Excitation spectrum for vibrations on a percolating network: Effective-medium approximation

Ora Entin-Wohlman,* S. Alexander,† R. Orbach, and Kin-Wah Yu
Department of Physics, University of California, Los Angeles, California 90024
(Received 14 October 1983)

The excitation spectrum for vibrations on a (bond-) percolating network are calculated with the use of an effective-medium approximation. For $2<d<4$, where d is the Euclidean dimensionality of the embedding space, we find a nearly linear relationship between frequency and wave vector for $\omega<\omega_c$, where ω_c represents the critical frequency separating phonon and fracton regimes as calculated previously by Derrida, Orbach, and Yu. The imaginary part of ω is small for $\omega<\omega_c$, signifying the correctness of a phonon eigenstate description in that regime. As the wave vector increases beyond the value corresponding to ω_c, a plane-wave extended-state representation fails, signaled by a rapidly growing imaginary part of the frequency. It is interesting that an effective-medium approximation can sense the transition between extended and localized states. We calculate the ω dependence of what we characterize as the localization length $l(\omega)$. We find $\omega\sim l^{-2}$ for $\omega<\omega_c$ in agreement with the scaling form generated by Alexander and Orbach. The length $l(\omega)$ diverges for $\omega<\omega_c$, as it should for wavelike excitations. Finally, we calculate the excitation spectrum for $1<d<2$, where Derrida *et al.* have shown that no sharp crossover occurs between phonon and fracton regimes. We expect both regimes to be localized. We find a smooth degradation of phonon character as ω increases, and a gradual transition to states with fracton character.

I. INTRODUCTION

In a recent article, Alexander *et al.*[1] have suggested that the thermal properties of Epoxy resin and other amorphous materials can be understood on the basis of a crossover from phonon excitations to amomalous vibrational excitations, referred to as fractons.[2] Their treatment assumes the existence of a crossover length and frequency $\omega_{c.o.}$, such that the appropriate eigenstates for $\omega<\omega_{c.o.}$ (long length scales) are phonons. At frequencies above $\omega_{c.o.}$ (short length scales) fractal geometry[3] leads to "anomalous" localized fracton states. This idea has been further developed by Tua *et al.*[4] who suggested a double-valued form for the excitation spectrum near $\omega_{c.o.}$. This leads to a jump in the vibrational density of states $N(\omega)$ at the crossover frequency $\omega_{c.o.}$. This description was based on a scaling analysis of the vibrational spectrum for a percolation model of disorder.[1,2,5] No direct studies of the spectrum and density of states are available, however. Derrida *et al.*[6] have very recently studied the density of states of lattice vibrations for a bond-percolation model within the effective-medium approximation (EMA).[7–9] Their results confirm both the general crossover ideas of Ref. 1 and the matching form of Tua *et al.*[4] The density-of-states curve resulting from their calculations exhibits a rapid increase in $N(\omega)$ at a critical frequency ω_c, which we shall identify with the crossover frequency $\omega_{c.o.}$, consistent with the proposal of Ref. 4.

As predicted,[1,5] Derrida *et al.* found the density of states to be phononlike at frequencies below ω_c, with $N(\omega)\propto\omega^{d-1}$ (d is the Euclidean dimensionality of the embedding space). A rapid increase in $N(\omega)$ at ω_c (for $2<d<4$) is observed, with $N(\omega)$ approaching ω^{d-1} at higher frequencies. The crossover frequency scales as $p-p_c$, where p and p_c are the percolation concentration and the critical percolation concentration, respectively.

The agreement of the phenomenological form for the vibrational density of states of Tua *et al.*[4] with the EMA results of Derrida *et al.*[6] leads one to ask whether the dispersion of the vibrational modes (ω vs q,l^{-1}) is also reproduced within the EMA (l is a measure of the fracton localization length; see below). To our surprise, we find this to be the case. The low-frequency modes have phononlike dispersion ($\omega\propto q$), and can be well described by a wave vector. They exist up to a crossover wave vector for which $q_{c.o.}\xi_p\cong 1$ [$\xi_p\propto(p-p_c)^{-1/2}$, the percolation correlation length]. At higher frequencies (shorter length scales), the modes have a different (fracton) dispersion, and are localized. While the crossover (within the EMA) is continuous, it is consistent with the crossover form hypothesized by Tua *et al.*[4] The fact that the EMA exhibits these crossover effects, from extended phonons to localized fractons with a different dispersion, is the main result of this paper.

We calculate here the dispersion law for vibrational excitations within the EMA (for bond percolation). This amounts to solving equations of the form

$$\omega^2=2W(-\omega^2)[1-\cos(qa)]$$

for the relationship between q and ω, where $W(-\omega^2)$ is determined from the EMA equations. In practice, we either keep q real (extended states) and solve for the (in general) complex ω^2, or keep ω^2 real and allow q to be complex (localized states). Throughout we restrict ourselves to small q ($qa\ll 1$) for which a phonon description would ordinarily be adequate.

We find that within the EMA for $2<d<4$, the plane-wave representation works well for excitations with

$\omega < \omega_c$. For q higher than the wave vector corresponding to ω_c ($q\xi_p > 1$), the "damping" becomes excessive, indicating a poor choice of representation. Conversely, $l(\omega)$ appears to diverge in the (extended-state) phonon regime, leading to a sensible relation between ω and l^{-1} only in the fracton regime. We stress that the "overdamping" of the plane-wave form is not a lifetime effect. Rather, it is simply an indication that we have chosen a poor (i.e., extended) representation for the excitations. Similar statements can be made conversely in the (localized-state) fracton regime. The overall structure of the "dispersion" law ω vs q, l^{-1} is found to be similar to the one proposed by Tua *et al.*[4]

In Sec. II we use the basic results of the EMA, as derived by Odagaki and Lax,[8] and by Webman,[9] to derive the dispersion law. We do this in detail for $2 < d < 4$, and comment upon the behavior for $1 < d < 2$. We also derive the frequency-dependent localization length at $d = 3$. We summarize our results in Sec. III.

II. THE EXCITATION SPECTRUM

The EMA (Refs. 6 and 7) provides us with a self-consistent solution for the probability density $P(\vec{s},t \mid \vec{s}_0)$ that a diffusing particle, initially at site $\vec{s}_0$, will be at site $\vec{s}$ at time t. The detailed calculation of this quantity has been carried out by Odagaki and Lax[8] and by Webman.[9] Here, we outline their results. It is assumed that $P(\vec{s},t \mid \vec{s}_0)$ obeys the following master equation:

$$\frac{\partial P(\vec{s},t \mid \vec{s}_0)}{\partial t} = -\sum_{\vec{s}'} W_{\vec{s}',\vec{s}} P(\vec{s},t \mid \vec{s}_0) + \sum_{\vec{s}'} W_{\vec{s},\vec{s}'} P(\vec{s}',t \mid \vec{s}_0) , \quad (1)$$

where the transition rate $W_{\vec{s},\vec{s}'}$ is a random quantity. Although the EMA is formulated in terms of the diffusion problem, it maps upon a variety of physical problems,[10,11] including that of lattice vibrations. In the vibrational problem, the transition rate transforms into the force constant divided by the mass, and the Laplace-transform parameter u corresponding to t is replaced by $-\omega^2$ upon analytic continuation.[10] We note that the model we consider here is the random force-constant problem. All (atomic) masses are implicitly assumed equal.

In the symmetric case, where $W_{\vec{s},\vec{s}'} = W_{\vec{s}',\vec{s}}$, and for bond percolation, the transition rate is distributed according to

$$f(W_{\vec{s},\vec{s}'}) = p\delta(W_{\vec{s},\vec{s}'} - W_0) + (1-p)\delta(W_{\vec{s},\vec{s}'}) , \quad (2)$$

where p is the percolation concentration. The EMA yields the propagator $\bar{P}(\vec{s},u \mid \vec{s}_0)$, the Laplace transform of the probability density in the effective (periodic) medium (for a d-dimensional simple-cubic lattice)

$$\bar{P}(\vec{s},u \mid \vec{s}_0) = \frac{1}{(2\pi)^d} \int dq_1 \cdots dq_d \frac{\exp[-i\vec{q}\cdot(\vec{s}-\vec{s}_0)]}{u + 2\overline{W} \sum_{i=1}^{d} (1-\cos q_i)} . \quad (3)$$

Here u is the spectral parameter, d is the dimension of the Euclidean space, and $\overline{W}$ is the effective coherent transition rate, $\overline{W} = \overline{W}(u)$. The last quantity is determined self-consistently from

$$(1/d)(\overline{W}-1)[1 - u\bar{P}(\vec{s},u \mid \vec{s}_0)] - \overline{W} + p = 0 , \quad (4)$$

where $\overline{W}(u)$ and u are measured in units of W_0.

Equations (3) and (4) are the basic results of the EMA, from which various quantities can be calculated. Here we shall use them to determine first the vibrational excitation dispersion law ω vs q. This is accomplished as follows. The vibrational propagator is obtained from $\bar{P}(\vec{s},u \mid \vec{s}_0)$ by analytic continuation $u \to -\omega^2$. We then find that the (spatial) Fourier transform of the propagator, from Eq. (3), is

$$\bar{P}(q,\omega) = \left[-\omega^2 + 2\overline{W}(\omega) \sum_{i=1}^{d} (1-\cos q_i) \right]^{-1} . \quad (5)$$

The poles of this propagator (in the lower half of the complex frequency plane) yield the frequency and apparent decay rate of the excitations.

To simplify the calculations, we use the Debye approximation

$$2 \sum_{i=1}^{d} (1-\cos q_i) \cong q^2 . \quad (6)$$

Then the poles of the propagator (5) are given by

$$\omega(q) = \omega_q - i\delta_q , \quad \omega^2(q) = q^2 \overline{W}(\omega(q)) , \quad (7)$$

where ω_w and δ_q denote the wave-vector-dependent frequency and the decay rate of the wavelike excitations, respectively.

To obtain $\omega(q)$ from Eq. (7), we need an explicit expression for $\overline{W}(u)$, to be obtained from Eq. (4) in conjunction with Eq. (3). From Eq. (4) it is seen that $\overline{W}/(p-p_c)$ remains finite as $p \to p_c$, where, within the EMA,

$$p_c = 1/d . \quad (8)$$

However, the frequencies which will be of interest, as is shown below, are of the order $p - p_c$. That is, $u/\overline{W} \propto p - p_c$. Consequently,[6] $u/\overline{W} \ll 1$. Expanding $P(\vec{s},u \mid \vec{s})$ [Eq. (3)] in powers of $u/\overline{W}$, we find[6]

$$\bar{P}(\vec{s},u \mid \vec{s}) \cong \frac{1}{2\overline{W}} \psi_d - \frac{1}{u} \left[\frac{u}{\overline{W}} \right]^{d/2} \chi_d , \quad 2 < d < 4 \quad (9)$$

and

$$\bar{P}(\vec{s},u \mid \vec{s}) = \frac{1}{u} \left[\frac{u}{\overline{W}} \right]^{d/2} \tilde{\chi}_d , \quad 1 < d < 2 , \quad (10)$$

where ψ_d, χ_d, and $\tilde{\chi}_d$ are numbers of order unity. Their explicit values are given in Ref. 6. Insertion of Eqs. (9) and (10) into Eq. (4) yields the desired equation for $\overline{W}$.

We shall now restrict our detailed discussion to the region $2<d<4$ [Eq. (9)]. From Eqs. (4) and (9) we have

$$p_c\left[\frac{\omega^2(q)}{2\overline{W}}\psi_d+\left[-\frac{\omega^2(q)}{\overline{W}}\right]^{d/2}\chi_d\right]=p-p_c-(1-p_c)\overline{W}\ . \tag{11}$$

In the regime $2<d<4$, $(-1)^{d/2}$ is a complex number. We accordingly denote

$$\chi_d(-1)^{d/2}=\chi_d^{(1)}+i\chi_d^{(2)}\ . \tag{12}$$

Substituting $\overline{W}=\omega^2(q)/q^2$, from Eq. (7) we find

$$\chi_d^{(2)}q^{d+2}=2[(1-p_c)/p_c]\omega_q\delta_q \tag{13}$$

and

$$(1-p_c)\omega_q^4-\left[(p-p_c)q^2-p_c\frac{\psi_d}{2}q^4-p_c\chi_d^{(1)}q^{d+2}\right]\omega_q^2 -p_c^2(\chi_d^{(2)})^2q^{2d+4}/4(1-p_c)=0\ . \tag{14}$$

Equation (14) gives us ω_q, the real part of $\omega(q)$, as a function of q. Inserting it into Eq. (13) yields the decay rate δ_q.

In the small-q limit we find that

$$\omega_q^2\sim\frac{p-p_c}{1-p_c}q^2,\quad p>p_c$$
$$\frac{\delta_q}{\omega_q}\sim\frac{q^d}{p-p_c},\quad p>p_c\ . \tag{15}$$

That is, the dispersion law for small q (large length scale) is that of phonons, $\omega_q\sim cq$, with a sound velocity c scaling as $(p-p_c)^{1/2}$. The decay rate in this limit is much smaller than ω_q so that the excitations are well defined. However, this holds only for $p>p_c$ and for small q such that

$$q^2<q_c^2,\quad q_c^2=2(p-p_c)/(p_c\psi_d)\ . \tag{16}$$

We depict the full range of ω vs q in Fig. 1 for the case $d=3$ and $p\sim p_c$. We see that ω_c, as defined by Derrida *et al.*,[6] represents the maximum value of ω in the phonon regime (i.e., for $q<q_c$). Explicitly,

$$\omega_c^2=(p-p_c)^2/[2\psi_d p_c(1-p_c)]\ . \tag{17}$$

Furthermore, the decay rate [i.e., the imaginary part of $\omega(q)$] increases remarkably fast. For $q>q_c$, $\delta_q\gg\omega_q$. This means that the poles of the propagator (5) do not lie close to the real axis, and hence do not describe a physically accessible excitation spectrum. The frequency ω_c was shown[6] to mark the crossover between the phonon and the fracton portions of the vibrational density of states within the EMA. Our calculation shows that at frequencies less than ω_c, or $q<q_c$, the plane-wave representation which yields a dispersion law of $\omega=cq$ is indeed meaningful. However, as q increases beyond q_c, the decay rate δ_q increases sharply. Consequently, the plane-wave representation in which the propagator (5) was derived is no longer adequate. We note that this breakdown of the plane-wave representation occurs at q of the order of q_c, which scales as $(p-p_c)^{1/2}$. As shown in Ref. 6, this is related within the EMA to the excitation length scale equaling the connectivity length of the percolation problem, in accordance with the conjecture of Ref. 2. That is, $1/q_c\sim\xi_p$, the percolation correlation length. Hence, ξ_p sets the length scale. For $1/q>\xi_p$, one is in the phonon regime. For $1/q<\xi_p$, one is in the fracton regime. The critical frequency ω_c scales as $p-p_c$ [Eq. (17)]. Noting that c scales as $(p-p_c)^{1/2}$, we see that ω_c scales as cq_c, so that ω_c can be identified with $\omega_{\text{c.o.}}$ of Refs. 1 and 2.

In order to see what excitation spectrum applies at frequencies higher than ω_c, we return to Eq. (3). In $d=3$ and within the Debye approximation, the integration is easily carried out. The result is

$$\bar{P}(\vec{s},-\omega^2\,|\,\vec{s}_0)=\frac{1}{4\pi R}\frac{1}{\overline{W}(\omega)}e^{i\{\omega/[\overline{W}(\omega)]^{1/2}\}R},\qquad R=|\vec{s}-\vec{s}_0|\ . \tag{18}$$

The absolute value of $\mathrm{Im}\{\omega/[\overline{W}(\omega)^{1/2}\}$ defines an inverse length scale, which we denote by $l(\omega)$,

$$\frac{1}{2l(\omega)}=\left|\mathrm{Im}\left[\frac{\omega}{[\overline{W}(\omega)]^{1/2}}\right]\right|\ . \tag{19}$$

Here, $l(\omega)$ represents a mean free path which, in the $q>q_c$ regime, we interpret as a frequency-dependent localization length. At low frequencies it is very large, in accordance with the validity of a plane-wave representation at that length scale. However, at frequencies greater than ω_c, $l(\omega)$ becomes substantially smaller. From Eqs. (4) and (9), it can be shown that, to leading order,

$$\left|\mathrm{Im}\left[\frac{\omega}{[\overline{W}(\omega)]^{1/2}}\right]\right|^2=\left[\frac{1-p_c}{2\psi_d p_c}\right]^{1/2}(\omega-\omega_c)\ ,\qquad \omega>\omega_c\ . \tag{20}$$

This leads to the relationship

$$\omega=\omega_c+\left[\frac{\psi_d p_c}{8(1-p_c)}\right]^{1/2}\left[\frac{1}{l^2}\right]\ . \tag{21}$$

It is very interesting to compare this result with the dispersion law at small length scales [fracton (fr) regime], as conjectured from scaling.[2] One finds

$$\omega_{\text{fr}}(l)\propto l^{-(2+\theta)/2}\ . \tag{22}$$

Thus, Eq. (21) and[2,6] $\omega_c\propto(p-p_c)\propto\xi_p^{-2}$ both imply consistently that $\theta=2$ in the EMA. We note that this is different from the mean-field result[2] (valid for $d\geq 6$) which gives $\theta=4$.

In order to demonstrate the full dispersion law we note the following. From Eq. (18), we see that the quantity $\omega/[\overline{W}(\omega)]^{1/2}$ represents an inverse length such that its real part corresponds to the wave vector in the phonon regime (where its imaginary part is small) and its imaginary part corresponds to l^{-1} in the fractal regime (where the real part is of the order of the imaginary part). Consequently, we plot, in Fig. 2, ω versus the imaginary and real parts of $\omega/[\overline{W}(\omega)]^{1/2}$. We see that this yields the phonon dispersion law at low frequencies, and the fracton

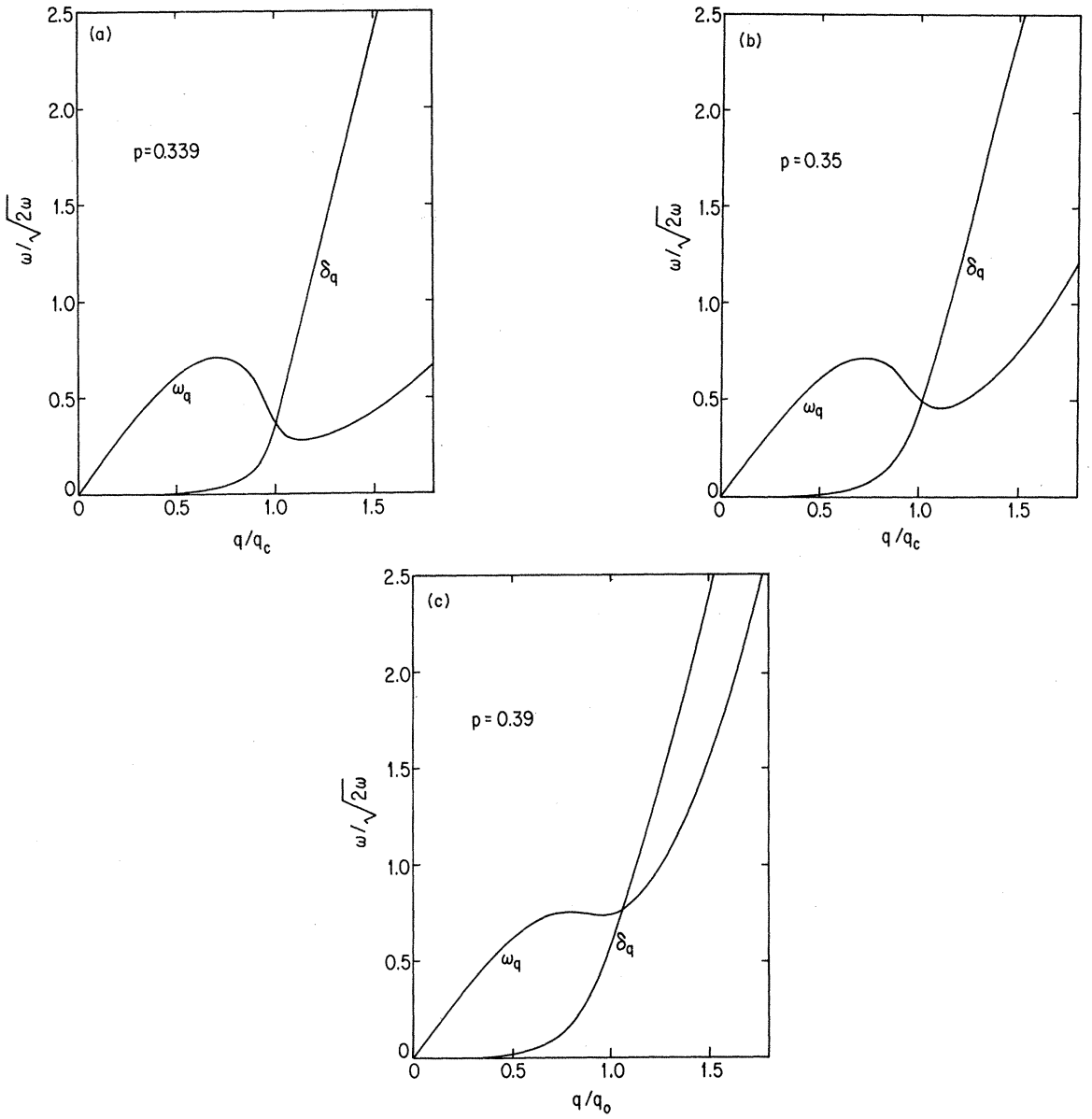

FIG. 1. Plot of the real and imaginary parts of ω, ω_q, and δ_q, respectively, vs q for $d=3$ from Eqs. (13) and (14). The normalizing quantities ω_c and q_c are defined by Eqs. (17) and (16), respectively. The critical percolation concentration $p_c=\frac{1}{3}$. (a) $p=0.339$: $(p-p_c)/p_c=0.017$, $\omega_c=0.0104$, and $q_c=0.226$. (b) $p=0.35$: $(p-p_c)/p_c=0.05$, $\omega_c=0.0306$, and $q_c=0.387$. (c) $p=0.39$: $(p-p_c)/p_c=0.17$, $\omega_c=0.104$, and $q_c=0.714$.

dispersion law [Eq. (21)] at high frequencies, with a crossover at ω_c.

A word should be said about the comparison of Figs. 1 and 2. The zeros of the expression in large parentheses on the right-hand side of Eq. (5) represent a mapping of the complex q plane onto the complex ω^2 (or ω) plane. If one had a truly periodic problem, real q would map onto real (positive) ω^2, giving the usual type of q,ω dispersion relation. For the density of states, and for dispersion, one is interested of course in real ω but not necessarily extended states (real q). Within the EMA one can always associate a complex q with real ω, or vice versa. In Fig. 1, q is taken as real and ω is allowed to be complex. This is appropriate if one is interested in plane-wave solutions. Figure 2 refers, however, to the converse: ω is taken as real and the inverse length $\omega/[\overline{W}(\omega)]^{1/2}$ is complex. We identify the localization length, $1/[2l(\omega)]$, with $\mathrm{Im}\{\omega/[\overline{W}(\omega)]^{1/2}\}$. At $\omega<\omega_c$, $1/[2l(\omega)]$ is very small, implying extended states appropriate to the phonon regime. At $\omega>\omega_c$, $1/[2l(\omega)]$ is quite finite, and can be identified with the range of the envelope function which specifies the characteristic length of the localized fracton state with energy ω. For $\omega<\omega_c$, $\mathrm{Re}\{\omega/[\overline{W}(\omega)]^{1/2}\}$ is well behaved, and is just q. That is, the eigenstate is in the phonon regime. For $\omega>\omega_c$, $\mathrm{Re}\{\omega/[\overline{W}(\omega)]^{1/2}\}$ roughly equals $1/[2l(\omega)]$, a condition appropriate to critical damping of the fracton modes. A convenient way to plot a significant relationship between ω and $\omega/[\overline{W}(\omega)]^{1/2}$ is to keep ω real and plot it against $|\omega/[\overline{W}(\omega)]^{1/2}|$. This is done in Fig. 3. One sees this is a reasonable way to inter-

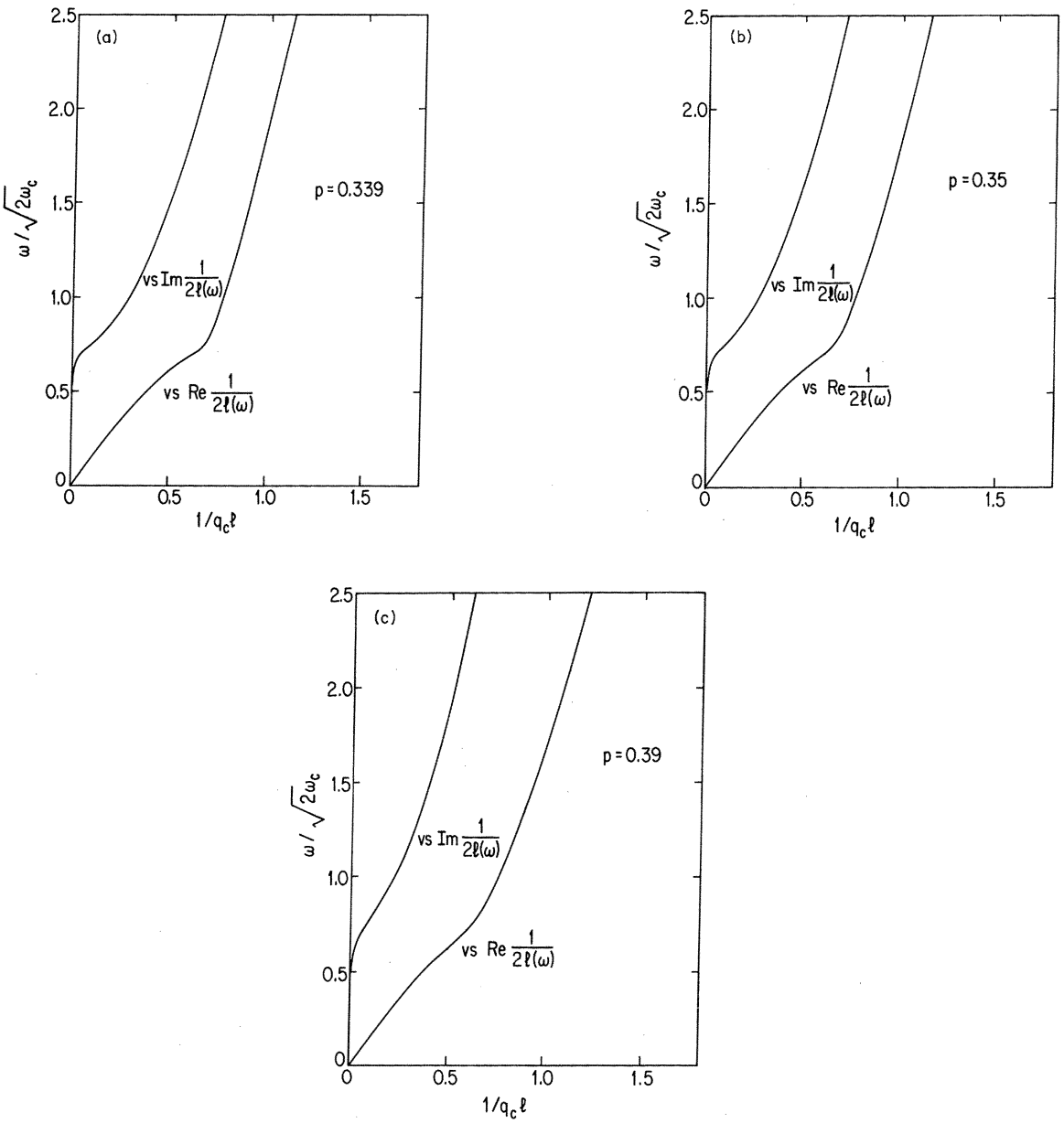

FIG. 2. Plot of ω vs the real and imaginary parts of $\omega/[\overline{W}(\omega)]^{1/2}$, the localization length, for $d=3$ from Eqs. (4) and (9). The normalizing quantities ω_c and q_c are given in the caption of Fig. 1. (a) $p=0.339$, (b) $p=0.35$, and (c) $p=0.39$.

polate between the phonon and fracton regimes.

Finally, we consider the dispersion law in the case $1<d\leq 2$. From Eqs. (4) and (10) we obtain

$$-p_c\left[-\frac{\omega^2(q)}{\overline{W}}\right]^{d/2}\tilde{\chi}_d=p-p_c-(1-p_c)\overline{W}\ . \tag{23}$$

Denoting

$$\tilde{\chi}_d(-1)^{d/2}=\tilde{\chi}_d^{(1)}-i\tilde{\chi}_d^{(2)} \tag{24}$$

and inserting $\overline{W}=\omega^2(q)/q^2$ from Eq. (7), we find

$$\tilde{\chi}_d^{(2)}q^{d+2}=2\frac{1-p_c}{p_c}\omega_q\delta_q \tag{25}$$

and

$$(1-p_c)\omega_q^4-[(p-p_c)q^2+p_c\tilde{\chi}_d^{(1)}q^{2+d}]\omega_q^2 -p_c^2(\tilde{\chi}_d^{(2)})^2q^{2d+4}/4(1-p_c)=0\ . \tag{26}$$

It is seen that for $p>p_c$, ω_q is a monotonically increasing function of q. Namely, the "double-valuedness" feature of the frequency versus q (see Fig. 1) is missing here. This is reflected in the fact that the density of states at dimension $d\leq 2$ does not show a rapid increase[6] at ω_c as opposed to the regime $2<d<4$. The ratio between the decay rate and the frequency ω_q from Eqs. (25) and (26) is

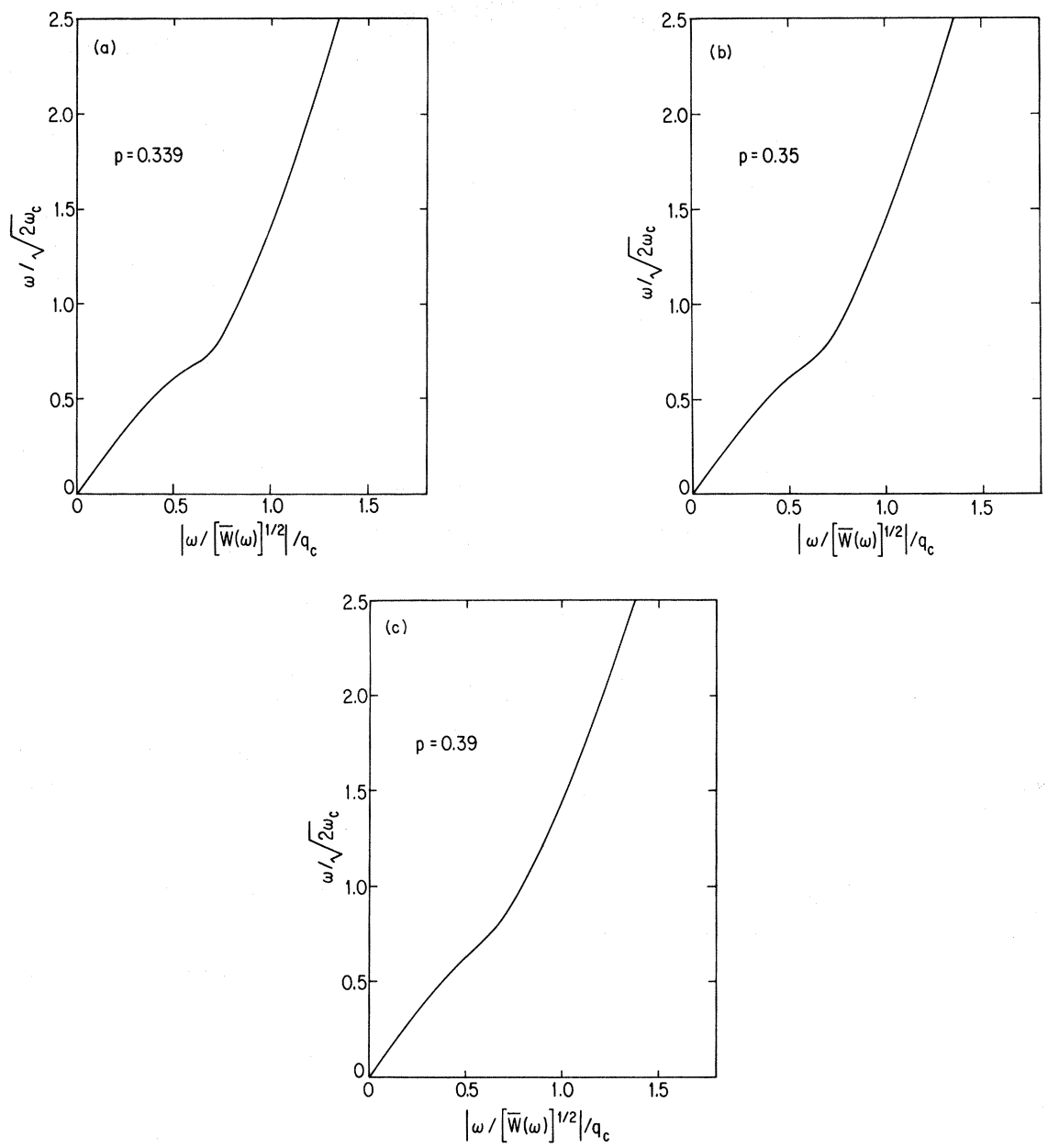

FIG. 3. Plot of ω vs the absolute value of $\omega/[\bar{W}(\omega)]^{1/2}$, the localization length, for $d=3$ from Eqs. (4) and (9). The normalizing quantities ω_c and q_c are given in the caption of Fig. 1. (a) $p=0.339$, (b) $p=0.35$, and (c) $p=0.39$.

$$\frac{\delta_q}{\omega_q}=\tilde{\chi}_d^{(2)}p_c q^d\{p-p_c+p_c\tilde{\chi}_d^{(1)}q^d + [(p-p_c+p_c\tilde{\chi}_d^{(1)}q^d)^2 + p_c^2(\tilde{\chi}_d^{(2)})^2 q^{2d}]^{1/2}\}^{-1} . \quad (27)$$

The right-hand side is seen here as a monotonically increasing function of q^d. This means that as q increases, the poles of the propagator (5) move away from the real axis. However, a natural crossover frequency does not seem to exist, in contrast with the case for $2<d<4$. We note that at small q, Eq. (26) does imply a phononlike dispersion law, with the sound velocity scaling as $(p-p_c)^{1/2}$.

III. CONCLUSIONS

We have shown, within the EMA, that the general form for ω vs $q, 1/l$ hypothesized earlier[1,4,5] does in fact obtain for vibrations on a percolating network. Forcing a plane-wave representation yields sensible results only for small q (large length scales), or $\omega<\omega_c$, the critical frequency exhibited by Derrida *et al.*[6] Indeed, ω vs q exhibits zero slope at the value $\omega=\omega_c$, suggestive of a feature in $N(\omega)$, as indeed found by Derrida *et al.*[6] For wave vectors higher than q_c, corresponding to $\omega>\omega_c$ (shorter length

scales) within the plane-wave representation, the decay rate δ_q increases precipitously, implying that the plane-wave representation fails. We then adopted a localized representation in terms of an inverse localization length $1/l(\omega)$. We found for $d=3$ that $\omega \sim 1/l^2$ in complete agreement with the scaling results,[1,2,4,5,12] $\omega \sim l^{-\bar{d}/\bar{d}}$ when the EMA value $\theta=2$ (for $d=3$) is used. Again, for $\omega<\omega_c$, $l(\omega)$ diverges, implying that a localized picture within the EMA is inappropriate, and that the excitations are more appropriately extended phonon states. We have exhibited in Fig. 3, therefore, an ω-vs-$|\omega/[\overline{W}(\omega)]^{1/2}|$ excitation spectrum which can be thought of as generating the $N(\omega)$ derived previously by Derrida *et al.*[5]

For $1<d<2$, Derrida *et al.*[6] do not find a rapid increase in $N(\omega)$ at ω_c. We too find no feature in the character of our solutions across q space, consistent with their findings.

It is remarkable that the EMA which, after all, replaces all unoccupied and occupied bonds with a uniform frequency-dependent bond, can mirror the character of the vibrational eigenstates so clearly. It would be of great value now to simulate the vibrating percolating network directly to obtain explicit eigenfunctions in order to test the EMA conclusions. It would also be of great interest to extend experimental phonon dispersion and "lifetime" measurements in amorphous systems into the frequency regime near ω_c to see if our predicted sharp increase in δ_q is found. This is predicted in Refs. 1 and 4. The experiments of Dietsche and Kinder[13] in fused silica appear to come close to ω_c from the estimates of Ref. 1.

ACKNOWLEDGMENTS

The authors wish especially to thank Dr. B. Derrida for many stimulating conversations and criticisms. We are also indebted to Dr. I. Webman and Professor T. Holstein for informative discussions. This research was supported by the National Science Foundation under Grant No. DMR 81-15542.

---

*Permanent address: Physics Department, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel.

†Permanent address: The Racah Institute, The Hebrew University, Jerusalem 91904, Israel.

[1]S. Alexander, C. Laermans, R. Orbach, and H. M. Rosenberg, Phys. Rev. B 28, 4615 (1983).

[2]S. Alexander and R. Orbach, J. Phys. (Paris) Lett. 43, L625 (1982).

[3]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[4]P. Tua, S. Putterman, and R. Orbach, Phys. Lett. 98A, 357 (1983).

[5]S. Alexander, Ann. Isr. Phys. Soc. 5, 144 (1983).

[6]B. Derrida, R. Orbach, and Kin-Wah Yu, Phys. Rev. B (to be published).

[7]S. Kirkpatrick, Rev. Mod. Phys. 45, 574 (1973).

[8]T. Odagaki and M. Lax, Phys. Rev. B 24, 5284 (1982).

[9]I. Webman, Phys. Rev. Lett. 47, 1496 (1981).

[10]S. Alexander, J. Bernasconi, W. Schneider, and R. Orbach, Rev. Mod. Phys. 53, 175 (1981).

[11]S. Alexander, Phys. Rev. B 27, 1541 (1983).

[12]R. Rammal and G. Toulouse, J. Phys. (Paris) Lett. 44, L13 (1983).

[13]W. Dietsche and H. Kinder, Phys. Rev. Lett. 43, 1413 (1979).