Lower critical dimensionality of Heisenberg spin-glasses

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Results of numerical investigations of the scaling properties of the equilibrium sensitivity to changes in the boundary conditions for two-dimensional and four-dimensional Heisenberg spin-glasses at $T=0$ are presented. The results are consistent with an algebraic size dependence of this sensitivity. They suggest that the two-dimensional system has a zero-temperature phase transition and that the four-dimensional system behaves as if it were at its lower critical dimensionality.

Recently, Banavar and Cieplak[1] introduced the concept of a scaling stiffness for spin-glasses. The scaling stiffness energy ΔE_s is a measure of the coupling energy between subsystems of the spin-glass. It has been suggested, therefore, that the scaling properties of ΔE_s can yield information about the nature of ordering and, in particular, about the lower critical dimensionality (LCD) of the system. To determine the scaling stiffness, one studies the sensitivity of a block of AL spins to changes in the boundary conditions. Here, L denotes the length and A ($\sim L^{d-1}$) the area of the block. For the sake of conceptual simplicity it is convenient to apply two different boundary conditions in the longitudinal direction and to retain the same boundary conditions (for example, periodic boundary conditions) in the transverse directions.

The scaling stiffness can be determined either in a metastable fashion, valid for short times, or in an equilibrium calculation. In the former situation one first evaluates the free energy with one type of boundary condition, or "wall potential," and then one envisions letting the system evolve to adjust to slightly modified boundary conditions. The corresponding change ΔF in the free energy is then evaluated. However, for the equlibrium calculation, the two boundary conditions need not be "close" to each other and the equilibrium free energy (or, at $T=0$, the true ground-state energy) is determined independently for both types of boundary conditions.

In a frustrated system, such as a spin-glass, there is nothing to choose between the two boundary conditions, so that in both the metastable and the equilibrium situations ΔF is equally likely to be positive or negative. The scaling stiffness energy is then defined as the root-mean-square ΔF over the distribution of the exchange constants. In a metastable calculation one should also average over different regions of the phase space (or, at $T=0$, over different ground states, for a given sample). It is clear that ΔE_s is proportional to $A^{1/2}$. The length dependence, however, requires a special analysis.

It has been shown in Ref. 1 that for the three-dimensional (3D) Heisenberg spin-glass with nearest-neighbor couplings

$$(\Delta E_s)_{\text{short times}} \sim A^{1/2}/L \sim L^{(d-3)/2} \quad . \tag{1}$$

Equation (1) has been interpreted as indicating that for short times the system behaves as if its apparent LCD were three. One may speculate that Eq. (1) should hold up to some finite freezing temperature T_f. Above T_f the system should behave as a paramagnet even on short time scales, and ΔE_s should decay exponentially with L. The short time properties of ΔE_s for the system would seem then analogous to those found in the $d=2$ XY model.[2]

On the other hand, numerical calculations[3] of the equilibrium behavior of ΔE_s for the 3D Heisenberg spin-glass at $T=0$ are consistent with

$$(\Delta E_s)_{\text{equilibrium}} \sim A^{1/2}/L^X \sim L^{(d-2x-1)/2} \quad , \tag{2}$$

where, numerically, $x \approx 2$. This suggests that the 3D system is below its LCD, implying, in turn, a $T=0$ phase transition. At nonzero temperatures $(\Delta E_s)_{\text{equilibrium}}$ should be an exponential function of L. The crossover between the algebraic and exponential laws has been demonstrated[4] exactly for a toy model of an Ising spin-glass in which the spins are located on the Sierpinski gasket. Similar crossover (but not necessarily at $T=0$) in unfrustrated random systems is discussed in Ref. 5.

In Ref. 3 the two different boundary conditions were chosen to be periodic and antiperiodic, respectively. In this case, $\Delta F = F_{\text{AP}} - F_{\text{P}}$, where F_{AP} and F_{P} denote the corresponding free energies. If one defines the characteristic free-energy difference per spin as

$$\gamma_w = \langle (\Delta F)^2 \rangle_c^{1/2} / AL \quad , \tag{3}$$

where $\langle \cdots \rangle_c$ denotes the configurational average over the distribution of the exchange constants, then

$$(\Delta E_s)_{\text{equilibrium}} = AL\gamma_w \quad . \tag{4}$$

Equation (2), written for γ_w, reads

$$\gamma_w = \sigma/(A^{1/2}L^p) \quad , \tag{5}$$

where $p = x+1$, with $p \approx 3$ in $d=3$. The parameter σ and the exponent p are quantities which, in general, could depend on the specifics of the system, including the dimensionality d and the temperature T. If p, or x, did not depend on the dimensionality and were equal to its $d=3$ value then the LCD of the system would be equal to 5. In this Brief Report we study the size dependence of ΔE_s, calculated in an equilibrium fashion, for the system in $d=2$ and $d=4$.

Our results of numerical calculations of γ_w for the two-dimensional (2D) system at $T=0$ confirm the algebraic form of γ_w, and the exponent p is found to be ≈ 3, as in the 3D system. While this in itself might suggest an exponent p independent of the dimensionality of the system, our results for the four-dimensional (4D) spin-glass show that this may not be the case. The data for the 4D system are consistent with a $p \approx \frac{5}{2}$. This, in turn, suggests that the 4D system behaves as if it were at its LCD. This result is somewhat surprising. A ferromagnet is characterized by an exponent p equal to 2. A spin-glass, on the other hand, due to the multiplicity of ground states, is better able to adjust to changes in the boundary conditions resulting in a value of p necessarily greater than or equal to 2. A decrease in the value of p on increasing the dimensionality seems to imply that the multiple ground states do not play as important a role in the 4D system. We have attempted to verify that this is not an artifact caused by the small sizes of the systems investigated. (It should be noted that the largest size studied had 6^4 spins.) The reason why p decreases on going to 4 dimensions and trends on increasing the dimensionality further are yet to be elucidated. For $d > 4$, one may speculate on two simple scenarios: Either p may remain fixed at $\frac{5}{2}$ and the spin-glass may well be above its LCD. On the other hand, p may change further with d keeping the spin-glass at its equilibrium LCD.

Two-dimensional Heisenberg spin-glass. Consider the 2D Heisenberg system given by the Hamiltonian

$$H = -\sum_{i=1}^{L-1}\sum_{j=1}^{A} J^{j}_{i,i+1}\vec{S}_{i,j}\cdot\vec{S}_{i+1,j} - \sum_{i=1}^{L}\sum_{j=1}^{A-1} J^{i}_{j,j+1}\vec{S}_{i,j}\cdot\vec{S}_{i,j+1} + H_B \ , \tag{6}$$

with

$$H_B = -\eta \sum_{j}^{A} J^{j}_{L,1}\vec{S}_{L,j}\cdot\vec{S}_{1,j} - \sum_{i=1}^{L} J^{i}_{A,1}\vec{S}_{i,A}\cdot\vec{S}_{i,1} \ . \tag{7}$$

Here η takes on the values of ± 1 for periodic and antiperiodic boundary conditions, respectively, and in the transverse direction periodic boundary conditions are imposed. The spins $\vec{s}_{ij}$ are classical unit vectors located on a square lattice. The exchange constants couple nearest neighbors, and their values are determined from a Gaussian distribution characterized by unit variance and zero mean value.

In order to determine the L dependence of γ_w at $T=0$ we investigated systems with $A=32$ and $L=4$, 6, 8, and 12. For $L=4$ and 6 we generated 40 samples (sets of the exchange couplings), and 30 samples for $L=8$ and 12. On the other hand, the A dependence was obtained for $L=8$ and $A=12$, 16, 24, and 32. For the three lowest A's we generated 45 samples. The bigger number of samples studied for the smaller-sized systems is to lower the statistical error related to the fewer spins in these systems.

Following Walker and Walstedt[6] and proceeding as in Ref. 3, "ground states" of the system for a given boundary condition were determined by starting from a random configuration of spins and aligning them sequentially in the direction of their instantaneous local fields. In the study of the L dependence, 40 initial configurations were found to be sufficient in selecting the true ground-state energy with adequate accuracy for each L. For $A=12$ there were 25 initial configurations and for $A=16$ and 24 we used 30 initial states. Sufficient equilibration was usually achieved within 500 to 1300 alignments per spin.

The quantity γ_w was obtained by calculating the root mean square of $F_{Ap}-F_p$ over the samples, where F_p and F_{Ap} denote the true ground-state energies for periodic and antiperiodic boundary conditions, respectively. The results of the calculations are shown in Fig. 1, where the error bars indicate the size of statistical error due to the finite number of samples considered. It is seen that the data points are consistent with the algebraic law described by Eq. (5) with $p \approx 3$. This implies that ΔE_s decreases to zero as $L \to \infty$, suggesting a $T=0$ phase transition.

It should be pointed out that for each value of L studied, γ_w covers several typical energy differences per spin between the ground states. As we have argued in Ref. 7, this indicates that the regime of the asymptotically large L's has already been reached for the sizes studied. For a 2D Ising spin-glass with Gaussian couplings[7] and A equal to 32, the asymptotic regime was reached at $L=8$. This difference in behavior can be explained as follows. Due to the discrete nature of the Ising spin states the energy differences between the Ising system ground states are larger, for small L's, than the ground-state separation in the Heisenberg spin-glass. On the other hand, the γ_w's for the discrete and continuous symmetry systems are comparable with each other.

Four-dimensional Heisenberg spin-glass. The 4D Heisenberg system is described by the Hamiltonian as in Eq. (7)

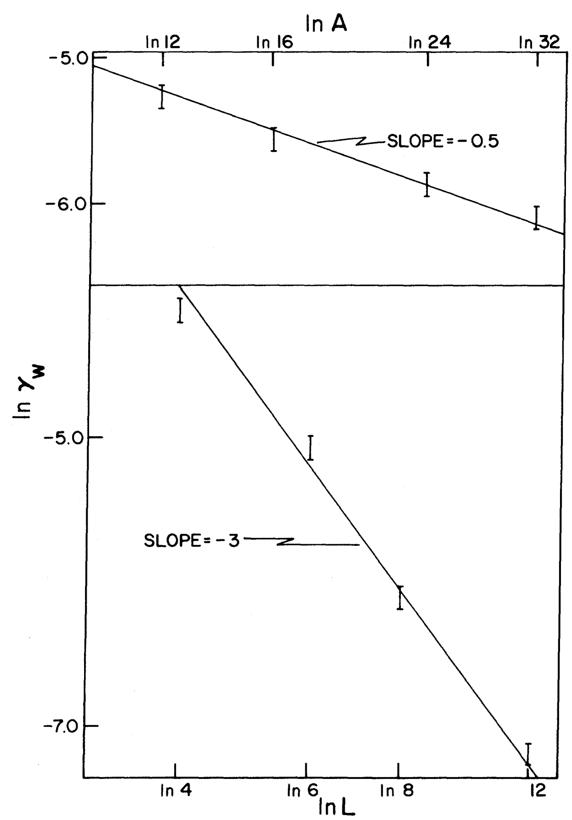

FIG. 1. Plot of $\ln\gamma_w$ vs $\ln L$ and vs $\ln A$ for the 2D Heisenberg spin-glass with Gaussian couplings. The L dependence is for $A=32$ and the A dependence is for $L=8$.

but the spins are located on the sites of a four-dimensional hypercubic lattice. Since the $A^{-1/2}$ law for γ_w is expected on general grounds and has been confirmed for many spin-glass systems, we have studied only the length dependence.

We investigated systems with $A=6\times6\times6$ and $L=4$, 5, and 6. For $L=4$ and 6 we generated 25 samples, in the manner described earlier. For $L=5$ we considered 15 samples. We selected the ground-state energy, for a given boundary condition, by starting from at least 75 initial configurations. Adequate equilibration was usually achieved within 1500 alignments per spin.

The results of the calculations are shown in Fig. 2. The error bars for $L=5$ are bigger than for the other two lengths because of the smaller number of samples which were taken into account. The results are again consistent with Eq. (5) but with $p\approx\frac{5}{2}$, suggesting that the system is at its LCD (ΔE_s is scale invariant). As shown in Fig. 2, the data points seem to rule out the algebraic decay with $p=3$. For each of the three values of L, γ_w is several times bigger than a typical difference in energy per spin between the lowest-lying "ground states." Judging at least by this criterion, the systems studied seem to be large enough to obtain the asymptotic L dependence.

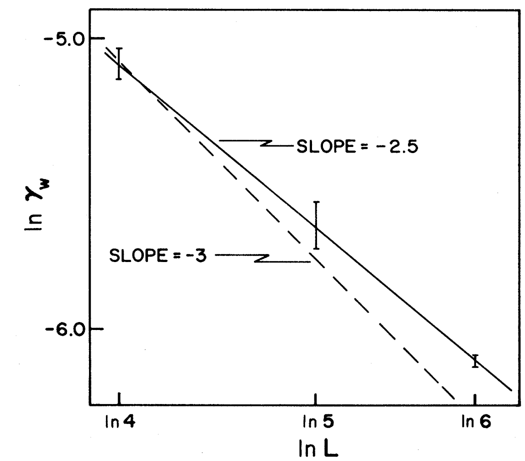

FIG. 2. Plot of $\ln\gamma_w$ vs $\ln L$ for the 4D Heisenberg spin-glass with Gaussian couplings. The L dependence is for $A=6\times6\times6$. The data points are consistent with $P=\frac{5}{2}$ (solid line). The broken line represents the behavior of γ_w if p were equal to 3.

The basic idea of probing the nature of ordering by studying the sensitivity of systems to changes in boundary conditions may be applicable to a variety of magnetic systems. Such a technique may be useful even in instances when there is no obvious ordering but only an algebraic decay of correlations. The crucial point is that the boundary conditions should couple to the long-range correlations. The ultimate value of such a picture would be if a scaling relation for σ appearing in Eq. (5) could be worked out in analogy with the work of Abrahams, Anderson, Licciardello, and Ramakrishnan,[8] who worked out a scaling description of the localization transition.

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