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Domain-wall renormalization-group study of the two-dimensional random Ising model

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The distribution of domain-wall free energies is calculated with the use of a transfer-matrix approach for finite lattices. A renormalization-group transformation is set up which preserves the domain-wall free-energy distribution as well as the susceptibility when the lattice parameter is changed. The fixed points, critical exponents, and phase diagram are determined for the two-dimensional random Ising model with a Gaussian distribution of nearest-neighbor interactions.

I. INTRODUCTION

The two-dimensional random Ising model has been studied by Monte Carlo,[1,2] transfer-matrix,[3] and cluster-quench[4] simulation techniques. A phenomenological scaling theory has been presented by the author.[5]

In this paper a macroscopic renormalization-group (RG) method is set up in which one calculates macroscopic physical quantities as a function of lattice parameter. The Hamiltonian parameters are then renormalized to preserve the physical quantities as the lattice parameter is varied. This approach is in contrast to the usual microscopic RG methods, such as the block-spin method,[6] in which one specifies a transformation of the microscopic spin variables. The physical quantities which we choose to preserve are the distribution of domain-wall free energies (actually, the mean and variance) and the mean susceptibility. For the two-dimensional random Ising model these quantities can be calculated numerically using a transfer-matrix technique.[7] Once the RG transformation has been set up, we use the standard RG methodology to find the fixed points, the critical exponents, and the phase diagram. Near the fixed points this approach is related to finite-size scaling[8] on the same physical quantities.

II. DOMAIN-WALL RG

Consider an $n\times n$ lattice of Ising spins with lattice spacing a and lattice size $L=an$. In the x direction the lattice is replicated periodically to form a long strip suitable for the transfer-matrix method. In the y direction we choose either periodic or antiperiodic boundary conditions. The Hamiltonian is

$$H=-\sum_{i,j}J_{ij}S_iS_j-h\sum_i \mu S_i \ , \tag{1}$$

with nearest-neighbor interactions J_{ij} chosen from a Gaussian distribution with mean $\bar{J}$ and variance $\tilde{J}$. The partition function

$$Z=\sum_{\{S_i=\pm 1\}}\exp(-H/T) \tag{2}$$

can be found from the largest eigenvalue of the transfer matrix.

We can set up a domain wall by using antiperiodic boundary conditions in the y direction; this forces a spin reversal over length scale L. Within a block-spin picture, the domain-wall free energy is proportional to the interaction free energy of block spins of length scale L. Clearly, the domain-wall free energy is a relevant physical quantity that one wants to preserve under the RG transformation. The free energy of a domain wall in zero field is

$$W_n(\bar{K},\tilde{K})\equiv -T\ln(Z_a/Z_p) \ , \tag{3}$$

where subscripts p or a indicate periodic or antiperiodic boundary conditions and the dimensionless Hamiltonian parameters are $\bar{K}=\bar{J}/T$ and $\tilde{K}=\tilde{J}/T$. The susceptibility is

$$\chi_n(\bar{K},\tilde{K})\equiv -\frac{T}{2}\frac{\partial^2}{\partial h^2}\ln(Z_p) \ . \tag{4}$$

For the pure Ising model ($\tilde{J}=0$) these physical quantities are unique. However, for the random Ising model ($\tilde{J}>0$) on a finite lattice, the physical quantities depend on the interactions J_{ij} and the quantities are distributed. We find the mean $\bar{W}$ and variance $\tilde{W}$ of the domain-wall free-energy distribution and the mean susceptibility $\bar{\chi}$ by calculating the physical quantities for several thousand configurations of interactions and taking the appropriate averages.

We now set up the RG transformation in a small field. Consider two lattices, the first an $n\times n$ lattice with lattice spacing a and length $L=an$ with Hamiltonian parameters $\bar{K}$, $\tilde{K}$, and $\mu h/T$, and the second an $n'\times n'$ lattice with lattice spacing $a'=ba$ ($b>1$) and the same length $L=a'n'$ with Hamiltonian parameters $\bar{K}'$, $\tilde{K}'$, and $\mu' h/T$. We require that the two lattices represent the same physical problem with different microscopic length scales a and a'. We therefore demand that the Hamiltonian parameters of the second lattice be chosen so that the physical properties are preserved,

$$\bar{W}_{n'}(\bar{K}',\tilde{K}')=\bar{W}_n(\bar{K},\tilde{K}) \ , \tag{5}$$

$$\tilde{W}_{n'}(\bar{K}',\tilde{K}')=\tilde{W}_n(\bar{K},\tilde{K}) \ , \tag{6}$$

$$\chi_{n'}(\bar{K}',\tilde{K}',\mu')=\chi_n(\bar{K},\tilde{K},\mu) \ . \tag{7}$$

Equations (5)–(7) define implicitly the RG transformation from Hamiltonian parameters $\bar{K}$, $\tilde{K}$, and μ at length scale a to Hamiltonian parameters, $\bar{K}'$, $\tilde{K}'$, and μ' at length scale a'. Note that since the temperature appears explicitly in the expressions for domain-wall free energy and susceptibility, the temperature is held fixed and the Hamiltonian parameters $\bar{J}$, $\tilde{J}$, and μ are renormalized.

III. RESULTS

We first test the domain-wall RG (DWRG) method by applying it to the pure Ising model ($\tilde{J}=0$) and comparing the results with the exact Onsager[9] solution. We first find the fixed point $\bar{K}^*$ such that

$$\bar{W}_{n'}(\bar{K}^*,0)=\bar{W}_n(\bar{K}^*,0) . \tag{8}$$

Linearizing about the fixed point $\bar{K}_{n'}=\bar{K}^*+\delta\bar{K}_{n'}$ we find

$$\delta\bar{K}_{n'}=b^{\lambda_1}\delta\bar{K}_n \tag{9}$$

and

$$\chi_{n'}=b^{4-\eta}\chi_n . \tag{10}$$

The thermal eigenvalue λ_1 is the inverse of the correlation-length exponent ν. Table I gives the transition temperature and exponents for several values of (n,n'). For large n and n', T_c and the exponents approach the Onsager values; however, the contributions from irrelevant eigenvalues (corrections to scaling) are not negligible. We can obtain more accurate exponents by including corrections to scaling. We first fit $\bar{W}_n(\bar{K},0)$ for a range of n to

$$\bar{W}_n(\bar{K},0)=A_0+A_1n^{\lambda_1}+A_2n^{\lambda_2} , \tag{11}$$

where λ_1 is the leading eigenvalue. At the critical temperature A_1 vanishes. Linearizing about the critical temperature

$$\frac{dW_n}{d\bar{K}}=B_0+B_1n^{\lambda_1}+B_3(-1)^n n^{\lambda_3} . \tag{12}$$

The alternating term in (12) is necessary in order to obtain a reasonably accurate fit; a period-4 contribution is also present in the data. The alternating term is not present at T_c. The susceptibility versus n is fitted by

$$\chi_n=C_1n^{\lambda_1'}+C_2n^{\lambda_2'} , \tag{13}$$

where $\lambda_1'=4-\eta$. Fitting (11)–(13) for $5\le n\le 12$ we find the exponents listed in Table I. The DWRG method works well to calculate the critical behavior of the pure Ising model. However, in order to obtain accurate exponents, one must include corrections to scaling; the presence of alternating terms limits the accuracy of the thermal eigenvalue of 0.2%.

TABLE I. Properties of the ferromagnet-paramagnet critical point of the pure Ising model.

n'/n	$T_c/\bar{J}$	ν	η
3/4	2.219 16	0.8902	0.0286
4/6	2.250 81	0.9678	0.1371
6/8	2.263 44	0.9840	0.2007
8/12	2.267 27	0.9901	0.2272
$5\le n\le 12$	2.269 02	0.9981	0.2500
Exact	2.269 19	1.0000	0.2500

IV. RANDOM ISING MODEL

We first examine the pure spin-glass model with $\bar{J}=0$. The average domain-wall free energy vanishes and we have a one-parameter RG transformation given implicitly by

$$\tilde{W}_{n'}(\tilde{K}')=\tilde{W}_n(\tilde{K}) . \tag{14}$$

We calculate $\tilde{W}_n(\tilde{K})$ for a range of temperatures and fit $\ln[\tilde{W}_n(\tilde{K})]$ to a Taylor series in T^2. The Taylor series provides an extrapolation to zero temperature; round-off errors prevent numerical work at very low temperatures. For $n=3$ and 4 we average over $N=90\,000$ configurations for six temperatures in the range $0.2\tilde{J}\le T\le 0.7\tilde{J}$; for $n=6$, $N=45\,000$ and $0.25\tilde{J}\le T\le 0.75\tilde{J}$; for $n=8$, $N=20\,000$ and $0.3\tilde{J}\le T\le 0.8\tilde{J}$. These data establish the temperature dependence. The lowest-temperature data points are then repeated with 6 times the number of configurations to establish a more accurate absolute magnitude for the low-temperature domain-wall free energy. The extrapolation to zero temperature is insensitive to the order of the polynomial and does not appear to introduce any significant error. The zero-temperature data can be fitted within statistical accuracy to the simple scaling form

$$\tilde{W}_n(\infty)=A_1n^{-\lambda_1} , \tag{15}$$

with $\lambda_1=0.281\pm0.005$. There is a "phase transition at zero temperature" with the spin-glass correlation length diverging as $T^{-\nu}$ at low temperature with $\nu=1/\lambda_1=3.56\pm0.06$. This exponent is somewhat larger than the value $\nu=2.96\pm0.22$ found from the transfer-matrix method.[3] The present value is believed to be more reliable. The susceptibility is not affected by the interactions and the mean-square magnetic moment of a block spin is equal to the sum of the squares of the spin moments. One can rewrite the Taylor-series expansion

$$\tilde{W}_n^2(\tilde{K})=D_n\tilde{K}^2(1-E_n/\tilde{K}^2+\cdots) . \tag{16}$$

In the strong coupling regime the RG transformation is then

$$\tilde{K}'^2=(D_{n'}/D_n)\tilde{K}^2[1-(E_n-E_{n'}D_{n'}/D_n)/\tilde{K}^2] . \tag{17}$$

This result confirms the form assumed in the author's phenomenological scaling theory.[5] The coefficient of the $\tilde{K}^{-2}$ correction in (17) is approximately 0.55 for $(n',n)=(4,8)$.

We next study the critical surface for ferromagnetism. We defined reduced variables, $r=\tilde{J}/\bar{J}=\tilde{K}/\bar{K}$ and $t=\bar{K}_c/\bar{K}=T\bar{K}_c/\bar{J}$, where $\bar{K}_c=1/2.269\,185$ is the critical coupling constant for the pure Ising model. With these variables the phase diagram and the RG flows can be represented on the same graph. We first find the functions $\bar{W}_4$, $\tilde{W}_4$, $\bar{W}_8$, and $\tilde{W}_8$ on a grid of points in (r,t) space. We then fit each function with a third- or fourthorder polynomial in r and t (or r and t^2 near the ferromagnet—spin-glass transition). The RG equations are then solved for the RG flows using the polynomial expressions. We typically use $N=20\,000$ configurations for $n=4$ and 2000 configurations for $n=8$. The critical surface and typical flows are shown in Fig. 1.

There are three fixed points. (1) The ferromagnet-paramagnet fixed point is at $(r,t)=(0,0.995)$. The eigenvalues and exponents are $\lambda_1=1.023$, $\nu=0.997$, $\lambda_2=-0.004$, and $\eta=0.16$. The second eigenvalue is approximately zero. We could argue that the model has no small parameters and that the exact eigenvalue is either of order unity or precisely zero; we would then conclude that the random interaction is a marginal operator. It is marginally irrelevant since the RG flow is toward the fixed point. It is already known, however, that this is the case.[10] (2) The ferromagnet—spin-glass fixed point is at zero temperature $(r,t)=(0.967,0)$. The eigenvalues and exponents are $\lambda_1=0.694$, $\nu=1.44$, and $\lambda_2=-0.356$. The low-temperature susceptibility of the strip is strongly temperature dependent and does not provide a usable estimate for η. The spin-glass phase exists only at zero temperature. (3) There is a bicritical point at $(r,t)=(0.97,0.47)$. The eigenvalues and exponents are $\lambda_1=0.633$ and $\lambda_2=0.258$. Since relatively few configurations are used for $n=8$, these results are not highly accurate. In order to assess the error, the computations were repeated with an independent set of configurations. The critical surface shifted by as much as $\Delta r=0.025$ at low temperature. The bicritical point occurred at $(r,t)=(0.93,0.51)$ with eigenvalues $\lambda_1=0.703$ and $\lambda_2=0.251$. The critical surface was reentrant by an amount $\Delta r=0.0050$ in the first case and by $\Delta r=0.0013$ in the second. We conclude that the model is on the borderline between reentrant and nonreentrant behavior. Surprisingly, the Migdal approximation

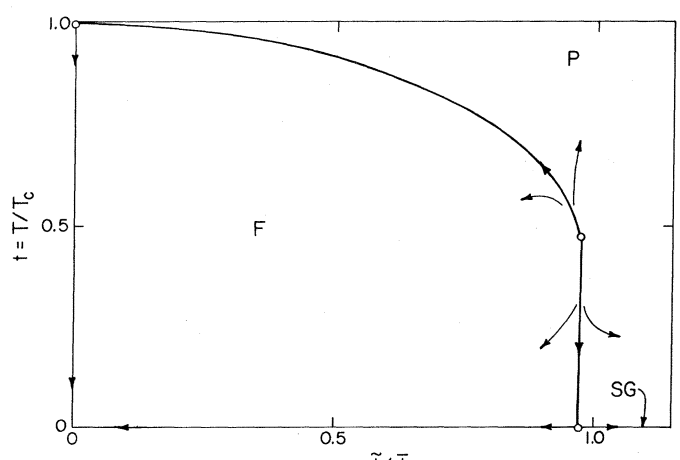

FIG. 1. Phase diagram of the two-dimensional random Ising model. The variables are reduced temperature $t=T/T_c$, where T_c is the transition temperature of the pure Ising model, and $r=\tilde{J}/\bar{J}$, the ratio of variance to mean of the interaction distribution function. The fixed points are shown as open circles and typical RG flows are shown. The ferromagnetic (F), paramagnetic (P), and spin-glass (SG) phases are indicated.

produces the right topology of the critical surface, including the bicritical point.[1,3] For temperatures less than the bicritical temperature the RG flow is toward the ferromagnet—spin-glass fixed point and the critical behavior of that part of the critical surface is governed by that fixed point. For temperatures greater than the bicritical temperature the flow is toward the ferromagnet-paramagnet fixed point and the critical behavior of that part of the critical surface is governed by that fixed point. This is not strictly true since the flow near the fixed point is marginal.

Finally, we reexamine the ferromagnet—spin-glass fixed point with better statistical accuracy. We are interested in the ratio

$$R_n(r,t)\equiv \widetilde{W}_n(r,t)/\overline{W}_n(r,t) \ . \tag{18}$$

We calculate this ratio for $t=0.2$ and 0.3, and $r=0.96$ and 0.98 using $N=120\,000$, $120\,000$, $64\,000$, and $24\,000$ configurations for $n=3$, 4, 6, and 8, respectively. From the earlier but more extensive data, the ratio is found to be weakly and approximately linearly dependent upon temperature; we extrapolate the present data to zero temperature linearly. The data can be fitted to the simple scaling form

$$R_n(r,0)=R_c+A_1(r-r_c)n^{\lambda_1} \tag{19}$$

within statistical accuracy, yielding $r_c=0.961\pm0.010$, $\lambda_1=0.713\pm0.042$, and $\nu=1.42\pm0.08$.

V. CONCLUSIONS

We have studied the critical behavior of the two-dimensional random Ising model using a macroscopic RG method. The macroscopic physical variables were chosen to be the distribution function of the domain-wall free energy and the mean susceptibility. These quantities were calculated numerically with use of transfer-matrix techniques. There were no uncontrolled approximations. There were systematic errors introduced in using finite lattice sizes and in extrapolating to zero temperature; there were statistical errors introduced in estimating the means and variances using finite samples. The overall quality of the calculation is good and the critical exponents are believed to be accurate to a few percent. The equilibrium behavior of the two-dimensional random Ising model is now well understood. Quantitative thermodynamic properties can be obtained from the RG method if desired.

The physical properties of the model are as follows. For $\tilde{J}>0.96\bar{J}$ there is a spin-glass phase at zero temperature, but no spin-glass—paramagnet phase transition at finite temperature. At zero temperature there is a continuous spin-glass—ferromagnet phase transition at $\tilde{J}=0.96\bar{J}$. There is a bicritical point on the ferromagnet-paramagnet critical surface which splits the critical surface into two portions. The phase transition is continuous on both portions with the critical behavior controlled by the ferromagnet—spin-glass fixed point on the low-temperature portion, and by the pure Ising fixed point on the high-temperature portion.

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