and W. L. McMillan, Phys. Rev. A 10, 1591 (1974); H. J. Kim, C. I. Um, S. T. Choh, and A. Isihara, Phys. Rev. B 23, 4498 (1981). The energy dispersion in relation with the structure factor has been discussed in E. Feenberg, Phys. Rev. Lett. 26, 301 (1971); T. Samulski and A. Isihara, Physica A 86, 257 (1977), etc. For more detailed literature, see H. J. Maris, Rev. Mod. Phys. 49, 341 (1977).

[4]D. S. Greywall, Phys. Rev. B 18, 2127 (1978); 21, 1329(E) (1979); N. E. Phillips, C. G. Waterfield, and J. K. Hoffer, Phys. Rev. Lett. 25, 1260 (1970).

[5]E. C. Svensson, D. Martel, and A. D. B. Woods, Phys. Lett. 55A, 152 (1975); R. A. Cowley and A. D. B. Woods, Can. J. Phys. 49, 177 (1971); R. B. Hallock, in *Low Temperature Physics LT-13,* Proceedings of the 13th International Conference on Low Temperature Physics, 1972, Boulder, Colorado, edited by W. J. O'Sullivan, K. D. Timmerhaus, and E. F. Hammel (Plenum, New York, 1974).

[6]J. E. Berthold, H. N. Hanson, H. J. Maris, and G. M. Seidel, Phys. Rev. B 14, 1902 (1976).

[7]J. Jackle and K. W. Kehr, Phys. Rev. Lett. 27, 654 (1971); Phys. Rev. A 2, 1757 (1974); R. C. Dynes and V. Narayanamurti, Phys. Rev. B 12, 1720 (1975).

[8]L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 19, 637, 709 (1949); I. M. Khalatnikov, *ibid.* 20, 243 (1950); 23, 821 (1952). See also, J. Wilks, *Properties of Liquid and Solid Helium* (Oxford Univ. Press, Oxford, 1967), Chap. 7.

[9]A. Isihara, S. T. Choh, and C. I. Um, Phys. Rev. B 20, 4482 (1979).

[10]A. Isihara and C. I. Um, Phys. Rev. B 19, 5725 (1979); C. I. Um, W. H. Kahang, S. T. Choh, and A. Isihara, Physica B 100, 74 (1980).

[11]C. H. Aldrich, III, C. J. Pethick, and D. Pines, J. Low Temp. Phys. 25, 677 (1976); 25, 691 (1976).

[12]W. W. Whitney and C. E. Chase, Phys. Rev. Lett. 9, 243 (1962).

[13]T. Kebukawa, Prog. Theor. Phys. 49, 388 (1973).