PHYSICAL REVIEW B VOLUME 29, NUMBER 9 1 MAY 1984

Percolation, clusters, and phase transitions in spin models

Chin-Kun Hu
Institute of Physics, Academia Sinica, Nankang, Taipei, Taiwan 115, Republic of China
(Received 4 October 1982; revised manuscript received 10 November 1983)

The phase transition in the Ising model and the percolation transition in the lattice percolation model have many common characteristics which have motivated researchers to explore whether the former is a percolation transition of a correlated percolation model. Previous attempts to draw such a connection have been either unsuccessful or unsatisfactory. Considering each lattice site with an Ising spin occupied and the nearest-neighbor (NN) coupling between occupied sites as a bond with a bond probability p depending on the NN coupling constant J and the temperature T, we formally show that the partition function of the Ising model is the generating function of bond-correlated percolation model (BCPM) with a bond probability $p=1-\exp(-2J/kT)$. The BCPM has the Ising critical temperature and exponents, including ν, ν', η, β, α, α', and γ (perhaps also γ'). From the connection between the Ising model and the BCPM, we also derive and hence give a geometrical meaning of the finite-size scaling and broadening at first-order phase transitions of the Ising model. Our approach may easily be extended to many spin models and give geometrical meaning to other properties of spin models.

I. INTRODUCTION

The phase transition in the Ising model[1-3] and the percolation transition in the lattice percolation model[4-11] have many characteristics in common. The singular behavior in the latter is clearly related to the onset of the appearance of the percolating cluster in the system. It is of interest to know whether the phase transition in the Ising model has a mechanism similar to that of the percolation transition, i.e., the onset of the appearance of the percolating cluster. To answer this question positively, one must map the Ising model into a percolation model such that the latter has the same critical point and exponents as those of the former and the variables, e.g., the bond probability, of the latter are specific functions of the variables of the former. The purpose of this paper is to propose a new and elegant approach to accomplish such mapping. Our approach simultaneously solves a closely related problem, i.e., to define in a precise way a cluster model which reproduces the critical point and exponents of the Ising model.[12-15] Both problems have attracted the attention of many researchers for the past decades.[12-28]

In the previous approaches[12-27] to both problems, the lattice site with an Ising spin of one sign is considered occupied in the corresponding cluster[12-15] or percolation model[16-27] and the lattice site with an Ising spin of an opposite sign is considered unoccupied. The ferromagnetic interactions between spins induce correlation in the occupation of lattice sites and the external magnetic field is used to control the concentration of occupied sites. Using this idea, Fisher proposed a semiphenomenological cluster model[12] which was found to be inconsistent with Monte Carlo data.[13,14] In the previous attempts to draw the connection between the Ising model and the correlated percolation model, the nearest-neighbor occupied sites are considered to be always in the same cluster, which is called the Ising cluster, of a site-correlated percolation model[16-24] (SCPM) or to be in the same cluster with a bond probability p_B of a site-bond-correlated percolation model[25] (SBCPM). However, it was found that the critical point[16,19] of the SCPM in $d=3$ is different from the Ising critical point in $d=3$, and the cluster-size exponent of the SCPM in $d=2$ is larger than the Ising susceptibility exponent in $d=2$. Coniglio and Klein[25] found that their SBCPM does have the Ising critical point in any dimensions for a particular choice of $p_B=1-\exp(-2K)$, where $K=J/kT$. They found by the Migdal-Kadanoff renormalization-group calculation[25,29] that the mean-cluster-size exponent γ_p of the SBCPM in $d=2$ equals the Ising magnetic susceptibility exponent γ in $d=2$, provided that $p_B=1-\exp(-2K)$. They also found that[25,29] ν_p of the SBCPM equals ν of the Ising model for any p_B. In the same manner, Coniglio and Peruggi[28] found that the Potts model was related to a polychromatic SBCPM with similar results.

Thus Coniglio and Klein's SBCPM (Ref. 25) is a good "candidate" for the correlated percolation model corresponding to the Ising model. However, their approach[25] is not quite satisfactory. Firstly, in the derivation of the connection between the Ising model and the SBCPM, Potts spin variables and their coupling constant [see Eqs. (1)—(3) in Ref. 25 and Eq. (2) in Ref. 28] are introduced. Thus the derivation is not elegant. Secondly, there is no physical interpretation for the introduction of the bond probability p_B between occupied sites in the Ising clusters; i.e., p_B is introduced artificially, Coniglio and Peruggi[28] stated: "Note that the bonds are only introduced to define the connectivity between two nearest-neighbor particles and do not affect their interacting energy and therefore the particle distribution." Thus, strictly speaking, Coniglio and Klein did not prove that the mechanism of percolation transition can be applied to the phase transition in the Ising model. Finally, the results $\nu_p=\nu$ and $\gamma_p=\gamma$ were obtained by an approximate calculation.

In our opinion, we should abandon the previous idea that the lattice site with an Ising spin of one sign should be considered occupied in the corresponding percolation problem and the site with a spin of an opposite sign should be considered unoccupied. Therefore, in order to establish the connection between the Ising model (and other Ising-like spin models) and the correlated percolation model, we propose that one should consider the sites with a spin occupied and only the sites without a spin unoccupied. One should also regard the coupling between spins as a bond with a bond probability p depending on the coupling strength J and the temperature T. Only the sites connected by attached bonds are defined to be in the same cluster. Using these ideas, we will show formally in Sec. II that the partition function of the spin-$\frac{1}{2}$ simple Ising model with normalized nearest-neighbor (NN) coupling constant $K=J/kT$ is the generating function of a bond-correlated percolation model[30] (BCPM) with $p=1-\exp(-2K)$, and the BCPM has the Ising critical point T_c and exponents,[31] i.e., the cluster in the BCPM is just the Ising droplet.[25] This connection has been derived by us indirectly in a previous paper.[30(b)] Here we derive such a connection in a direct and simple way. Our approach can be easily extended to many spin models.[32–34] For example, in another paper[32] we show that a sublattice dilute q-state (q represents an even positive integer) Potts model (SDQPM) is equivalent to the SBCPM defined in that paper, where the sites with (without) a Potts spin in the SDQPM correspond to the occupied (unoccupied) sites in the SBCPM. Other examples will be mentioned at the end of this paper.

Our approach not only unifies the thermal phase transition and the percolation transition but also provides geometrical explanations for many properties of spin models, such as the finite-size scaling (FSS) of the magnetic susceptibility at the first-order phase transitions. The FSS was obtained by many different methods,[35] but its geometrical explanation is still absent. In Sec. III the connection between the BCPM and the Ising model is used to obtain the FSS of the magnetic susceptibility and the broadening of the transition region at the first-order phase transitions of the Ising model.

Based on the results of Secs. II and III, a unified physical picture for the first- and second-order phase transitions and for the FSS at first-order phase transitions is given in Sec. IV. In Sec. V we discuss some theoretical problems related to this work and point out the extension of this work to other spin models.

II. CONNECTION BETWEEN ISING MODEL AND A BOND-CORRELATED PERCOLATION MODEL

In a previous paper[30(a)] we found that a sublattice dilute q-state Potts model at $T\to 0$ is equivalent to a correlated percolation model. Since a sublattice dilute q-state Potts model on a decorated lattice [i.e., case 1 lattice in Ref. 30(a)] at $T\to 0$ is equivalent[30(b)] to the q-state Potts model on the original undecorated lattice at a finite T, it can be shown that the latter model also corresponds to the correlated percolation model of Ref. 30(a). For the sake of simplicity, we will derive such connections directly and restrict these to the case $q=2$, which corresponds to the spin-$\frac{1}{2}$ Ising model. The partition function of the simple Ising model on the graph G with N sites and N_b bonds and with only the NN interactions is given by

$$Z_N(K,B)=\sum_{\sigma}\exp[-\mathscr{H}(\sigma)/kT]$$
$$=\sum_{\sigma}\exp\left[\sum_{\langle ij\rangle}K(\sigma_i\sigma_j+1)+B\sum_i\sigma_i-2KN_b\right], \tag{1a}$$

where $\sigma_i,\sigma_j=\pm 1$, $K=J/kT$, and $B=H/kT$. Equation (1a) can be rewritten as

$$Z_N(K,B)=\sum_{\sigma}\prod_{\langle ij\rangle}[1+(e^{2K}-1)\delta(\sigma_i,\sigma_j)]$$
$$\times\prod_i\exp(B\sigma_i)\exp(-2KN_b)\ . \tag{1b}$$

Here $\prod_{\langle ij\rangle}$ and $\prod_i$ extend over all NN bonds and sites in G, respectively. $\delta(\sigma_i,\sigma_j)$ equals 0 when $\sigma_i\neq\sigma_j$ and equals 1 when $\sigma_i=\sigma_j$. The energy of the system at $T=0$ is chosen to be 0 as in Ref. 36. Now we expand the first product in (1b) and use the subgraphs $G'\subseteq G$ to represent the terms in the expansion. A subgraph G' of a 5×5 lattice is shown in Fig. 1 as an illustration. For each NN pair of sites $\langle ij\rangle$ there occurs in (1b) the two terms 1 and $[\exp(2K)-1]\delta(\sigma_i,\sigma_j)$; subgraphs G' with no $\langle ij\rangle$ bond correspond to the former and those with an $\langle ij\rangle$ bond to the latter. There are $e(G')$ bonds in the subgraph G' $0\le e(G')\le N_b$. If a particular bond $\langle ij\rangle$ is attached by the factor $[\exp(2K)-1]\delta(\sigma_i,\sigma_j)$, then $\sigma_i=\sigma_j$ after sum over spin states and i and j are said to be in the same cluster. In general, if two sites can be connected through a series of bonds, they are said to be in the same cluster. A given G' usually contains a large number of independent clusters, including isolated sites which do not connect with any other sites via bonds. For a given G', we can carry out the configuration summation of all spin states, and in such a summation only the terms where all spins in the same cluster have the same spin component have nonzero contributions. Thus (1b) can be written as

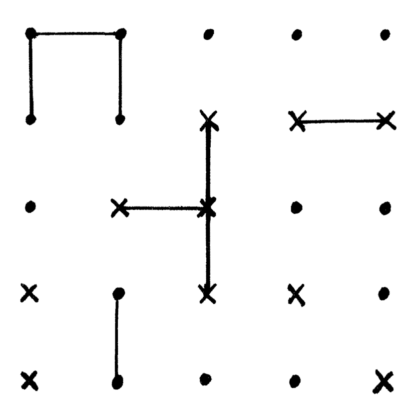

FIG. 1. Subgraph G' of a 5×5 lattice G and a spin state on the G'. The solid lines represent bonds in the G'; the sites with $\times$ or $\bullet$ represent occupied spin of one sign or an opposite sign, respectively.

$$Z_N(K,B)=\sum_{G'\subseteq G} p^{e(G')}(1-p)^{N_b-e(G')} \times \prod_c 2\cosh[Bn_c(G')] , \qquad (2)$$

where $\prod_c$ extends over all clusters c in G', $n_c(G')$ is the number of sites in cluster c, and $p=1-\exp(-2K)$. When $B=0$, Eq. (2) reduces to the result of Ref. 36 obtained by a different approach. From (2), it is easy to derive expressions for the spontaneous magnetization M and the magnetic susceptibility χ for the Ising model at $B=0$. In the thermodynamic limit, they are as follows:

$$M(G,p)=\lim_{B\to 0^+}\lim_{N\to\infty}\frac{\partial}{\partial B}f_N(K,B) = \lim_{N\to\infty} W^{-1}\sum_{G'\subseteq G}\Pi(G',p)[N^*(G')/N] , \qquad (3)$$

$$\chi(G,p)=\lim_{B\to 0^+}\lim_{N\to\infty}\frac{\partial^2}{\partial B^2}f_N(K,B) = \lim_{N\to\infty} W^{-1}\sum_{G'\subseteq G}\Pi(G',p)\sum_c^f n_c^2(G')/N+\lim_{N\to\infty}W^{-2}\sum_{G'\subseteq G}\sum_{G''\subseteq G}\Pi(G',p)\Pi(G'',p)[N^*(G')-N^*(G'')]^2/2N , \qquad (4)$$

where $f_N(K,B)=\ln Z_N(K,B)/N$,

$$\Pi(G',p)=p^{e(G')}(1-p)^{N_b-e(G')}2^{n_f(G')} , \qquad (5)$$

$W=\sum_{G'\subseteq G}\Pi(G',p)$, and $\sum_c^f$ is a sum over all finite clusters in G', except the percolating clusters defined below. $N^*(G')$ of (3) and (4) is the total number of sites which belong to one of the "percolating clusters" in G', which extend from one side of G to another and become infinite clusters as $N\to\infty$. The $n_f(G')$ of (5) is the total number of finite clusters in G', which is essentially the same as the total number of clusters, including percolating cluster, for $T>0$ and $N\to\infty$. From (3) and (5), it is obvious that $M(G,p)$ is just the percolation probability of the following (BCPM) on G:

(1) All sites of G are occupied and each bond of G is attached with the bond probability $p=1-\exp(-2K)$.

(2) The overall probability of $G'\subseteq G$ is enhanced by a factor of 2 for each finite cluster.

The first term in (4) is the mean cluster size for the BCPM. For $p<p_c=1-\exp(-2K_c)$, the second term in (4) is always 0, and $\chi(G,p)$ is just the mean cluster size for the BCPM. It is possible, but not proved, that for $p>p_c$ the second term of (4) makes only a finite contribution to $\chi(G,p)$; in that case the singular part of $\chi(G,p)$ would be given by the mean cluster size of the BCPM and they would have the same critical exponent. This conjecture is based on the idea that, as $N\to\infty$, the contribution to the summation in the second term of (4) can be expected to peak sharply on certain rather similar subgraphs having values of N^* differing by order of $\sqrt{N}$ or less. From (2), one can show easily that the internal energy and specific heat of the Ising model are proportional to the mean number of $e(G')$ and the fluctuations of $e(G')$, respectively, in the BCPM.

Based on (1b), the two-spin correlation function of the Ising model, $\langle\sigma_a\sigma_b\rangle$, for two spins at sites a and b of G, respectively, can be written as[2]

$$\langle\sigma_a\sigma_b\rangle=\frac{1}{Z_N(K,B)}\sum_\sigma\prod_{\langle ij\rangle}[1+(e^{2K}-1)\delta(\sigma_i,\sigma_j)]\sigma_a\sigma_b \times\prod_i\exp(B\sigma_i) . \qquad (6)$$

Using a procedure similar to that used to derive (2), we expand the first product in (6) and use the subgraphs $G'\subseteq G$ to represent the terms in the expansion. The criteria of assigning bonds and defining clusters in subgraphs G' are the same as those used before. For a given G', we can carry out the configuration summation of all spin states and in such a summation, only the terms, where all spins in the same cluster have the same spin component, can have nonzero contributions. If the sites a and b locate in the same cluster of G', σ_a and σ_b always have the same spin component (hence $\sigma_a\sigma_b$ always equals 1), and the contributions from such G' to the summation of (6) are the same as those to the summation of (2). However, if the sites a and b locate in different clusters of G', σ_a and σ_b have equal probability of having the same or different spin components (hence $\sigma_a\sigma_b$ equals $+1$ or -1 with a equal probability) and the contribution from such G' to the summation of (6) is 0. Thus $\langle\sigma_a\sigma_b\rangle$ of (6) in the thermodynamic limit and at $B=0$, denoted by $\langle\sigma_a\sigma_b\rangle_0$, can be written as

$$\langle\sigma_a\sigma_b\rangle_0=\lim_{N\to\infty}\left[\sum_{G'\subseteq G}p^{e(G')}(1-p)^{N_b-e(G')}2^{n_f(G')}\delta(c_a,c_b)\right]\Big/ W , \qquad (7)$$

where $\delta(c_a,c_b)$ equals 1 when a and b are in the same cluster of G' and equals 0 when a and b are in different clusters of G'. The right-hand side of (7) is just the pair-connectedness function of the sites a and b in the BCPM. Thus the exponents ν_p and ν_p' for the correlation length near p_c and η_p for the pair-connectedness function at p_c in the BCPM are the same as the corresponding values for the Ising model.[29]

In summary, the clusters in the BCPM have the desired properties that[25] (i) they diverge at the Ising critical point, (ii) their linear dimension diverges as the Ising correlation length, and (iii) the mean cluster size diverges as the Ising susceptibility. Besides these, we also found that the percolation probability in the BCPM is given by the Ising spontaneous magnetization, and the fluctuations of the number of bonds in the BCPM diverge as the Ising specific heat.

III. FINITE-SIZE SCALING

Now we turn to the case of the large but finite N. In this case, the magnetic susceptibility per spin $\chi_N(G,p)$ is given by

$$\chi_N(G,p)=\lim_{B\to 0}\left[\frac{\partial}{\partial B}\ln Z_N(K,B)-\frac{\partial}{\partial B}\ln Z_N(K,-B)\right]\Big/(2BN)$$
$$=W^{-1}\sum_{G'\subseteq G}\Pi(G',p)\sum_{c}^{\text{all}} n_c^2/N\ , \tag{8}$$

where $\sum_c^{\text{all}}$ is a sum over all clusters, including percolating clusters in G', and the $n_f(G')$ appearing in $\Pi(G',p)$ should include the total number of percolating clusters in G'. In Eq. (8), we have included the contribution from the smearing of the Dirac δ function arising from the nonzero spontaneous magnetization along the $B=0$ axis for $p>p_c$ (i.e., $T<T_c$) in the limit $N\to\infty$, while in (4) the contribution from the δ function has been excluded. Namely, if (8) is used to calculate the magnetic susceptibility in the limit $N\to\infty$, $\chi_\infty(G,p)$, then we have

$$\chi_\infty(G,p)=\lim_{B\to 0^+}[M_\infty(T,B)-M_\infty(T,-B)]/2B$$
$$=\lim_{B\to 0^+}\frac{M_\infty(T,0^+)}{B}+\lim_{B\to 0^+}\frac{M_\infty(T,B)-M_\infty(T,0^+)}{B}$$
$$=M_\infty(T,0^+)\delta(0)+\chi(G,p)\ , \tag{9}$$

where $M_\infty(T,B)$ is the magnetization of the Ising model in the limit $N\to\infty$, $\delta(0)$ is the Dirac δ function, and $\chi(G,p)$ is the magnetic susceptibility given by Eq. (4).

Suppose G is a graph in a d-dimensional space. When $T<T_c$ and N is very large, the contribution to the summation in Eq. (8) is expected to peak sharply on certain rather similar subgraphs G^*. The spanning cluster c^* in each G^*, which is defined to be the percolating cluster extending simultaneously over the d orthogonal directions through the lattice, is expected to have the site content (i.e., the total number of sites) $n_{c^*}=NM_\infty(T,0^+)$. This follows from the result that there is at most one c^* cluster in each G^* for $d=2$ for topological reasons, and it is reasonable that this is also the case for $d=3$ in the very-large-N limit.[37] The contributions of the c^* clusters in all G^* constitute the leading term of $\chi_N(G,p)$, which is therefore given by

$$\chi_N(G,p)=NM_\infty^2(T,0^+)\ , \tag{10}$$

when $M_\infty(T,0^+)\neq 0$. This is just the finite-size scaling (FSS) at first-order phase transitions,[35] which has been reached by different approaches.

Equation (10) can also be obtained from the finite-size broadening of the first-order phase-transition region. In the thermodynamic limit $N\to\infty$ and for $T<T_c$, it follows from (2) that a vanishing small magnetic field $B\to 0^+$ is sufficient to maintain the percolating clusters in the direction of the magnetic field; i.e., the separation between the spin-up phase and spin-down phase is vanishing small. In the system of finite N, it follows from (2) that a magnetic field with magnitude of the order of

$$\Delta B=1/n_{c^*}=1/[M_\infty(T,0^+)N] \tag{11}$$

is necessary to maintain the percolating clusters in the direction of the magnetic field. The Dirac δ function of (9) also broadens into a function with width of the order of Eq. (11) and magnitude of the order of $1/\Delta B$ (so that the integral of the function remains the same). Hence, for a finite system, the first term of (9) becomes

$$M_\infty(T,0^+)\frac{1}{\Delta B}=M_\infty^2(T,0^+)N\ , \tag{12}$$

which is the same as that given by Eq. (10). Note that the broadening of the magnetic field, given by Eq. (11), is similar to Imry's result[38] about the finite-size broadening of the transition temperature at a first-order phase transition. Equation (11) is obtained directly from the formulation of Sec. II, while Imry's result is based on the assumption that the temperature fluctuation in a finite system plays the role of the intrinsic temperature uncertainty.[38]

In sufficiently high dimensions $d\gg 2$, it was suspected that an infinite number of percolating clusters can occur.[37(a)] If this is indeed the case, then the sum of the sites in these percolating clusters must be approximately equal to $NM_\infty(T,0^+)$ and the leading term of $\chi_N(G,p)$ is no longer as large as that given in Eq. (10) or (12).

For a semi-infinite $n\times\infty$ strip of spins, the spontaneous magnetization is still 0 and Eq. (8) is still valid. In such a system, for $0<T<T_c$, there is no cluster in G^*, which can extend all the way along the direction of the infinite number of spins. Instead, there are many large clusters of equal importance, whose linear dimensions along the direction of finite number of spins are $a_0 n$ (a_0 is the lattice spacing) and along the direction of the infinite number of spins are approximately equal to the same value, for example, $a_0 l(n,T)$. Let N_g be the total number of such large clusters (actually $N_g\to\infty$ for semi-infinite systems for $0<T<T_c$) in a G^*. The contributions of the N_g clusters in all G^* constitute the leading term of the magnetic susceptibility $\chi_{n\times\infty}(G,p)$ for the $n\times\infty$ system at $T<T_c$:

$$\chi_{n\times\infty}(G,p)=N_g[M_\infty(T,0^+)N/N_g]^2/N$$
$$=M_\infty^2(T,0^+)N/N_g$$
$$=M_\infty^2(T,0^+)nl(n,T)\ , \tag{13}$$

which is much smaller than $n\times\infty$ as suggested by Eq. (10).

IV. PHYSICAL PICTURE OF THE CONNECTION

Here we give a physical picture for the expansion in (2) and hence the connection between the Ising model and BCPM. In ordinary descriptions of the thermodynamic properties of the Ising model, as embodied in (1a), the Ising spins are coupled with each other via their NN interactions. Thus the motion of each spin will influence its neighbors. In (2) such coupled motion is decomposed with the p into "normal modes" of motion, where each normal mode is represented by a subgraph G'. The probability for a given subgraph to appear is proportional to $\Pi(G',p)$ of (5), which depends on T. Each subgraph usually contains many clusters; all spins in the same cluster move up and down as a single entity, but different clusters move independently; i.e., there is no interaction between different clusters. It should be noted that the meaning of the cluster in this BCPM is different from that in ordinary Monte Carlo (MC) simulations[14–17] or series-expansion (SE) studies[19] based on (1a). For example, suppose we have a subgraph G' of a 5×5 lattice as shown in Fig. 1, where the bonds are represented by solid lines and a spin state is also shown, with $\times$ representing spin of one sign and $\bullet$ representing spin of opposite sign. According to the definition of BCPM, only the sites connected by solid lines are in the same cluster, so there are two four-site clusters, two two-site clusters, and thirteen one-site clusters in this G'. However in ordinary MC simulation or SE studies, the nearest-neighbor occupied sites belong to a cluster, so that there are one seven-site cluster, one two-site cluster, and one one-site cluster in Fig. 1 if the sites with $\times$ are considered occupied. Furthermore, the spin states generated in MC simulations based on (1a) usually contain a superposition of many subgraphs and therefore it is difficult to reveal the cluster contents of the BCPM in such MC simulations. Hence, it is desirable to generate directly the distribution of subgraphs G' according to the probability factor $\Pi(G,p)$ by MC simulation or SE techniques and study the behavior of clusters in such subgraphs. This could be a useful new way to study the thermodynamic properties of the Ising model.

We will now summarize, in a simple physical picture, what we have said. At any T, there is a distribution of $G'\subseteq G$ according to the probability factor $\Pi(G',p)$, each G' usually contains many independent clusters. At $T\rightarrow\infty$, the G' with N single-site clusters dominates. As T decreases, larger and larger clusters begin to form. When $T\rightarrow T_c$, infinite clusters begin to appear and there is a very large fluctuation in cluster sizes, which accounts for the singular behavior of the second-order phase transition at T_c. For $T<T_c$, the dominant subgraphs contain a spanning cluster c^* with a finite fraction of total lattice sites. The first-order phase transitions and nonzero spontaneous magnetization along the axis $B=0$ and $T<T_c$ are due to the response of such c^* clusters to the external field. For the finite and large N it is also due to the contributions of c^* clusters that the magnetic susceptibility follows the FSS at first-order phase transitions.

V. DISCUSSION

The subgraph expansion of the Ising partition function with $B=0$ [i.e., Eq. (2) in this paper with $B=0$] has been derived by Kasteleyn and Fortuin[36] some time ago with a different method. However, since they did not include the magnetic field and hence did not derive an equation for the spontaneous magnetization [i.e., Eq. (3) in this paper], the connection between the Ising model and the BCPM was not noticed by them. Essam[39] has used Kasteleyn and Fortuin's result to draw a connection between the Ising model and a bond random percolation model (BRPM) with a bond probability $p_B=1-\exp(-2K)$. In such a connection, the Ising partition function is the mean value of $2^{n(G')}$ in the BRPM, where $n(G')$ is the number of clusters in G'. However, the Ising critical point K_c is different from the percolation point for the BRPM, $K_{c,p}$, which satisfies $1-\exp(-2K_{c,p})=p_{B,C}$, with $p_{B,C}$ being the critical probability of the BRPM. Thus we may not use such a connection to establish that the percolation transition is the mechanism of the phase transition in the Ising model.

The results for the simple Ising model obtained in Secs. II–IV can be easily extended to many dilute and undilute spin models of half-integer spins, including various q-state Potts models (QPM) (e.g., the sublattice dilute QPM (SDQPM),[32,40] dilute QPM,[32,41] and QPM with NN, distant-neighbor, and multispin interactions), the Ising model with antiferromagnetic and ferromagnetic competing interactions, and the Baxter model. In each case, the sites with (without) a spin in the spin model correspond to occupied (unoccupied) sites in the percolation model, and the percolation transition has the same critical point and properties as the spin model. We can also derive the FSS and broadening at first-order transitions by a similar procedure. We can also show that a model for the hydrogen bonding in water molecules on a lattice is equivalent to a bond-correlated percolation model,[33] which can be used to explain some peculiar behavior of supercooled water.[42] All the above and more results are planned to be presented later in detail.[34]

Note added in proof. By using the idea of this paper, the author has established the connection between the QPM and a q-state bond-correlated percolation model. Based on such a connection, the author has proposed a geometrical condition of phase transitions and given geometrical reasons for the variation of the specific-heat exponent α with q, the changeover from second-order to first-order phase transition as q increases, and the finite-size scaling of the specific heat at the thermal first-order phase-transition point [Chin. J. Phys. (Taipei) (in press)].

ACKNOWLEDGMENTS

The author thanks Professor J. P. Valleau, Professor S. G. Whittington, and Dr. S. L. Carnie for useful discussions and a critical reading of the paper. He is indebted to Professor M. E. Fisher for useful discussion, and Dr. C. C. Chiang, Dr. C. Y. Tseng, and Dr. W. D. Chen for suggestions on the presentation of the paper. The work was originally supported by the Natural Sciences and Engineering Research Council of Canada. The author was supported by the National Science Council of the Republic of China when the final version of the paper was completed.

[1]L. Onsager, Phys. Rev. 65, 117 (1944).

[2]H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, New York, 1971).

[3]*Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1972—1976), Vol. 1—6.

[4]S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. 53, 629 (1957); J. M. Hammersley, *ibid.* 53, 642 (1957).

[5]H. L. Frisch and J. M. Hammersley, J. Soc. Indust. Appl. Math. 11, 894 (1963).

[6]V. K. S. Shante and S. Kirkpatrick, Adv. Phys. 20, 325 (1971).

[7]J. W. Essam, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1973), Vol. 2, pp. 197—270.

[8]J. W. Essam, Rep. Prog. Phys. 43, 833 (1980).

[9]D. Stauffer, Phys. Rep. 54, 1 (1979).

[10]D. Stauffer, A. Coniglio, and M. Adam, Adv. Polym. Sci. 44, 105 (1982).

[11]*Percolation Structures and Processes,* edited by G. Deutscher, R. Zallen, and J. Adler (AIP, New York, 1983).

[12]M. E. Fisher, J. Appl. Phys. 38, 981 (1967); Physics 3, 255 (1967); Rep. Prog. Phys. 30, 615 (1967).

[13]E. Stoll, K. Binder, and T. Schneider, Phys. Rev. B 7, 2777 (1972).

[14]H. Müller-Krumbhaar, Phys. Lett. 48A, 459 (1974).

[15]K. Binder, Ann. Phys. (N.Y.) 98, 390 (1976).

[16]H. Müller-Krumbhaar, Phys. Lett. 50A, 27 (1974).

[17]T. Odagaki, N. Ogita, and H. Matsuda, J. Phys. Soc. Jpn. 39, 618 (1975).

[18]A. Coniglio, J. Phys. A 8, 1773 (1975).

[19]M. F. Sykes and D. S. Gaunt, J. Phys. A 9, 2131 (1976).

[20]A. Coniglio, C. R. Nappi, F. Perruggi, and L. Russo, J. Phys. A 10, 205 (1977), and references therein.

[21]C. Domb and E. Stoll, J. Phys. A 10, 1141 (1977); E. Stoll and C. Domb, *ibid.* 11, L57 (1978).

[22]W. Klein, H. E. Stanley, P. J. Reynolds, and A. Coniglio, Phys. Rev. Lett. 41, 1145 (1978).

[23]A. Coniglio, H. E. Stanley, and W. Klein, Phys. Rev. Lett. 42, 518 (1979).

[24]K. K. Murata, J. Phys. A 12, 81 (1979).

[25]A. Coniglio and W. Klein, J. Phys. A 13, 2775 (1980).

[26]A. Coniglio, F. di Liberto, and G. Monroy, J. Phys. A 14, 3017 (1981).

[27]G. F. Tuthill and W. Klein, J. Phys. A 15, L377 (1982).

[28]A. Coniglio and F. Peruggi, J. Phys. A 15, 1873 (1982).

[29]For definitions of critical exponents in percolation problems, see Refs. 7—9; for definitions of critical exponents in spin models, see Ref. 2. We use subscript p to denote exponents for the percolation model.

[30](a) C.-K. Hu, Physica 116A, 265 (1982); (b) 119A, 609 (1983).

[31]The exponents include β_p for the percolation probability, α_p and α'_p for the fluctuations of the number of bonds, γ_p for the mean cluster size exponent for $p < p_c$ (perhaps also γ'_p for $p > p_c$) and the exponents (ν_p, ν'_p, and η_p) related to the pair-connectedness function near or at p_c.

[32]C.-K. Hu, following paper, Phys. Rev. B 29, 5709 (1984).

[33]C.-K. Hu, J. Phys. A 16, L321 (1983).

[34]C.-K. Hu (unpublished).

[35]C.-K. Hu and P. Kleban, Bull. Am. Phys. Soc. 26, 241(A) (1981); P. Kleban and C.-K. Hu, *ibid.* 27, 92(A), (1982); 27, 325(A) (1982), and unpublished; M. E. Fisher and A. N. Berker, Phys. Rev. 26, 2507 (1982); H. W. J. Blöte and M. P. Nightingale, Physica 112A, 405 (1982); C.-K. Hu and P. Kleban, Annu. Rep. Inst. Phys. Acad. Sin. (Taiwan) 13, 3 (1983).

[36]P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. Suppl. 26, 11 (1969); C. M. Fortuin and P. W. Kasteleyn, Physica 57, 536 (1972).

[37](a) C. M. Newman and L. S. Schulman, J. Phys. A 14, 1735 (1981), and references therein; (b) H. Nakanishi and H. E. Stanley, *ibid.* 14, 693 (1981) (see footnote on page 695 of this reference). We thank Professor S. G. Whittington for locating these references.

[38]Y. Imry, Phys. Rev. B 21, 2042 (1980).

[39]See J. W. Essam, in *Phase Transitions and Critical Phenomena,* Ref. 7, page 266.

[40]C.-K. Hu and P. Kleban, Phys. Rev. B 25, 6760 (1982).

[41]F. Y. Wu, J. Phys. A 14, L39 (1981).

[42]H. E. Stanley, J. Phys. A 12, L329 (1979); H. E. Stanley and J. Teixeira, J. Chem. Phys. 73, 3404 (1980); H. E. Stanley, J. Teixeira, A. Geiger, and R. L. Blumberg, Physica 106A, 260 (1981).