Further comment on "Dislocations and melting in two dimensions: The critical region"

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(Received 31 May 1983)

An examination of the reduced-temperature range for which the scaling form for the correlation length in the critical region of a two-dimensional phase transition is valid is extended to smaller dislocation (vortex) core energies. There exists a range in which this scaling form is valid for small core energy to transition-temperature ratios. The temperature ranges in which correlation lengths in superfluids have been studied are examined.

Experiments[1-8] which have probed the correlation length ξ_+ above the critical temperature in two-dimensional systems have been analyzed by using an equation of the form[9]

$$\xi_+ \propto \exp(bt^{-\bar{\nu}}) \ , \tag{1}$$

where $t=(T-T_c)/T_c$ is the reduced temperature. Greif, Goodstein, and Silva-Moreira[10] and Cardy[11] have pointed out that for two-dimensional crystals the range of validity of this expression is restricted to $t \leq 10^{-2}$, and that for t much less than 10^{-2} the correlation length is much larger than the size of experimental samples, thereby casting doubt on whether the asymptotic form (1) can be observed directly for crystals. These authors examined parameters appropriate for a crystal composed of a monolayer of helium atoms on a graphite substrate. The range of validity of Eq. (1) depends only on the ratio E_c/T_c, where E_c is the core energy of a dislocation in a lattice or of a vortex in the case of superfluids. Since the ratio E_c/T_c varies with the form of the interatomic potential and its magnitude is difficult to estimate exactly, it is useful to extend the analysis of Refs. 10 and 11 to a range of values of E_c/T_c. We present graphically the temperature ranges for which Eq. (1) is approximately correct and for which ξ_+ is less than a typical sample size as a function of E_c/T_c for superfluids and for triangular lattices. The regions in which ξ_+ has been studied in superfluids are indicated.

According to the Kosterlitz-Thouless,[12] Halperin-Nelson,[13] and Young[14] theory, a two-dimensional crystal melts as dislocation pairs, which exist as thermal excitations in the lattice, break up to form free dislocations above T_c. For $T>T_c$, spatial correlations are assumed to decay over a characteristic length ξ_+ which is of the order of the spacing between free dislocations. It can be shown by scaling arguments that

$$\xi_+ = \xi_0 e^{l^*} \ , \tag{2}$$

where l is a parameter describing the renormalization equations, and l^* is the value of l at which $\xi_0(l)$ approaches a constant value ξ_0.

We will use the notation of Ref. 11. The renormalization trajectories are functions of two variables x and y and are given by[10]

$$(y-m_-x)^{\bar{\nu}}(y-m_+x)^{(1-\bar{\nu})} = \kappa \sim \lambda t^{\bar{\nu}} \ , \tag{3}$$

which is a solution of the renormalization-group equations

$$\frac{dx}{dl} = 12\pi^2 A y^2 \ , \tag{4}$$

$$\frac{dy}{dl} = 2xy + 2\pi B y^2 \ . \tag{5}$$

Here A, B, and λ are constants, $\bar{\nu}=6\pi^2 A m_-^2/(1+6\pi^2 A m_-^2)$, and the slopes of the separatricies are given by

$$m_\pm = [B \pm (B^2+24A)^{1/2}]/12\pi A \ . \tag{6}$$

The value of l^* is given by

$$l^* = \int_0^{l^*} dl = (12\pi^2 A)^{-1} \int_{x_1}^{x_2(l^*)} y^{-2} dx \ , \tag{7}$$

where $x_1<0$ is the starting point, i.e., $l=0$ at $x=x_1$. The starting point at $t=0$ lies on the separatrix and is given by $y_0=\exp(-E_c/T_c)$, $x_0=y_0/m_-$. The coordinates of the starting point for small values of $|t|$ are

$$y_1 = \exp\left(\frac{-E_c}{T}\right) \simeq y_0 + \frac{\delta y}{\delta t} t \simeq y_0 + \left(y_0 \frac{E_c}{T_c}\right) t \ , \tag{8}$$

$$x_1 = x_0 + \frac{\delta x}{\delta t} t = x_0 + (1+x_0) t \ . \tag{9}$$

The last equation follows from the definition of $x=dT-1$, where d is a constant which is universally related to the coupling between vortex pairs. The parameter λ is obtained by substituting the values x_1 and y_1 into Eq. (3):

$$\lambda \simeq [y_0(m_+-m_-)/|m_-|]^{(1-\bar{\nu})}\{[(E_c/T_c)-1]y_0+|m_-|\}^{\bar{\nu}} \ . \tag{10}$$

There are two systems of general interest which we examine separately: superfluids and freely floating triangular crystals, i.e., crystals for which the Lamé constants are not influenced by the substrate.

(a) Superfluids. For these systems the parameters are $A=\frac{2}{3}$, $B=0$, $\bar{\nu}=\frac{1}{2}$, and $m_\pm = \pm 1/2\pi$. The value of l^* given in Eq. (7) is

$$l^* = (12\pi^2 A m_+ \lambda t^{1/2})^{-1}[\tan^{-1}(m_+x_2/\lambda t^{1/2}) + \tan^{-1}(m_-x_1/\lambda t^{1/2})] \ . \tag{11}$$

The end point is usually taken to be $y_2 \simeq m_+x_2 \simeq 0.1$, since for small t, $\xi_0(l)$ is independent of y_2 for $y_2 \gtrsim 0.1$, and the renormalization-group equations are valid only for small values of y. A typical value of λ for superfluids is $\lambda \simeq 10^{-1}$ so that for small values of $t \ll 10^{-1}$, l^* may be written as[15]

$$l^* \simeq bt^{-1/2}[\tfrac{1}{2} + \pi^{-1}\tan^{-1}(m_{-}x_1/\lambda t^{1/2})] \ , \qquad (12)$$

where $b = 1/4\lambda$.

The coefficient of $t^{-1/2}$ in Eq. (12) varies smoothly from b to $b/2$ as t is increased. Experiments which explore a small range of reduced temperature can be fitted with the use of Eq. (1) and will yield an effective value of b with $b/2 \leq b_{\text{eff}} \leq b$.

The range of validity of Eq. (1) can more easily be observed by expanding Eq. (12) for small values of $u = (\lambda/m_{-}x_1)^2 t$ as

$$l^* = bt^{-\bar{\nu}} - \frac{1-\bar{\nu}}{2|x_1|}\sum_{n=0}^{\infty}\frac{(-1)^n u^n}{n+\bar{\nu}} \ . \qquad (13)$$

We write the parameter u in a more general form as

$$u = \frac{2(\lambda|x_1|)^{1/\bar{\nu}} t}{|m_{-}|(m_{+}-m_{-})^{(1-\bar{\nu})/\bar{\nu}}} \ . \qquad (14)$$

The first term in the summation, $1/(2|x_1|)$, is approximately constant so that only the terms for $n > 0$ give a deviation from Eq. (1). Thus Eq. (1) is approximately correct for $u < 1$.

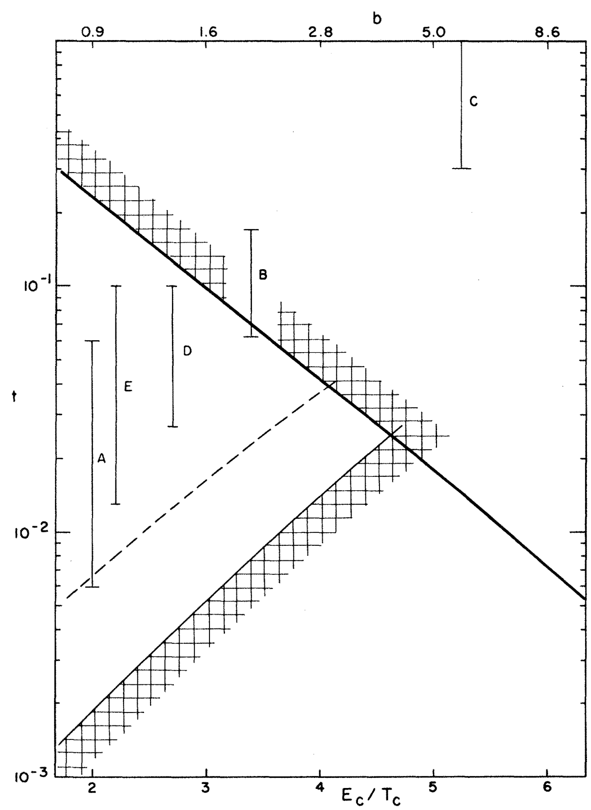

FIG. 1. Range of validity for $\xi_{+} \propto \exp(bt^{-\bar{\nu}})$ for superfluid films. The heavy line represents the values of t for which $m_{-}x_1/\lambda t^{1/2} = 1$. The fine solid and dashed lines represent, respectively, the value of t for which $\xi_{+}/a = 10^8$ and $\xi_{+}/a = 10^4$. The vertical lines denote the temperatures investigated for A-helium films (Ref. 4) and for superconductivity; B (Ref. 2); C (Ref. 3); D (Ref. 5) (sample 6); and E (Ref. 8).

In Fig. 1, the values of t for which $u = 1$ are plotted as a function of E_c/T_c. Also shown are lines representing $\xi_{+}/a = 10^4$ and 10^8, where the Debye-Hückel estimate of $\xi_0/a \simeq (16\pi y_2)^{-1}$ at $y_2 = 0.1$ was used, and a is the vortex core radius. The scale at the top of the graph gives the value of b. The vertical lines represent the range of t for experiments performed on a helium film and on superconducting films and arrays with the horizontal position of the line located at the value of b quoted in the references. Experiments on two-dimensional arrays of Josephson junctions[6] and weak links[7] were performed in a temperature range outside of the limits of Fig. 1.

(b) Freely floating triangular crystals. The parameters entering the renormalization-group equations for freely floating triangular crystals are

$$A = 21.937 \ ,$$
$$B = 6.195 \ ,$$
$$\bar{\nu} = 0.37 \ ,$$
$$m_{+} = +0.0362 \ ,$$
$$m_{-} = -0.0212 \ .$$

Cardy[11] has evaluated l^* for this case in the limit $y_1/\lambda t^{\bar{\nu}}$ and $y_2/\lambda t^{\bar{\nu}}$ large. For $y_2/\lambda t^{\bar{\nu}} \gg 1$ an approximate value of l^* is given by Eq. (13) with u defined by Eq. (14). The form of the second term on the right-hand side of Eq. (13)

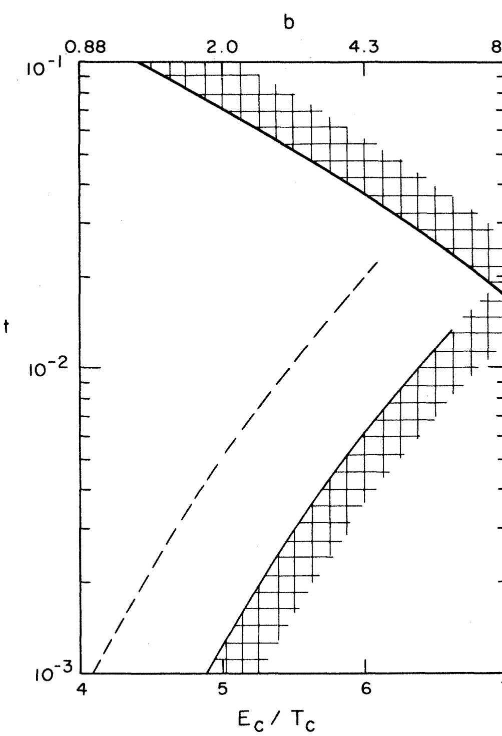

FIG. 2. Range of validity for $\xi_{+} \propto \exp(bt^{-\bar{\nu}})$ for freely floating triangular crystals. The heavy line represents the values of t for which $u = 1$. The fine solid and dashed lines represent, respectively, the values of t for which $\xi_{+}/a = 10^8$ and $\xi_{+}/a = 10^4$.

is obtained by writing Eq. (3) as

$$y^2-m_-^2x^2=\frac{\lambda^{1/\bar{\nu}}t(y+m_-x)}{(y-m_+x)^{(1-\bar{\nu})/\bar{\nu}}}\simeq\frac{2t\lambda^{1/\bar{\nu}}|m_-||x|^{-1/\bar{\nu}}}{(m_+-m_-)^{(1-\bar{\nu})/\bar{\nu}}} \quad (15)$$

and substituting this value of y^2 into Eq. (7). The value of b is given by[11]

$$bt^{-\bar{\nu}}=(12\pi^2A)^{-1}\int_{-\infty}^{\infty}y^{-2}dx \ . \quad (16)$$

It can be observed from Eqs. (3) and (16) that b scales as λ^{-1}. From Cardy's numerical evaluation of Eq. (16) one obtains $b\simeq 0.051/\lambda$. Values of λ for lattices are estimated to be $\simeq 10^{-2}$.

The values of t for which $u=1$ are presented as a function of E_c/T_c in Fig. 2 for freely floating triangular crystals. Lines are also drawn which represent reduced temperatures for which $\xi/a=10^4$ and 10^8. Here a is a lattice spacing. No experiments have been reported to date which measure ξ_+ in two-dimensional crystals.

To examine deviations from Eq. (1) it is convenient to define

$$f\equiv(l^*-bt^{-\bar{\nu}}+\text{const})/bt^{-\bar{\nu}} \ ,$$

which with the use of Eqs. (13) and (14) can be written as

$$f=cu^{\bar{\nu}}\sum_{n=1}^{\infty}\frac{(-1)^n u^n}{n+\bar{\nu}} \ , \quad (17)$$

with

$$c=(m_+-m_-)^{(1-\bar{\nu})}(|m_-|/2)^{\bar{\nu}}(1-\bar{\nu})/2b\lambda \ .$$

The value of c is $1/2\pi$ for superfluids and 0.19 for freely floating crystals. The value of f is less than 0.1 for $u=1$. Thus Eq. (1) is approximately correct for the temperature range below the heavy solid lines in the figures, and the scaling form of the correlation length can be investigated below these lines and above the region where ξ_+ is equal to the sample size. This regime is represented in both figures for a sample of size 10^8a as the region to the left of the hatched area.

We have shown that the scaling form for the correlation length above T_c can be tested on experimental samples of two-dimensional lattices for a range of ratios E_c/T_c and that corrections should be applied in extracting the parameter b from some experimental data on two-dimensional superconducting systems.

ACKNOWLEDGMENTS

I would like to thank M. A. Stan for assistance with numerical calculations. This work was supported in part by the National Science Foundation under Grant No. DMR-82-13581.

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