Long-range correlated percolation

Abel Weinrib*
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 15 August 1983)

This paper is a study of the percolation problem with long-range correlations in the site or bond occupations. An extension of the Harris criterion for the relevance of the correlations is derived for the case that the correlations decay as x^{-a} for large distances x. For $a<d$ the correlations are relevant if $a\nu-2<0$ (where ν is the percolation correlation-length exponent), while for $a>d$ the correlations are relevant if $d\nu-2<0$. Applying this criterion to the behavior that results when the correlations are relevant, we argue that the new behavior will have $\nu_{\rm long}=2/a$. It is shown that the correlated bond percolation problem is equivalent to a q-state Potts model with quenched disorder in the limit $q\rightarrow 1$. With the use of this result, a renormalization-group study of the problem is presented, expanding in $\epsilon=6-d$ and in $\delta=4-a$. In addition to the normal percolation fixed point, we find a new long-range fixed point. The crossover to this new fixed point follows the extended Harris criterion, and the fixed point has exponents $\nu_{\rm long}=2/a$ (as predicted) and $\eta_{\rm long}=\frac{1}{11}(\delta-\epsilon)$. Finally, several results on the percolation properties of the Ising model at its critical point are shown to be in agreement with the predictions of this paper.

I. INTRODUCTION

Most previous work on the percolation problem[1] assumes no spatial correlations for the site or bond occupations.[2] Renormalization-group (RG) ideas and the analogy with thermal critical phenomena suggest that the critical properties of all percolation problems with correlations of sufficiently short range will be the same as those of the uncorrelated percolation problem. This paper discusses the effects of correlations on the percolation critical properties, what range of correlations are in fact relevant, and what happens when they are.

The work presented here is closely related to a study[3] by Weinrib and Halperin (hereafter called WH) of the critical properties of a system with quenched disorder which has long-range spatial correlations. In that paper we observed that some of our results apply to the correlated percolation problem, and examined previous investigations of the percolation properties of spin clusters of the triangular Ising model in light of these results. The present study includes a more detailed discussion of the percolation results of WH, including an extension of the analysis of the percolation of the Ising model to arbitrary dimension.

In addition, it will be shown that the correlated percolation problem can be formulated in terms of a two-step process involving correlated occupation probabilities, and thus that the bond percolation version is equivalent to a q-state Potts model with correlated quenched coupling-constant disorder as $q\rightarrow 1$. This result then serves as the foundation for a RG study of the correlated percolation problem.

As in WH, we derive a generalization of the Harris criterion[4] for the correlated percolation problem, which indicates the relevance of the correlations in the bond or site occupations, i.e., whether the correlations change the percolation critical behavior, or not. We find that correlations with correlation function $g(r)$ which fall off at large distances r faster than r^{-d} (where d is the spatial dimension) are the same (insofar as the critical behavior is concerned) as no spatial correlations at all. Thus it is natural to consider correlations, which fall off as a power law at large distances, $g(r)\sim r^{-a}$ with $a<d$. Choosing such a scale-invariant form for the correlation function at large distances has the additional advantage that it allows for the possibility of new scaling behavior described by new fixed points of the RG when the correlations are relevant.

If the correlations are short ranged (or fall off faster than r^{-d}), the normal Harris criterion[4] applies: The correlations are relevant if $d\nu-2<0$, where ν is the correlation-length exponent for the uncorrelated percolation problem. For the percolation problem $d\nu-2$ is always positive, so short-range correlations do not change the critical behavior. This result is to be expected since the uncorrelated percolation problem already has nonzero autocorrelation of site or bond occupations. Thus, the introduction of other short-range correlations should not be relevant. With a power-law form for the correlation function $g(r)$ we derive the extended Harris criterion, which states that the long-range nature of the correlations are relevant if $a\nu-2<0$. As in the thermal case considered in WH, we apply the criterion to the system when the correlations are relevant in order to argue that the new correlation-length exponent obeys the scaling law

$$\nu_{\rm long}=2/a\ . \tag{1.1}$$

We now define the problem we wish to consider. For the sake of definiteness we shall discuss correlated site percolation; correlated bond percolation can be defined completely analogously. The site percolation problem can be defined by site-occupation variables θ_i at the sites $\{i\}$ of a regular lattice of dimension d. The θ_i take on the values 1 and 0 corresponding to occupied and vacant sites, respectively. (For the bond problem, the $\{i\}$ label bonds, and an occupied bond is a conducting one.) We shall characterize[5] the system by the site-occupation probability

$$p=\langle \theta_i \rangle \ , \tag{1.2}$$

and the site-occupation correlation function

$$g_\theta(|\vec{r}_i-\vec{r}_j|)=\langle \theta_i\theta_j\rangle^c \equiv \langle \theta_i\theta_j\rangle - \langle \theta_i\rangle\langle\theta_j\rangle \ , \tag{1.3}$$

where $\langle \ \rangle$ is an average over realizations of the random variables $\{\theta_i\}$. We consider a statistically homogeneous and isotropic system so that p is the same for all sites and g_θ is a function only of the distance $|\vec{r}_i-\vec{r}_j|$ between the sites at positions $\vec{r}_i$ and $\vec{r}_j$. In the normal, uncorrelated percolation problem the correlation function $g_\theta(r)=p(1-p)\delta_{r,0}$. This paper is concerned with more general g_θ, particularly with ones that fall off as a power law at large distances

$$g_\theta(r)\sim r^{-a} \ . \tag{1.4}$$

The system will percolate, i.e., there will be an infinite cluster of nearest-neighbor occupied sites, when the site-occupation probability exceeds the percolation threshold p^*. In general p^* will depend on the correlations as well as on the geometry of the lattice,[6] but in this paper we shall concentrate upon the scaling behavior of the system for p near p^*. In particular, the mean size of a cluster of nearest-neighbor occupied sites diverges at the threshold

$$\xi \sim (p^*-p)^{-\nu} \ . \tag{1.5}$$

So far, we have taken the statistical properties of the $\{\theta_i\}$ as given. Now we shall discuss two ways of generating them with power-law correlations. The first is to take the site-occupation variables $\{\theta_i\}$ to be Ising lattice-gas variables at the critical temperature, $T=T_c$. In this case $g_\theta(r)\sim r^{-(d-2+\eta_T)}$, where η_T is the usual Ising thermal critical exponent. If, in addition, the site percolation problem is also at its percolation threshold, then the $\{\theta_i\}$ define a correlated percolation problem with $a=d-2+\eta_T$. As will be discussed in greater detail in Sec. V, in two dimensions the Ising thermal and percolation problems are critical at the same point, while for $d>2$ the thermal and percolation problems can be made to be critical at the same point by randomly removing active bonds in defining the percolation clusters.[7–12] This problem, the percolation of Ising-spin clusters, has received considerable attention.[7–16] In WH we showed that previous results[13] on the percolation of the triangular Ising model were in accord with our scaling law $\nu=2/a$. Section V extends this analysis, finding that the scaling law is consistent with results in other dimensions as well.

A second, more general, approach to generating the $\{\theta_i\}$ is a two-step process[17] in which the site or bond occupations are independently determined by occupation probabilities which vary from site to site. The occupation variables can be made to have correlations by introducing appropriate correlations in the occupation probabilities. Random occupation variables with mean p and correlation function $g_\theta(r)$ are created by placing an occupation probability p_i (with $p_i\in[0,1]$) at each site (or bond) such that $\langle p_i\rangle_{\rm av}=p$ and $\langle p_ip_j\rangle^c_{\rm av}=g_\theta(|\vec{r}_i-\vec{r}_j|)$, where $\langle \ \rangle_{\rm av}$ is an average over the $\{p_i\}$. We can then consider the percolation problem defined by

$$\theta_i=\Theta(p_i-x_i) \ , \tag{1.6}$$

where each site is independently occupied with probability p_i since the $\{x_i\}$ are independent random variables uniformly distributed on the unit interval. Here, Θ is the unit-step function: $\Theta(x)=1$ if $x>0$ and zero otherwise. With these definitions the θ_i will have the desired statistical properties. Note that the correlated bond percolation problem can be defined by the same sort of two-step process. We define correlated bond probabilities $p_{i,j}$, and then bond-occupation variables

$$\theta_{i,j}=\Theta(p_{i,j}-x_{i,j}) \tag{1.7}$$

with the $\{x_{i,j}\}$ independent random variables on the unit interval.

Of course, the percolation problem is only completely defined by specifying all of the higher moments of the $\{\theta_i\}$, and thus of the $\{p_i\}$. Since $\theta_i=\theta_i^2$ but in general $p_i\neq p_i^2$ the percolation problem is specified by all of the different-point correlation functions, while to specify the $\{p_i\}$ one requires all of the same-point correlation functions as well. However, we only explicitly consider the two-point correlation function g_θ because we believe[5] that it is the behavior of g_θ which determines the universality class of the critical behavior.

For a given realization of the $\{p_i\}$ the properties of the correlated percolation problem can be expressed as averages over an ensemble defined by the $\{x_i\}$. Then, as for quenched thermodynamic systems, extensive quantities can be averaged over the $\{p_i\}$ to find macroscopic properties of the system.

In studying the thermal critical phenomena of inhomogeneous materials, one distinguishes[18,19] annealed disorder (which comes to thermal equilibrium in experimental time scales), from quenched disorder (which is essentially fixed for a given sample). While annealed disorder is simply described by a more complex Hamiltonian which must be used in calculating thermodynamic quantities, in the case of quenched disorder one must calculate such quantities for a given realization of the disorder. Extensive quantities can then be averaged over the quenched disorder to find the large system observables. The Ising percolation problem is similar to an annealed disorder problem in that we are using the thermal distribution of the Ising spins in defining the percolation problem. Percolation quantities of interest, such as cluster statistics, are calculated by averaging over the Ising thermal disorder. In fact, in Sec. V we shall find that the Ising percolation problem maps into a model with coupled Potts and Ising variables. We have introduced an analog of quenched disorder in defining the correlated percolation problem involving a two-step averaging procedure. The occupation probabilities give rise to an ensemble of occupation variables with the required correlations, and then percolation quantities can be averaged over the "quenched" probabilities. In Sec. III we show that this formulation of the correlated percolation problem maps into a Potts model with quenched correlated coupling-constant disorder.

For the percolation problem the introduction of the average over the independent $\{x_i\}$ in addition to the average over the correlated $\{p_i\}$ appears superfluous; we are introducing an extra unnecessary step. In fact, for the uncorrelated case, where the $\{p_i\}$ are uncorrelated random variables, it is easy to see that the resulting $\{\theta_i\}$ are exactly those that would be obtained by replacing the random p_i by a single, homogeneous, occupation probability $p=\langle p_i\rangle_{\rm av}$; thus, for the uncorrelated case the second average over the $\{p_i\}$ is irrelevant.

However, the introduction of the two-step process in defining the correlated percolation problem is useful in that we show in Sec. III that the bond problem defined in this way is equivalent to a q-state Potts model with quenched inhomogeneous coupling constants in the limit $q\to 1$. The couplings $K_{i,j}$ are related to the bond probabilities $p_{i,j}$ by $p_{i,j}=1-e^{-K_{i,j}}$. If there are long-range correlations in the $\{p_{i,j}\}$ then there will also be such correlations in the $\{K_{i,j}\}$. This equivalence allows us to study the correlated percolation problem for spatial dimension d near 6 by applying the RG to the Potts model with correlated quenched disorder. As discussed above, we do not expect short-range correlations in the occupations to change the percolation critical behavior. This expectation is confirmed in a study of the $q\to 1$ state Potts model for which short-range correlated coupling-constant disorder is irrelevant. With a long-ranged coupling-constant—disorder correlation function $g_K(r)\sim r^{-a}$ we find that we must carry out a double expansion[20] in $\epsilon=6-d$ and in $\delta=4-a$ with δ of order ϵ. Expanding to lowest order in ϵ and δ, we find crossover, to new critical behavior described by a new stable fixed point of the RG, according to the extended Harris criterion. When the long-range correlations are relevant, the new critical exponents are

$$\nu_{\rm long}=2/a\ ,\quad \eta_{\rm long}=(\delta-\epsilon)/11\ , \tag{1.8}$$

and the scaling law Eq. (1.1) is in fact obeyed. We shall show that these results are identical to those of Coniglio and Lubensky[11] for the special case of Ising percolation with d near 6 for which $a=d-2$.

The remainder of this paper is organized as follows. In the next section we apply the Harris criterion to the correlated percolation problem. In Sec. III we display the equivalence of the correlated percolation problem and a disordered q-state Potts model with $q\to 1$, which is utilized in Sec. IV for a RG study of the percolation problem. Finally, Sec. V contains a discussion of the percolation of like-pointing Ising spins.

II. EXTENDED HARRIS CRITERION

The derivation of the Harris criterion[4] for the correlated percolation problem is very similar to that introduced in WH for the thermal system with correlated disorder, and the results for the two problems are identical. We start with the correlated percolation problem as in the Introduction (we shall consider the site problem for definiteness, analogous arguments can be made for the bond problem), defined by site-occupation variables $\theta_i=1,0$ with mean $\langle\theta_i\rangle=p$ and connected correlation function $g_\theta(|\vec r_i-\vec r_j|)=\langle\theta_i\theta_j\rangle^c$.

Our basic assumption is that a (large) region of the system of size $V=L^d$ will contain largest clusters of size ξ_V, with

$$\xi_V\sim(p^*-p_V)^{-\nu}\ , \tag{2.1}$$

where p^* and ν are the percolation threshold and correlation-length exponent of the infinite system, and p_V is the average site occupation of the region,

$$p_V=\frac{1}{V}\sum_{i\in V}\theta_i\ . \tag{2.2}$$

Clearly, in order for Eq. (2.1) to be true a cluster must fit into the region, so we must take $L\geq\xi_V$. In the following we shall take $L=\xi_V$, since this choice results in the most stringent criterion.

We now ask if a uniform percolation transition with the pure correlation-length exponent ν is consistent. In a uniform transition the correlation length diverges uniformly across the system, so that the ξ_V for different regions will all diverge at the same point as $p\to p^*$. If a uniform transition with the pure correlation-length exponent is inconsistent, then the long-range nature of the correlations are relevant and we expect the system to exhibit new critical behavior. From Eq. (2.1), a uniform transition will occur only if the variations in p_V are small as the threshold is approached. The variance of p_V is

$$\Delta^2\equiv\langle(p_V)^2\rangle^c=\frac{1}{V^2}\sum_{i,j\in V}\langle\theta_i\theta_j\rangle^c\sim\xi^{-d}\int_0^\xi g_\theta(r)r^{d-1}dr\ , \tag{2.3}$$

where we have taken the size of a region to be $V=\xi^d$ with ξ large as $p\to p^*$. A uniform transition is consistent only if $\Delta^2/(p^*-p)^2\to 0$ as $p\to p^*$. If $g_\theta(r)\sim r^{-a}$ for large r then, upon solving the integral, we find

$$\frac{\Delta^2}{(p^*-p)^2}\sim\begin{cases}(p^*-p)^{d\nu-2}\ , & a>d\\ (p^*-p)^{a\nu-2}\ , & a<d\ .\end{cases} \tag{2.4}$$

The result for $a>d$ is the same as that for purely short-range correlations. For the uncorrelated percolation problem $d\nu-2>0$ for all dimensions as we expect, short-range correlations are irrelevant. However, for $a<d$ our extended Harris criterion results. The correlations are relevant when

$$a\nu-2<0\ . \tag{2.5}$$

We now apply this criterion to the new behavior with correlation exponent $\nu_{\rm long}$, which occurs when the long-range nature of the correlations is relevant. Consider introducing additional correlations, which produce an additional power-law term in the correlation function,

$$g_\theta(r)\sim Ar^{-a}+Br^{-b}\ , \tag{2.6}$$

with $B\ll A$. Since it is the long-range nature of the correlation function which is relevant to the critical properties, if $b<a$ the B term will eventually dominate, and crossover to new behavior will result. However, if $b>a$ the A term continues to dominate, and the original behavior will be stable. Thus, applying the extended Harris criterion, we expect

$$b\nu_{\text{long}}-2>0 \text{ if } b>a \ ,$$
$$b\nu_{\text{long}}-2<0 \text{ if } b<a \ . \tag{2.7}$$

This can be satisfied for any b only if

$$\nu_{\text{long}}=2/a \ . \tag{2.8}$$

Below, we shall see that this scaling law is realized in the percolation of like-pointing Ising spins at their critical point and by the results of a RG study of correlated percolation near six dimensions.

III. EQUIVALENT POTTS MODEL

As is well known,[18,21–25] the uncorrelated bond percolation problem is equivalent to the q-state Potts model in the limit $q\to 1$. In this section we show that the correlated bond percolation problem maps onto a Potts model with quenched coupling-constant disorder. For small disorder (corresponding to weak correlations in the percolation problem) the coupling-constant—disorder correlation function g_K has the same form as the equivalent percolation problem's correlation function g_θ. These results serve as the foundation for the next section's $\epsilon=6-d$ expansion for the percolation problem, derived by applying the RG to the disordered Potts model with $q\to 1$.

Consider the correlated bond percolation problem defined by the two-step process as in the Introduction. Let us define the "free energy" of the problem[1] for a given realization of the $\{p_{ij}\}$ to be

$$f(h)=\sum_s K_s e^{-hs} \ , \tag{3.1}$$

where K_s is the average number of s-site clusters per site. $f(h)$ is a generating function for the cluster statistics of the percolation problem; for example, the probability that a site belongs to the infinite cluster

$$P(p)=1-\left.\frac{df}{dh}\right|_{h=0} \tag{3.2}$$

since the probability that a site belongs to an s-site cluster is sK_s. Defining graphs as subsets of the bonds of the underlying lattice of N sites, and introducing the probability of a graph G,

$$\pi(G)=\left[\prod_{\langle i,j\rangle\in G} p_{i,j}\right]\left[\prod_{\langle i,j\rangle\notin G}(1-p_{i,j})\right] , \tag{3.3}$$

where $p_{i,j}$ is the probability of bond $\langle i,j\rangle$, we can write $f(h)$ as

$$f(h)=\frac{1}{N}\sum_G \pi(G)\sum_s N_s e^{-hs} \ , \tag{3.4}$$

where the first sum is over all graphs of the lattice and N_s is the number of s-site clusters in graph G.

Closely following the derivation of the relation between percolation and the Potts model given by Lubensky[18] for the uncorrelated case, we now show that the partition function for the Potts model can be written as a cluster expansion involving graphs of the lattice. The free-energy of the percolation problem with bond probabilities $\{p_{i,j}\}$ is then related to the free energy of the Potts model with disordered coupling constants $\{K_{i,j}\}$ defined by the relation

$$p_{i,j}=1-e^{-K_{i,j}} \ . \tag{3.5}$$

For weak disorder in the $K_{i,j}$ (and thus in the $p_{i,j}$) the two correlation functions have the same form,

$$g_\theta(i,j;\,k,l)\equiv\langle p_{i,j}p_{k,l}\rangle^c_{\text{av}}\cong e^{-2K}g_K(i,j;\,k,l) \ , \tag{3.6}$$

where the coupling-constant disorder is characterized by mean and correlation function

$$K=\langle K_{i,j}\rangle_{\text{av}} \ , \quad g_K(i,j;k,l)=\langle K_{i,j}K_{k,l}\rangle^c_{\text{av}} \ , \tag{3.7}$$

with $\langle\ \rangle_{\text{av}}$ an average over the quenched disorder.

We start with a disordered Potts model defined on a regular lattice of N points with reduced Hamiltonian

$$-\beta H=\sum_{\langle i,j\rangle} K_{i,j}\delta_{\sigma_i,\sigma_j}+h\sum_i \delta_{\sigma_i,1} \ , \tag{3.8}$$

where $\beta=(k_BT)^{-1}$, k_B is Boltzmann's constant, and T is the temperature. The first sum is over nearest-neighbor bonds $\langle i,j\rangle$, the Potts variables σ_i take on q different values, and the quenched coupling-constant disorder $K_{i,j}$ is as above. The partition function for a given $\{K_{i,j}\}$ is

$$Z=\mathrm{Tr}_{\{\sigma\}}e^{-\beta H}$$
$$=\mathrm{Tr}_{\{\sigma\}}\left[\prod_{\langle i,j\rangle}[1+(e^{K_{i,j}}-1)\delta_{\sigma_i,\sigma_j}]\exp\left[h\sum_i\delta_{\sigma_i,1}\right]\right] . \tag{3.9}$$

Z can be expressed as an expansion over all graphs G of bonds on the lattice where a present bond corresponds to the term in the product $(e^{K_{i,j}}-1)\delta_{\sigma_i,\sigma_j}$ and an absent bond corresponds to the 1. Each connected cluster is forced to have the same value of σ by the $\delta_{\sigma_i,\sigma_j}$ factor. Thus, upon taking the trace over the $\{\sigma_i\}$, a connected cluster of s sites corresponds to a term

$$e^{hs}\prod_{\langle i,j\rangle}(e^{K_{i,j}}-1)+(q-1)\prod_{\langle i,j\rangle}(e^{K_{i,j}}-1)$$
$$=(e^{hs}+q-1)\prod_{\langle i,j\rangle}p_{i,j}(1-p_{i,j})^{-1} \ , \tag{3.10}$$

where $p_{i,j}$ is the bond probability defined in Eq. (3.5), and the products are over bonds of the cluster. The partition function is a sum over all graphs G of the product of the contributions from the different clusters. Using the definition (3.3) of the probability of a configuration $\pi(G)$ we can write Z as

$$Z=\left[\prod_{\langle i,j\rangle}(1-p_{i,j})^{-1}\right]\sum_G\pi(G)\left[\prod_{\text{clusters}}(e^{hs}+q-1)\right] , \tag{3.11}$$

where the first product is over all bonds of the lattice.

After some algebra and using Eq. (3.4), we find that the percolation free energy Eq. (3.1) is

$$f=\frac{1}{N}\frac{\partial F}{\partial q}\bigg|_{q=1}, \quad (3.12)$$

where $F=\ln Z$ is the free energy of the q-state Potts model in a magnetic field h with quenched coupling constants $\{K_{i,j}\}$. Finally, we can average Eq. (3.12) over the disorder, corresponding to averaging the Potts model free energy F over the quenched coupling constants $K_{i,j}$, and the percolation free energy f (and hence the cluster statistics) over the bond probabilities $p_{i,j}$.

Thus, in the next section we shall outline a RG analysis of the q-state Potts model with long-range correlated quenched coupling-constant disorder and identify the $q\to 1$ results as applying to the correlated percolation problem.

IV. RG STUDY

In the preceding section we showed that the correlated bond percolation problem is equivalent to a q-state Potts model with quenched correlated coupling-constant disorder in the limit $q\to 1$. We shall now analyze the critical properties of the percolation problem by applying the RG to the disordered Potts model. The case of weak disorder is considered so that the percolation problem's bond-occupation correlation function has the same form as the quenched coupling-constant correlation function of the related Potts model [see Eq. (3.6)]. We shall consider the case of a power-law form for these functions.

Following Priest and Lubensky's[23] study of the ordered Potts model, we study a "soft-spin" continuum version of the disordered Potts model introduced in the preceding section. The order parameter $Q_{i,j}(\vec{x})$ is a traceless, diagonal tensor of order q. The coupling-constant disorder of our discrete Potts model leads to inhomogeneous coefficients in the Landau-Ginzburg-Wilson expansion of the Hamiltonian of the continuum version. However, only the inhomogeneity in the "temperature" coefficient $r(\vec{x})$ is relevant near the upper critical dimension of 6. In addition, terms of fourth and higher order in $Q_{i,j}$ or of higher than second order in the gradient are irrelevant near $d=6$ and shall be neglected. Thus, we wish to consider the Hamiltonian

$$\beta H=\int d^dx\left[\tfrac{1}{4}r(\vec{x})\mathrm{Tr}Q^2+\tfrac{1}{4}\partial_k Q_{i,j}\partial_k Q_{j,i}-t\,\mathrm{Tr}Q^3\right], \quad (4.1)$$

with $r(\vec{x})$ satisfying

$$\langle r(\vec{x})\rangle_{\mathrm{av}}=r\,,\quad \langle r(\vec{x})r(\vec{y})\rangle^c_{\mathrm{av}}=g_r(|\vec{x}-\vec{y}|)\,, \quad (4.2)$$

where $\langle\ \rangle_{\mathrm{av}}$ is an average over the quenched disorder, and $g_r(x)\sim x^{-a}$ for large x.

We now wish to find the free energy of the Potts model averaged over the quenched disorder. A convenient formalism which accomplishes this is the replica technique,[26,27] which replicates the order parameter n times and takes the limit $n\to 0$ at the end. The averaged free energy is then the trace of a homogeneous effective Hamiltonian, which is expressed as cumulants of $(\beta H)^n$ averaged over the disorder. The RG is then applied to this effective Hamiltonian. These steps are presented in more detail in WH for the case of an m-vector model with the same type of disorder, and our derivation here follows that paper very closely. Expanding to the second cumulant,[5] the resulting effective Hamiltonian is

$$\beta H_{\mathrm{eff}}=\sum_\alpha\int d^d\left[\tfrac{1}{4}r\mathrm{Tr}(Q^\alpha)^2+\tfrac{1}{4}\partial_k Q^\alpha_{i,j}\partial_k Q^\alpha_{j,i}-t\,\mathrm{Tr}(Q^\alpha)^3\right]-\tfrac{1}{32}\sum_{\alpha,\beta}\int d^dx\,d^dy\,\mathrm{Tr}[Q^\alpha(\vec{x})]^2 g_r(|\vec{x}-\vec{y}|)\mathrm{Tr}[Q^\beta(\vec{y})]^2\,, \quad (4.3)$$

where $Q^\alpha_{i,j}$ is the replicated order parameter. Notice that in addition to the original three-point interaction t acting within a single replica there is a new four-point interaction $g_r(|\vec{x}-\vec{y}|)$ acting between replicas. Fourier-transforming yields the interactions t and w, which are illustrated graphically in Fig. 1. For small k, the Fourier-transformed correlation function $\frac{1}{32}g_r(k)\sim v+wk^{-(d-a)}$. However, near $d=6$ the v interaction is irrelevant and shall be ignored.

Applying the RG to this effective Hamiltonian, we find that we must carry out a double expansion[20] in $\epsilon=6-d$ and in $\delta=4-a$, with δ of order ϵ. This expansion is very similar to the expansion in WH, except that in that case the upper critical dimension was 4 rather than 6. We shall find fixed points of the RG with r, t^2, and w of order ϵ. Thus, we expand the RG recursion relations to order t^3, wt, and w^2 in order to determine the fixed points and exponents to lowest order in ϵ. In Fig. 2 we show the diagrams contributing to r', t', and w' in the limit $n\to 0$. The momentum dependence of the diagrams of Fig. 2(a) fix the anomalous dimension η to be

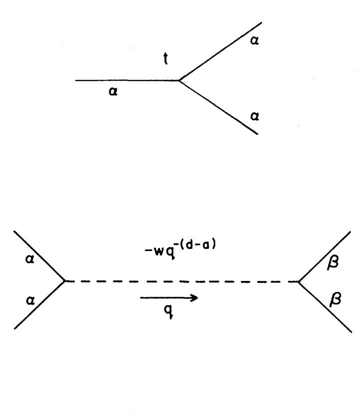

FIG. 1. Interactions t and w. Notice that t acts only within a single replica α and is momentum independent, while w acts between arbitrary replicas α and β and has momentum dependence $q^{-(d-a)}$.

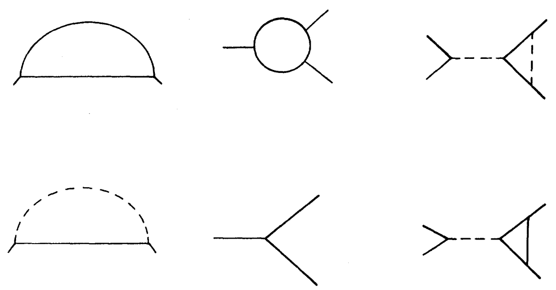

FIG. 2. Graphs that contribute to (a) r', (b) t', and (c) w'. The critical exponent η is determined by the momentum dependence of graphs (a).

$$\eta=2^4 3\left[1-\frac{2}{q}\right]t^{*2}+\frac{2^5}{3}w^* , \quad (4.4)$$

where t^* and w^* are the coupling values at the fixed point of the RG differential recursion relations. These relations in turn take the form

$$\frac{dr}{dl}=(2-\eta)r-\frac{2^4 3^2(q-2)}{q(1+r)^2}t^2-\frac{2^5}{1+r}w ,$$
$$\frac{dt}{dl}=\tfrac{1}{2}(\epsilon-3\eta)t+\frac{2^5 3^2(q-3)}{q(1+r)^3}t^3+\frac{2^5 3}{(1+r)^2}wt , \quad (4.5)$$
$$\frac{dw}{dl}=(\delta-2\eta)w+\frac{2^6}{(1+r)^2}w^2+\frac{2^6 3^2(q-2)}{q(1+r)^3}wt^2 ,$$

where we have absorbed irrelevant factors into a redefinition of the couplings, and $\epsilon=6-d$ and $\delta=4-a$. Notice the appearance of δ in the third recursion relation. For the correlated percolation problem we are interested in the special case $q\rightarrow 1$. Thus, for clarity we shall take $q=1$ from now on.

The fixed points of the recursion relations are listed in Table I. We find the Gaussian and pure fixed points found previously which describe the uncorrelated percolation problem for $d>6$ and $d<6$, respectively, an unphysical fixed point, which is unstable for $\delta<0$ and has an unphysical negative value of w^* for $\delta>0$, and a new long-range fixed point, which describes the system's critical behavior when the long-range nature of the correlations is relevant.

The eigenvalues of the various fixed points are shown in Table II. Notice that to order ϵ the scaling law Eq. (2.8) is satisfied since the long-range fixed point has relevant temperature eigenvalue $\lambda_r=\frac{1}{2}(4-\delta)=\frac{1}{2}a$, so

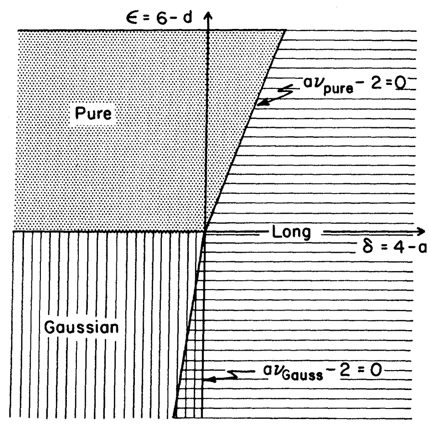

FIG. 3. Regions in the δ-ϵ plane, where the various types of critical behavior occur. Here, $\epsilon=6-d$ and $\delta=4-a$, where d is the spatial dimension and a is the power of falloff of the correlation function. The Gaussian and pure behaviors become unstable to the long-range behavior according to the extended Harris criterion. The crossover occurs when $a\nu-2$ goes negative, where ν is the Gaussian or pure correlation-length exponent, respectively. In the crosshatched region separating the long-range and Gaussian behaviors the long-range fixed point has a finite domain of attraction; those systems with Hamiltonians which lie outside of the domain of attraction exhibit Gaussian behavior.

$$\nu_{\text{long}}=2/a . \quad (4.6)$$

Also, we find for the new critical behavior

$$\eta_{\text{long}}=(\delta-\epsilon)/11 . \quad (4.7)$$

The crossover between the various fixed points is determined by the extended Harris criterion Eq. (2.5). With $\epsilon>0$ the pure fixed point exchanges stability with the long-range fixed point when

$$\delta-\tfrac{10}{21}\epsilon=(2/\nu_{\text{pure}})-a=0 , \quad (4.8)$$

while when $\epsilon<0$ the Gaussian fixed point is unstable to the long-range fixed point for all $\delta>0$, since $a\nu_{\text{Gauss}}-2<0$ for $a<4$ because $\nu_{\text{Gauss}}=\frac{1}{2}$. Figure 3 provides a summary of the regions where the various types of critical behavior occur.

The long-range fixed point has complex eigenvalues in the two irrelevant directions for $0.245<(\delta/\epsilon)<1.16$. This will lead to oscillating corrections to scaling[20,28] and to the possibility of Hopf bifurcation behavior.[3,29] In fact, with $\epsilon<0$ the long-range fixed point becomes unstable via

TABLE I. Fixed points of the recursion relations, Eq. (4.5).

Fixed point	r^*	$(t^*)^2$	w^*
(I) Gaussian	0	0	0
(II) Pure	$-\frac{1}{14}\epsilon$	$\frac{1}{1008}\epsilon$	0
(III) Unphysical	$-\frac{3}{8}\delta$	0	$-\frac{3}{128}\delta$
(IV) Long range	$\frac{3}{22}(\delta-\epsilon)$	$\frac{1}{3168}(15\delta-4\epsilon)$	$\frac{1}{704}(21\delta-10\epsilon)$

TABLE II. Eivenvalues of the fixed points of Table I.

Fixed point	$\lambda_r = 1/\nu$	η	λ_1	λ_2
(I) Gaussian	2	0	$\frac{1}{2}\epsilon$	δ
(II) Pure	$2-\frac{5}{21}\epsilon$	$-\frac{1}{21}\epsilon$	$-\frac{8}{7}\epsilon$	$\delta-\frac{10}{21}\epsilon$
(III) Unphysical	$2-\frac{1}{2}\delta$	$-\frac{1}{4}\delta$	$\frac{1}{8}(4\epsilon-15\delta)$	$-\frac{3}{2}\delta$
(IV) Long range	$2-\frac{1}{2}\delta$	$\frac{\delta-\epsilon}{11}$	$\frac{1}{22}[(6\epsilon-39\delta)\pm(-284\epsilon^2+1404\epsilon\delta-999\delta^2)^{1/2}]$	

a subcritical Hopf bifurcation[29] when $\delta<\frac{2}{13}\epsilon$. However, unlike the case of the m-vector model studied in WH, the correlated percolation problem does not exhibit runaway of the RG flows as a result of the bifurcation. The bifurcation occurs for δ negative so that the Gaussian fixed point is already stable. For $\frac{2}{13}\epsilon<\delta<0$ there are two stable fixed points and thus two possible critical behaviors, depending upon the explicit form of the Hamiltonian describing the system. The Hopf bifurcation is the vanishing of the domain of attraction of the long-range fixed point.

V. ISING CORRELATED PERCOLATION

The percolation of like-pointing Ising spins ($s_i=\pm 1$) has recently received considerable attention.[7–16] Considering the site percolation of "up" spins, so the site-occupation variable is

$$\theta_i=\tfrac{1}{2}(s_i+1)\ , \tag{5.1}$$

then the probability that a site is occupied and the site-occupation correlation function are, respectively,

$$\begin{aligned} p_s &\equiv \langle \theta_i \rangle = \tfrac{1}{2}(m+1)\ , \\ g_\theta(|\vec r_i-\vec r_j|) &\equiv \langle \theta_i\theta_j\rangle^c=\tfrac{1}{4}g_T(|\vec r_i-\vec r_j|)\ , \end{aligned} \tag{5.2}$$

with m and g_T the Ising magnetization and correlation function, respectively. $\langle\ \rangle$ denotes an Ising thermal average. At its critical temperature $T=T_c$ in zero field h the Ising model's spin correlation function has a power-law form $g_T(r)\sim r^{-(d-2+\eta_T)}$. Thus, if the resulting site percolation problem is also at its percolation threshold, we have an example of the power-law correlated percolation problem we wish to study.

The magnetization of the Ising model vanishes at its critical point, so for the percolation problem to be at threshold at the same time it must have percolation threshold $p_s^*=\frac{1}{2}$, in which case

$$p_s^*-p_s=-\tfrac{1}{2}m\ . \tag{5.3}$$

Coniglio *et al.*[14] proved that for Ising correlated percolation in two dimensions the percolation problem is at threshold at the Ising critical point. In WH we discussed the case of the triangular lattice, for which $p_s^*=\frac{1}{2}$, independent of correlations. We found that the scaling law Eq. (2.8) implies that the Ising correlation length ξ_T and the percolation correlation length ξ_p are proportional, in agreement with previous results for this model.[13] In general $p_s^*<\frac{1}{2}$ for lattices of spatial dimension $d>2$. However, randomly removing bonds of such a lattice raises the site percolation threshold. Specifically, for Ising percolation with $T=T_c$ and $h=0$ it has been argued[8–10] that if bonds are independently active with probability

$$p_b^*=1-\exp(-2J/k_BT_c)\ , \tag{5.4}$$

where J is the Ising coupling constant, then the site percolation problem is also at threshold, i.e., $p_s^*=\frac{1}{2}$. Note that the lattice is left intact insofar as the thermal Ising problem is concerned; it is only the definition of the percolation clusters as occupied sites connected by active bonds which is affected.

Applying the Harris criterion for the correlated site percolation problem derived in Sec. II, we find that the long-range correlations with power $d-2+\eta_T$ are relevant for $d<6$. Thus, applying our scaling law, we expect that $\nu_p=2/(d-2+\eta_T)$ for Ising correlated percolation, so the percolation correlation length

$$\xi_p\sim(p_s^*-p_s)^{-\nu_p}\sim|m|^{-\nu_p}\ , \tag{5.5}$$

where the second step is a result of Eq. (5.3). We imagine adding a small field h to the Ising model in order to induce a small magnetization m, thereby making both the thermal and percolation problems noncritical. Now, for $d<4$, scaling laws for the Ising exponents imply that $2\beta_T=\nu_T(d-2+\eta_T)$, and the thermal correlation length $\xi_T\sim|m|^{-\nu_T/\beta_T}$, so our scaling law predicts that the percolation and thermal correlation lengths are proportional as we take $h\to 0$: $\xi_p\propto\xi_T$. This prediction is in accord with our expectation that at the Ising critical point there should be only a single relevant length scale for $d<4$, and agrees with previous $d=2$ results[13] and with arguments[8–10] that the percolation critical behavior will be of Ising type in all dimensions $d\geq 2$. For $4<d<6$ we shall see that the scaling law is in agreement with other results on Ising percolation, although it is no longer true that $\xi_p\propto\xi_T$.

If $d\geq 4$ then $\eta_T=0$, so we predict $\nu_p=2/(d-2)$ with $\xi_p\sim|m|^{-\nu_p}$. Note that this prediction agrees with the $d=4$ behavior, where scaling still holds and the correlation lengths are proportional, and that for $d=6$, where the long-range correlations become irrelevant and ν_p takes on its mean-field value of $\frac{1}{2}$. However, for $d>4$, scaling of the Ising model breaks down and $\xi_T\sim|m|^{-1}$, independent of dimension, so there are two diverging length scales, ξ_p and ξ_T with $\xi_p<\xi_T$. The appearance of two length scales can be understood from the fact that the Ising model's upper critical dimension is 4, while that of the percolation problem is 6. For $d<4$, the fluctuations of both the Ising and percolation "fields" are relevant on all length scales, and there is only a single diverging length as the critical point is approached. However, for $4<d<6$, fluctuations of the Ising field are irrelevant and mean-field theory is correct, while fluctuations of the percolation fields are still relevant and the percolation correlation length is suppressed relative to the Ising one. Finally, for $d>6$ the two models decouple since the Ising correlations are irrelevant to the percolation problem, and both exhibit their respective mean-field behavior: $\xi_p \sim |m|^{-1/2}$, while $\xi_T \sim |m|^{-1}$.

We now show that this picture of the $4<d<6$ behavior is in accord with a study of Ising correlated percolation with $d \cong 6$ by Coniglio and Lubensky.[11,30] They develop a field-theoretic version of the problem, appropriate near the upper critical dimension of 6, from the mapping[8–12,15] of the problem onto a coupled Ising and Potts system with the number of Potts states $q \to 1$. The Hamiltonian in zero external fields is

$$\beta H = \beta H_I + \beta H_P + \beta H_C , \tag{5.6}$$

with Ising, Potts, and coupling Hamiltonians

$$\begin{aligned} \beta H_I &= \int d^d x \left[\tfrac{1}{2} r_0 \phi^2 + \tfrac{1}{2} (\vec{\nabla}\phi)^2 + u\phi^4 \right] , \\ \beta H_P &= \tfrac{1}{2} \int d^d x \left[r_1 \psi^2 + (\vec{\nabla}\psi)^2 - \tfrac{1}{3} w_1 \lambda_{ijk} \psi_i \psi_j \psi_k \right] , \\ \beta H_C &= -\tfrac{1}{2} w_2 \int d^d x \, \phi \psi^2 , \end{aligned} \tag{5.7}$$

where ϕ is the Ising field, ψ is a $(q-1)$-component Potts field, and λ_{ijk} is a tensor coupling of the ψ fields. The result of the RG analysis of the system in $d=6-\epsilon$ is that at the point that we are interested in, where both the thermal and percolation systems are critical ($r_0 = r_1 = 0$), the critical behavior is determined by a new fixed point of the RG. At this fixed point the scaling field r_0 relevant for the Ising model exhibits mean-field behavior as expected, while the other relevant scaling field has eigenvalue $(\nu_p')^{-1} = \frac{1}{2}(d-2)$ to first order in $\epsilon = 6-d$. Keeping the Ising model at $T=T_c$ ($r_0=0$), this other field is r_1 which corresponds to varying p_b the fraction of active *bonds*, so that [compare to Eq. (5.5)]

$$\xi_p \sim |r_1|^{-\nu_p'} \sim (p_b^* - p_b)^{-\nu_p'} . \tag{5.8}$$

In addition, for the percolation field, $\eta=0$ in $d=6-\epsilon$. This result is in agreement with that of the preceding section that $\eta = \frac{1}{11}(\delta-\epsilon)$, since $a=d-2$ and thus $\delta \equiv 4-a=\epsilon$ for $d>4$ Ising correlated percolation. Coniglio and Lubensky suggest that $\nu_p' = 2/(d-2)$ for dimensions $d<6$, although only so long as other terms in the expansion of the Hamiltonian (5.6) remain irrelevant. Also, under similar assumptions, Benzoni and Cardy[12] prove to all orders in ϵ that $\nu_p' = 2/(d-2+\eta_T)$. We shall argue below that if the Hamiltonian correctly describes the system then $\nu_p = \nu_p'$. Hence, our scaling law $\nu_p = 2/(d-2+\eta_T)$ is in agreement with these other results. Note, however, that in low dimensions the expansion of the Hamiltonian becomes invalid so ν_p and ν_p' need not, and will not, be the same. While ν_p continues to be determined by our scaling law, ν_p' does not.[31]

Consider the system described by the Hamiltonian equation (5.6) with $r_0 = r_1 = 0$, but with a small Ising magnetization $-m$. We now show that this situation is equivalent, with respect to the percolation scaling, to the system with no Ising magnetization but with percolation scaling field $r_1 \propto |m|$. This implies that $\nu_p' = \nu_p$ since $\xi_p \sim |r_1|^{-\nu_p'}$. The argument depends upon the irrelevance of higher-order terms of the expansion of the Hamiltonian. In lower dimensions the expansion breaks down and our argument for the equality of the two exponents is not valid.

Let us define $\langle \; \rangle_{I,m}$ to be an average over the Ising Hamiltonian H_I of Eq. (5.7) with Ising magnetization[32] $\langle \phi(\vec{x}) \rangle_{I,m} = -m$, i.e., for any function $F(\phi)$,

$$\begin{aligned} \langle F(\phi) \rangle_{I,m} &= Z_I(m)^{-1} \mathrm{Tr}_{\{\phi\}} F(\phi) e^{-\beta H_I(m)} , \\ Z_I(m) &= \mathrm{Tr}_{\{\phi\}} e^{-\beta H_I(m)} . \end{aligned} \tag{5.9}$$

Then, we can write

$$\begin{aligned} Z &\equiv \mathrm{Tr}_{\{\phi,\psi\}} e^{-\beta H} = Z_I(m) \mathrm{Tr}_{\{\psi\}} e^{-\beta H_P} \langle e^{-\beta H_C} \rangle_{I,m} \\ &= Z_I(m) \mathrm{Tr}_{\{\psi\}} e^{-\beta H_P - \beta H_C'(m)} , \end{aligned} \tag{5.10}$$

where $H_C'(m)$ is expressed as a cumulant expansion in H_C averaged over the Ising Hamiltonian. Let us display the first cumulant explicitly, by writing

$$\beta H_C'(m) = \tfrac{1}{2} m w_2 \int d^d x \, \psi^2 + \beta H_C''(m) , \tag{5.11}$$

where $H_C''(m)$ involves higher-order Ising correlation functions. For m small, these correlation functions will have their $m=0$ forms up to lengths of order $\xi_T \sim m^{-1}$. However, in determining the critical properties of the percolation system only the forms of the correlation functions up to ξ_p will be relevant. Thus, if $\xi_p \le \xi_T$, we can replace $H_C''(m)$ by $H_C''(0) = H_C'(0)$ in (5.11) without effect. In addition, notice that the first term of (5.11) is quadratic in ψ, so we can absorb it into H_P and thus generate a new nonzero value for the temperature: $r_1 = m w_2$. If we now carry out the steps of Eq. (5.10) in the reverse order, we recover the original coupling H_C and generate $H_I(0)$ rather than $H_I(m)$ so that

$$Z = Z_I(m) Z_I(0)^{-1} \mathrm{Tr}_{\{\psi\}} e^{-\beta \tilde{H}} , \tag{5.12}$$

with $\beta\tilde{H}$ the Hamiltonian of the system with $r_0 = m = 0$ but $r_1 = m w_2$. The prefactor $Z_I(m)/Z_I(0)$ gives the Ising-model behavior as $m \to 0$, while the new nonzero value of r_1 leads to a percolation correlation length $\xi_p \sim |m|^{-\nu_p'}$, so that $\nu_p' = \nu_p$. Notice that the assumption that $\xi_p \le \xi_T$ is satisfied, as it must be for the argument to be self-consistent.

ACKNOWLEDGMENTS

I wish to thank B. I. Halperin for many helpful conversations, and for a careful reading of an earlier version of this paper. It is also a pleasure to acknowledge a number of useful conversations with C. J. Lobb, H. Sompolinsky, D. R. Nelson, and E. Brézin. This work was supported by the National Science Foundation through the Harvard University Materials Research Laboratory and Grant No. DMR-82-07431.

*Present address: Laboratory of Atomic and Solid State Physics, Clark Hall, Cornell University, Ithaca, NY 14853.

[1]For recent reviews of percolation theory, see D. Stauffer, Phys. Rep. 54, 1 (1979); J. W. Essam, Rep. Prog. Phys. 43, 883 (1980), and references therein.

[2]There has been considerable work (see Refs. 7–16) on the percolation properties of the Ising model, which is a correlated percolation problem. Also, G. F. Tuthill and W. Klein [J. Phys. A 15, L377 (1982)] have developed a position-space renormalization-group approach to correlated percolation. In two dimensions they find the usual uncorrelated percolation fixed point and a new one which they identify with Ising correlated percolation. In addition, they suggest that there are only two universality classes for correlated percolation characterized by these two fixed points. This suggestion is in disagreement with the results of this paper.

[3]Abel Weinrib and B. I. Halperin, Phys. Rev. B 27, 413 (1983).

[4]A. B. Harris, J. Phys. C 7, 1671 (1974).

[5]We believe that the universality class of the critical behavior is determined by the two-point correlation function Eq. (1.3). This can be shown to be true near $d=6$ for quite general constraints on the higher-order correlation functions. (See Footnote 18 of Ref. 3 for a discussion of this point for the thermal problem.)

[6]For studies of the effects of correlations on the percolation threshold see, for example, A. Weinrib, Phys. Rev. B 26, 1352 (1982); A. Weinrib and B. I. Halperin, *ibid.* 26, 1362 (1982); G. E. Pike and C. H. Seager, *ibid.* 10, 1421 (1974); C. H. Seager and G. E. Pike, *ibid.* 10, 1435 (1974).

[7]A. Coniglio, H. E. Stanley, and W. Klein, Phys. Rev. Lett. 42, 518 (1979).

[8]A. Coniglio and W. Klein, J. Phys. A 13, 2775 (1980).

[9]A. Coniglio, F. di Liberto, and G. Monroy, J. Phys. A 14, 3017 (1981).

[10]A. Coniglio and F. Peruggi, J. Phys. A 15, 1873 (1982).

[11]A. Coniglio and T. C. Lubensky, J. Phys. A 13, 1783 (1980).

[12]J. Benzoni and J. L. Cardy, University of California at Santa Barbara Report No. TH-59, 1983 (unpublished).

[13]W. Klein, H. E. Stanley, P. J. Reynolds, and A. Coniglio, Phys. Rev. Lett. 41, 1145 (1978).

[14]A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo, J. Phys. A 10, 205 (1977).

[15]K. K. Murata, J. Phys. A 12, 81 (1979).

[16]E. Stoll and C. Domb, J. Phys. A 12, 1843 (1979).

[17]H. Sompolinsky and collaborators have used a similar approach in a real-space renormalization-group study of the correlated percolation problem [H. Sompolinsky (private communication)].

[18]See, for example, T. C. Lubensky, in *Ill Condensed Matter,* edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979), and Ref. 19.

[19]S.-k. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, 1976), Chap. X.

[20]The structure of the calculation is similar for thermal systems with long-range correlated quenched disorder. See Ref. 3; D. Boyanovsky and J. L. Cardy, Phys. Rev. B 26, 154 (1982); S. N. Dorogovtsev, Phys. Lett. 76A, 169 (1980); Fiz. Tverd. Tela (Leningrad) 22, 321 (1980); 23, 1803 (1981) [Sov. Phys.—Solid State 22, 3658 (1980); 23, 1050 (1981)]; Zh. Eksp.Teor. Fiz. 80, 2053 (1981) [Sov. Phys.—JETP 53, 1070 (1981)].

[21]P. W. Kastelyn and C. M. Fortuin, J. Phys. Soc. Jpn. Suppl. 26, 11 (1969); C. M. Fortuin and P. W. Kastelyn, Physica (Utrecht) 57, 536 (1972).

[22]A. B. Harris, T. C. Lubensky, W. K. Holcomb, and C. Dasgupta, Phys. Rev. Lett. 35, 327 (1975).

[23]R. G. Priest and T. C. Lubensky, Phys. Rev. B 13, 4159 (1976).

[24]D. J. Amit, J. Phys. A 9, 1441 (1976).

[25]M. J. Stephen, Phys. Rev. B 15, 5674 (1977).

[26]G. Grinstein and A. Luther, Phys. Rev. B 13, 1329 (1976).

[27]V. J. Emery, Phys. Rev. B 11, 239 (1975).

[28]D. E. Khmelnitskii, Phys. Lett. 67A, 59 (1978).

[29]Robert Gilmore, *Catastrophe Theory for Scientists and Engineers* (Wiley, New York, 1981); J. E. Marslen and J. McCracken, *Hopf Bifurcation and its Applications* (Springer, New York, 1976).

[30]See also Ref. 12.

[31]In Ref. 8 Coniglio and Klein calculate $\nu_p'=0.535$ while our scaling law (1.1) is satisfied: $\nu_p=2/\eta_T$.

[32]Note that the Ising model is unaffected by the coupling to the Potts model via H_C due to the $q\to 1$ limit.