Velocity dependence of the superfluid density in liquid ^{4}He

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The roton liquid theory of Bedell, Pines, and Fomin has been used to calculate the variation of the superfluid density with velocity and temperature. The results are in good quantitative agreement with recent experiments. In addition, we find a small suppression of the superfluid transition temperature with increasing velocity due to the quasiparticle interactions.

For a complete hydrodynamic description of a superfluid, three independent variables are needed.[1] In addition to the usual pressure and temperature, an appropriate third variable is the square of the Galilean-invariant velocity $w^2 = (\vec{v}_n - \vec{v}_s)^2$, where $\vec{v}_n$ and $\vec{v}_s$ are the velocities of the normal and superfluid components, respectively. All thermodynamic functions then become functions of these three variables. In particular, we are interested in the variation of the superfluid density ρ_s with w^2.

From an experimental point of view, the measurement of the velocity dependence of ρ_s is complicated due to the usually simultaneous generation of vortices[2,3] in the superfluid. This problem has apparently been eliminated in the experiments of Hess,[4] whose data we have used for comparison with theory.

To explain his data, Hess[4] used the expression derived by Khalatnikov[5] for the velocity dependence of the normal fluid density ρ_n. In the temperature range where rotons dominate ($T > 1.4$ K), this can be written to order w^2 as

$$\rho_n(T,w^2) = \rho_n^0(T)\left[1 + \frac{1}{10}\left(\frac{p_0 w}{T}\right)^2\right] , \tag{1}$$

with p_0 as the roton momentum and the Boltzmann constant $k_B = 1$. This relation applies to the noninteracting rotons for which the zero velocity normal density $\rho_n^0(T) \sim T^{-1/2}\exp(-\Delta_0/T)$ with Δ_0 being the zero-temperature roton gap. For temperatures around 2.0 K, the corrections arising from roton-roton interactions become significant and Eq. (1) will break down. To correct for this, Hess[4] simply replaced Δ_0 by $\Delta(T)$ in $\rho_n^0(T)$. It was found by Hess[4] that the $\Delta(T)$ needed to explain his data was different from the previous determinations from neutron scattering experiments.[6]

In this paper, we derive an expression for $\rho_n(T,w^2)$ [or equivalently $\rho_s(T,w^2)$] within the roton liquid theory of Bedell, Pines, and Fomin.[7] The expression we obtain takes into account the roton-roton interactions exactly to order w^2. To this order only the $l = 0, 1$, and 2 moments of the quasiparticle interaction will enter. From this calculation, several important features emerge. The first is that the simple replacement of Δ_0 by $\Delta(T)$ in Eq. (1) does not correctly describe the dependence of $\rho_n(T,w^2)$ on the interactions. The second point is that the experiments can be fitted, even for temperatures above 2.1 K, with a $\Delta(T)$ which fitted both the recent neutron scattering data[8-10] and the specific heat.[7] This $\Delta(T)$ is also different from the temperature-dependent gap measured in Ref. 6. In Bedell, Pines, and Fomin (BPF) it was shown that the $\Delta(T)$ in Ref. 6 violates a stability condition for $T < T_\lambda$, where T_λ is the superfluid transition temperature and thus is probably too small. Within the roton liquid theory of BPF, we obtain a consistent picture of the thermodynamic properties of ^{4}He for 1.4 K $< T < T_\lambda$ and the velocity dependence of the superfluid fraction. The calculation is described in the following.

In the roton liquid theory of BPF, the roton-roton interactions are treated in the spirit of the Landau theory of a normal Fermi liquid.[11] This theory has been shown[7] to accurately describe the thermodynamics of superfluid ^{4}He for 1.4 K $< T < T_\lambda$. The roton-roton interaction $f_{\vec{p}\vec{p}'}$ is defined as the second functional derivative of the energy with respect to the distribution function n_p:

$$f_{\vec{p}\vec{p}'} = \frac{\delta^2 E}{\delta n_{\vec{p}} \delta n_{\vec{p}'}} , \tag{2}$$

where (we put $k_B = 1$)

$$n_{\vec{p}} = \frac{1}{e^{\epsilon_p/T} - 1}$$

and

$$\epsilon_p = \epsilon_p^0 + \sum_{p'} f_{\vec{p}\vec{p}'} n_{\vec{p}'} . \tag{3}$$

The zero-temperature single-particle energy ϵ_p^0 is fitted to neutron scattering data and here we use the simple fit introduced by BPF. For temperatures small compared with Δ_0, the roton gap, the quasiparticles with momenta close to p_0 make the dominant contribution to the thermodynamics. In $f_{\vec{p}\vec{p}'}$ we then ignore the dependence on the magnitude of the momenta and expand in a Legendre series in $\cos\theta$ $(= \hat{p} \cdot \hat{p}')$,

$$f_{\vec{p}\vec{p}'} = \sum_l f_l P_l(\cos\theta) . \tag{4}$$

The normal component of the density is defined through the mass current in the suprefluid rest frame where

$$\vec{j}_n = \rho_n \vec{w} = \sum_{\vec{p}} \vec{p}\, n(\epsilon_p\{\vec{w}\} - \vec{p} \cdot \vec{w}) . \tag{5}$$

In the superfluid rest frame, the single-particle energy is itself a function of the velocity $\vec{w}$. If we expand the energy to second order in $\vec{w}$ we have

$$\epsilon_p\{\vec{w}\} - \vec{p} \cdot \vec{w} = \epsilon_p - a(\vec{p} \cdot \vec{w}) + b(\vec{p} \cdot \vec{w})^2 , \tag{6}$$

where ϵ_p depends on $w^2=|\vec{w}|^2$. To determine the velocity dependence of ρ_n we must first determine the coefficients a and b.

The coefficients a and b are determined by evaluating Eq. (3) in the superfluid rest frame, using Eq. (6) for the single-particle energies, then

$$\epsilon_p\{\vec{w}\}-\vec{p}\cdot\vec{w}=\epsilon_p^0-\vec{p}\cdot\vec{w}+\sum_{\vec{p}'} f_{\vec{p}\vec{p}'}n(\epsilon_{p'}\{\vec{w}\}-\vec{p}'\cdot\vec{w}) \ . \tag{7}$$

If we expand the distribution function to order w^2 in Eq. (7), only the moments f_0, f_1, and f_2 enter. If we now equate the coefficients in Eq. (7), we find

$$a=[1+F_1(T)/3]^{-1} \tag{8a}$$

and

$$b=\frac{a^2}{2T}\frac{F_2(T)/5}{1+F_2(T)/5} \ , \tag{8b}$$

where $F_l(T)=[N_r(T)/T]f_l$ and $N_r(T)$ is the number of rotons. Here we use the roton number as determined by BPF. The additional velocity dependence of ϵ_p in Eq. (6) comes from the fact that the $(\vec{p}\cdot\vec{w})^2$ term contains both $l=0$ and $l=2$ Legendre polynomials. For $|\vec{p}|=p_0$ then $\epsilon_{p_0}=\Delta(T,w^2)$, where to order w^2 we have

$$\frac{\Delta(T,w^2)-\Delta_0}{T}=F_0(T)+(Z^2/6)\left[\frac{F_0(T)-F_2(T)/5}{1+F_2(T)/5}\right] \ , \tag{9}$$

with $Z=(p_0wa/T)$. Here we note that the number of rotons used in Eqs. (8a), (8b), and (9) is given by

$$N_r(T)=\sum_{\vec{p}} n_{\vec{p}} \sim T^{1/2}e^{-\Delta(T)/T} \ ,$$

where $\Delta(T)$ is evaluated at $\vec{w}=0$, but it includes the interactions.

The expression for the normal fluid density can now be obtained by substituting Eq. (6) into Eq. (5). If we keep terms only to order w^2 we find

$$\frac{\rho_n(T,w^2)}{\rho_n(T)}=1+\frac{Z^2}{10}\left[\frac{1-5F_0(T)/3-F_2(T)/15}{1+F_2(T)/5}\right] \ . \tag{10}$$

The normal fluid density $\rho_n(T)$ is given by[7]

$$\rho_n(T)=\frac{\rho_n^{(0)}(T)}{1+F_1(T)/3} \ , \tag{11a}$$

with

$$\rho_n^{(0)}(T)=\frac{p_0^2}{3T}N_r(T) \ . \tag{11b}$$

Again in Eqs. (10), (11a), and (11b) the number of rotons is evaluated at $\vec{w}=0$.

The velocity dependence of the superfluid density $\rho_s(T,w^2)=\rho-\rho_n(T,w^2)$, where ρ is the mass density of ^{4}He, has been measured by Hess.[4] To determine $\rho_s(T,w^2)$, Hess[4] measured the difference in resonant frequencies for $\vec{w}=0$ and finite $\vec{w}$ in a Helmholtz resonantor filled with He II. The relationship between $\rho_s(T,w^2)$ and the frequency shift $\Delta f/f$ for a Helmholtz resonator depends on the geometrical shape of the channel connecting the two regions filled with He II. Here we are not interested in solving the detailed boundary value problem to find the geometrical shape factor, so we use the simple parametrization introduced by Hess.[4] The frequency shift is then given by[4]

$$\frac{\Delta f}{f}=\frac{3}{8}\gamma\frac{\rho_s(T,w^2)-\rho_s(T)}{\rho_s(T)} \ , \tag{12}$$

where γ depends on the geometry of the connecting channel. Expanding Eq. (12) to order w^2 and inserting the results from Eq. (10) we find

$$-10^4\frac{\Delta f}{f}=375\gamma Z^2\frac{\rho_n(T)}{\rho_s(T)}\left[\frac{1-\frac{5}{3}F_0(T)-F_2(T)/15}{1-F_2(T)/5}\right] \ , \tag{13}$$

where we use Eqs. (11a), (11b), and the definitions following Eqs. (8b) and (9).

Equation (13) is the main result of our paper and it should be compared with the similar expression for $(\Delta f/f)$ in Eq. (2) of Ref. 4. This equation was obtained from Eq. (12) using the Khalatnikov expression in Eq. (1) for $\rho_n(T,w^2)$ with Δ_0 replaced by $\Delta(T)$, as discussed earlier. This clearly ignores the corrections coming from the terms inside the large parentheses of Eq. (13), where the main contribution originates from the $-\frac{5}{3}F_0(T)$ term. At $T=2.1$ K, ignoring this term gives a result which is 80% smaller than the result which includes it. From the experiment, it is clear that the interaction corrections are significant and they should be included in a consistent manner (see Fig. 1).

To make comparisons with experiment, the interaction parameters f_0, f_1, and f_2 and the geometry-dependent parameter γ must be determined. The interaction parameters f_0 and f_1 were determined by BPF by a fit to the specific heat and superfluid density, but f_2 could not be determined from the thermodynamic quantities at $w=0$. Here we can estimate its order of magnitude and sign from the fits to $(\Delta f/w^2f)$. The value we find is $Nf_2\sim10$ K at standard volume and pressure (SVP), where N is the ^{4}He

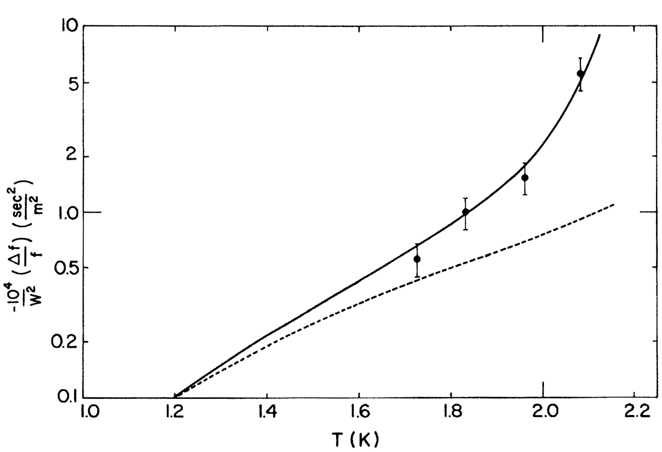

FIG. 1. Frequency shift and superfluid velocity-dependent quantity $(-10^4\Delta f/w^2f)$ vs temperature. The solid line is the calculation from Eq. (12) including the interaction [Eq. (13)]. The dashed line is also due to Eq. (12) but using the noninteracting density expression due to Khalatnikov [Eq. (1)]. The full circles and error bars are experimental data from Ref. 4. Note the importance of interaction as T approaches T_λ.

number density. This number is quite close to the value predicted by Bedell, Pines, and Zawadowski,[12] who calculated it microscopically from a model for the roton-roton pseudopotential introduced to explain the roton-roton bound states, the temperature dependence of the gap, and the roton viscosity. The model predicted a value of $Nf_2 = 9.5$ K.

For the geometry-dependent factor, we chose the best fit to the data, which yielded $\gamma = 0.61$. It turned out that incidentally it fits the value of $(\Delta f / w^2 f)$ at $T = 1.838$ K. In Fig. 1, we have plotted $(-10^4 \Delta f / w^2 f)$ for $\gamma = 0.61$ and $Nf_2 = 9.5$ K. The values of $Nf_0 = -9.7$ K and $Nf_1 = -1.3$ K were taken from BPF. In Ref. 4, a value of $\gamma = 1$ was chosen; however, this is probably incorrect. Firstly, the data point at $T = 1.43$ K in Fig. 3 of Ref. 4 is too high and should not be used in comparing with theory.[13] Secondly, the formula used for $\Delta f / f$, as noted above, is not correct. Taking these points into account gives $\gamma = 0.61$. We have also plotted in Fig. 1 the noninteracting results for $\gamma = 0.61$. In Fig. 2 we show the velocity dependence of $\Delta f / f$ for several temperatures. Clearly, both the velocity and temperature dependence are accurately described by Eq. (13). Since $\Delta f / (w^2 f)$ changes by an order of magnitude in the temperature range $1.4 \text{ K} < T < 2.1$ K, the uncertainties introduced by using Eq. (12), in lieu of an exact solution of the boundary value problem, are not significant.

Within the framework of the roton liquid theory of BPF we have derived an expression for $\rho_n(T, w^2)$ which is exact to order w^2. This result ignores the possible energy dependence of the parameters f_l. From the fit to the specific-heat data used by BPF and from the theoretical work of Bedell, Pines, and Zawadowski,[12] there appears to be only a very weak (if any) energy dependence in f_0 at SVP. It certainly can be ignored in comparison to the corrections we have included here. To further pin down the parameters γ and f_2, more accurate experiments are needed. A study of these effects with pressure would also be useful to further test the roton liquid theory of BPF. Finally, an additional feature of this calculation is that we can determine the amount by which the superfluid phase transition is suppressed. For a value of $(p_0 w / T) = 0.41$ we find that $\delta T_\lambda = [T_\lambda - t_\lambda(w^2)] / T_\lambda \simeq 1\%$. This is a small effect and it may be hard to detect given the difficulty of the experiment.

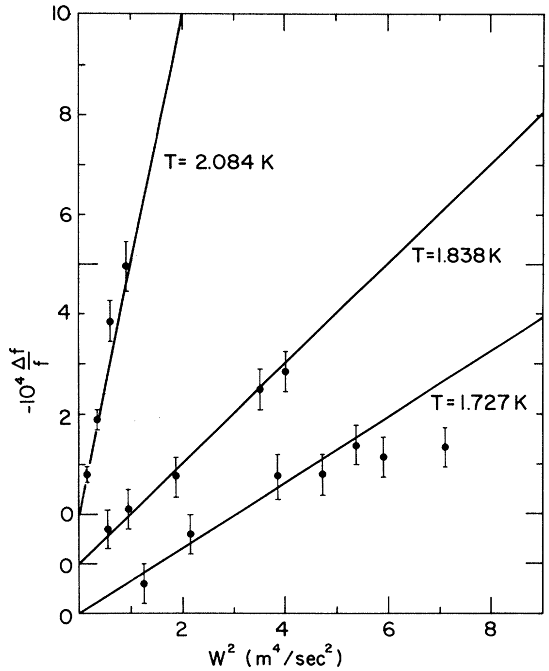

FIG. 2. Frequency shift $(-10^4 \Delta f / f)$ vs square of the superfluid fluid velocity. The full circles and error bars denote the experimental results from Ref. 4.

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