Exact solutions to the time-dependent Landau-Ginzburg model of phase transitions

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Using the Landau-Ginzburg Hamiltonian, which provides a phenomenological description of phase transition, we derive its Lagrangian and subsequently the Lagrange-Euler equations of motion for the order parameter. In the case of one-dimensional systems these equations appear in the form of a nonlinear Schrödinger equation and are solved exactly following the method of Hasse, using the assumption of constant envelope velocity. The resulting time-dependent solutions possess a permanent profile, and their role and range of existence are discussed. The temperature dependence of their velocities and widths is analyzed.

I. INTRODUCTION

This paper is intended to generalize the Landau-Ginzburg model of phase transitions and to investigate the resulting time-dependent solutions of the equation of motion of the order parameter η. We treat the order parameter as a complex time- and space-dependent function and utilize the Hamiltonian approach. The potential function appears as a power series up to order $2n$ in $|\eta|$. The equation of motion also includes a dissipative term. We begin by reviewing the original Landau-Ginzburg approach.

Landau[1] developed a phenomenological theory of second-order phase transitions by expanding the Gibbs free energy G of the system in a series of symmetry invariants of the low-temperature symmetry group g_0 which is also a subgroup of the high-temperature symmetry group g_1,

$$G(T,P)=G_0(T,P)\eta^2+A_4\eta^4+\cdots , \tag{1}$$

where P denotes the external field conjugated with the order parameter η and G_0 is the Gibbs free energy of the high-temperature phase. In order to obtain the desired transition at T_c Landau postulated that

$$A_2(T,P)=a(P)\epsilon , \tag{2}$$

where $\epsilon=(T-T_c)/T_c$ is the normalized temperature. The extremal value of η, $\bar{\eta}$, is found simply by minimizing G,

$$\left[\frac{\partial G}{\partial \eta}\right]_{\eta=\eta}=0 ,$$

and using the stability conditions

$$\left[\frac{\partial^2 G}{\partial \eta^2}\right]_{\eta=\eta}>0 ,$$

$$\bar{\eta}=\begin{cases}\pm[-(a/2A_4)\epsilon]^{1/2}, & T<T_c\\ 0, & T\geq T_c .\end{cases} \tag{3}$$

It is required that $A_4>0$ and $a>0$ for the low-temperature phase to be ordered.

When the expansion Eq. (1) is carried out to the next order,

$$G(T,P)=G_0(T,P)+a(P)\epsilon\eta^2+A_4\eta^4+A_6\eta^6+\cdots , \tag{4}$$

then $\bar{\eta}$ becomes

$$\bar{\eta}=\begin{cases}\dfrac{\pm[-A_4+(A_4^2-4aA_6\epsilon)^{1/2}]^{1/2}}{3A_6}, & T<T_c^*\\ 0, \quad T\geq T_c^* & \end{cases} \tag{5}$$

where the new transition temperature T_c^* is

$$T_c^*=T_c+(A_4^2/4aA_6) , \tag{6}$$

and the transition may be either of second or first order depending on whether $A_4>0$ or $A_4<0$, respectively.[2]

In order to include spatial inhomogeneities of the order parameter, when these inhomogeneities are slowly varying, Ginzburg[3] added another correction to the original Landau expression,

$$G=G_0+\int(A_2\eta^2+A_4\eta^4+D\,|\vec{\nabla}\eta\,|^2)dV , \tag{7}$$

where the sixth-order term of Eq. (4) has been dropped and D is called the dissipation constant. Minimizing this functional with respect to η one obtains[4]

$$2(a\epsilon+2A_4\eta^2-D\nabla^2)\eta=0 , \tag{8}$$

which has two trivial, spatially homogeneous solutions [those of Eq. (5)] and one nontrivial solution (in one-dimensional systems) representing a boundary wall between two possible ordered phases of opposite signs,

$$\eta=\eta_0\tanh[x/\sqrt{2}\xi(T)] , \tag{9}$$

where $\eta_0=\eta(x\rightarrow\infty)$ and $\xi^2(T)\equiv D/|A_2|$. Here ξ is the coherence length. The latter solution exists only below T_c and only if $D>0$.

In order to consider time-dependent phenomena one uses the Gibbs free energy G from Eq. (7) and arrives at the Langevin[5] equation (without the presence of external forces),

$$\dot{\eta}=(-2)\left[(A_2=2A_4\eta^2)\eta-D\frac{\partial^2\eta}{\partial x^2}\right], \quad (10)$$

where the constants, A_2, A_4, and D should be taken in units of the rate constant as Eq. (10) is a rate equation. This equation has been termed as "rather hopeless"[15] to be solved exactly. Therefore, a usual procedure has been to linearize it and use various approximations to perturb the linearized solution.[6] However, Parliński and Zieliński[7] discussed the first-order structural phase transitions in a recent paper and postulated two types of special solutions to Eq. (10). The Gibbs free energy they used can be represented as

$$G=G_0+\int\left[A_2\eta^2+A_{n+2}\eta^{n+2}+A_{2n+2}\eta^{2n+2}+\frac{D}{2}(\vec{\nabla}\eta)^2\right]dV, \quad (11)$$

where $n=2$ corresponds to the standard Landau functional and $n=1$ may be the case in systems where η is time reversal or space-inversion invariant. They obtained an equation of motion analogous to Eq. (10) and postulated the following solutions (which they checked by substitution). (i) A kink in the form of

$$\eta=\eta_0\{1+\exp[\delta(X-X_0-vt)]\}^{-1/n}, \quad (12)$$

where η_0 corresponds to $\bar{\eta}$ of Eq. (5), δ is the reciprocal width, and v is the velocity. (ii) A kink-antikink couple in the form of

$$\eta=\eta_0(1+\exp\{-\delta[X-X_A(t)]\})^{-1/n}\times(1+\exp\{\delta[X-X_B(t)]\})^{-1/n}. \quad (13)$$

The first solution is interpreted as an interface separating the ordered phase from the disordered one, whereas the other solution describes a nucleation center of the ordered phase. Both solutions have permanent profiles and travel with constant velocities. All of their parameters depend significantly on temperature.

It is our intention to provide a derivation of exact solutions of a generalized time-dependent Landau-Ginzburg model without restricting the order of the transition. Instead of the free Gibbs energy we will employ the Hamiltonian formalism and the order parameter will be a complex function. The resultant equation which replaces Eq. (10) is

$$-D\eta_{xx}+[2A_2+(n+2)A_{n+2}|\eta|^2+(2n+2)A_{2n+2}|\eta|^{2n}]\eta=-i\eta_t. \quad (14)$$

II. EQUATION OF MOTION

In this paper we follow the approach taken by Ma[8] and employ a block Hamiltonian for a system undergoing phase transition. The free energy of the system is therefore not used. The necessary steps leading to the final form of the equation of motion may be conveniently summarized as follows.

(1) We use the continuum approximation and include the possibility of inhomogeneities by writing an appropriate Hamiltonian H as

$$H=\int dx[(m/2)\dot{\eta}(x)^2+A_2|\eta|^2+A_{n+2}|\eta|^{n+2}+A_{2n+2}|\eta|^{2n+2}-(D/2)|\vec{\nabla}\eta|^2]\equiv\int dx\,\mathscr{H}(x), \quad (15)$$

where (x) is the Hamiltonian density.

(2) The Landau condition given by Eq. (2) is adopted.

(3) We invoke the Gaussian approximation with a probability density ρ given by

$$\rho=Z^{-1}\exp[-(\eta-\bar{\eta})^2/kT]. \quad (16)$$

Here Z is the appropriate normalization constant and $\bar{\eta}$ is the most probable value of η. Within this approximation $\bar{\eta}=\bar{n}$.

(4) Assuming that the kinetic energy term in Eq. (15) can be treated as a constant we replace the block Hamiltonian by an approximate equivalent Hamiltonian expressed as

$$H=\int dx[_0+A_2\bar{\eta}^2+A_{n+2}|\bar{\eta}|^{n+2}+A_{2n+2}|\bar{\eta}|^{2n+2}-(D/2)|\vec{\nabla}\bar{\eta}|^2]. \quad (17)$$

Here $\bar{\eta}=\bar{\eta}(x)$.

(5) We express $\bar{\eta}$ as a complex function depending upon both space (x) and time (t),

$$\bar{\eta}=|\eta(x,t)|e^{i\phi(x,t)}\equiv\eta e^{i\phi}, \quad (18)$$

where for the sake of convenience we have also denoted $\eta\equiv|\bar{\eta}|$.

(6) By using the Legendre transformation we convert the Hamiltonian density (x) into a Lagrangian density $L(\eta_x,\eta_t,\eta)$. This transformation is readily carried out as follows:

$$H=L-\pi\eta_t,\quad \pi=\frac{\partial L}{\partial\eta_t}, \quad (19a)$$

$$\eta_t=\frac{\partial\eta}{\partial t},\quad \eta_x=\frac{\partial\eta}{\partial x}. \quad (19b)$$

From Eqs. (19) it is easy to see that π is the momentum density conjugate to η. The Lagrangian density resulting from these calculations has the form

$$L=(i/2)(\eta\eta_t^*-\eta^*\eta_t)(D/2)\eta_x^2-A_2|\eta|^2-A_{n+2}|\eta|^{n+2}-A_{2n+2}|\eta|^{2n+2}-_0. \quad (20)$$

(7) The standard Lagrange-Euler equations of motion may be written as

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial\eta_x}+\frac{\partial}{\partial t}\frac{\partial L}{\partial\eta_t}-\frac{\partial L}{\partial\eta}=0. \quad (21)$$

(8) Substituting Eq. (20) in Eq. (21) we get

$$-D\eta_{xx}+[2A_2+(n+2)A_{n+2}|\eta|^n(2n+2)A_{2n+2}|\eta|^{2n}]\eta=-i\eta_t. \quad (22)$$

This is our final equation of motion and is an example of the well-known nonlinear Schrödinger equation.[9] Hasse[10] has presented a method which can be adapted for the exact solution of Eq. (22). In the next section we turn our attention to this aspect of the problem.

III. SOLUTION OF THE NONLINEAR SCHRÖDINGER EQUATION

Representing the potential term in Eq. (22) by

$$W(\eta)=2A_2(n+2)A_{n+2}|\eta|^n+(2n+2)A_{2n+2}|\eta|^{2n}, \tag{23}$$

and substituting $\bar{\eta}$ from Eq. (18) into Eq. (22), we obtain

$$i(\eta_t+i\eta\phi_t)=Dn_{xx}+2iD\eta_x\phi_x+iD\eta\phi_{xx}-D\eta\phi_x^2 - W(\eta)\eta . \tag{24}$$

We may now separate the real and imaginary parts as follows.

(i) The imaginary part yields

$$\eta_t=2D\eta_x\phi_x+D\eta\phi_{xx}, \tag{25}$$

which when multiplied through by 2η transforms into

$$\frac{\partial}{\partial t}(\eta^2)=\frac{\partial}{\partial x}(\eta^2)(2D\phi_x)+\eta^2(2D\phi_{xx}) . \tag{26}$$

Introducing the quantum velocity as

$$v=-2D\phi_x \tag{27}$$

and the density as

$$\rho(x,t)=\eta^2(x,t), \tag{28}$$

we rewrite Eq. (26) in the form of an equation of continuity,

$$\rho_t=-(\rho V)_x . \tag{29}$$

(ii) The real part yields

$$-\eta\phi_t=D\eta_{xx}-D\eta\phi_x^2-W(\eta)\eta, \tag{30}$$

which can be transformed into

$$\phi_t-W(\eta)-(1/4d)v^2+(D/\eta)\eta_{xx}=0 . \tag{31}$$

Integration of Eq. (27) yields

$$\phi(x,t)=(1/2D)\int_{x'=vt}^{x'=x}dx'v(x',t)+f(t), \tag{32}$$

where the function $f(t)$ is selected in the manner suggested by Hasse[10] such that the expectation value of the Hamiltonian gives the energy of the wave packet. To carry out this program we interpret $(2D)^{-1}$ as the effective mass of the quasiparticle described by the Schrödinger equation, Eq. (22): $m=(2D)^{-1}$, and we write

$$\langle H\rangle=-\langle\phi_t\rangle=mv^2/2=v^2/4D . \tag{33}$$

We now assume v to be constant and set $f(t)=v^2/4Dt$ to obtain

$$\phi(x,t)=(1/2D)v(x-vt)+(v^2/4D)t, \tag{34}$$

which yields $-\phi_t=-\langle\phi_t\rangle=v^2/4D$ as required by Eq. (33). Substituting this value of ϕ_t into Eq. (31) we arrive at

$$\eta_{xx}=[(1/2D)v^2+W(\eta)]\eta/D . \tag{35}$$

Because of the assumed constancy of v, Eq. (35) can be solved exactly upon introducing a new variable: $z\equiv(\eta_x)^2$. As a result we transform Eq. (35) into

$$\frac{dz}{d\eta}=\frac{2}{D}\eta\left[\frac{1}{2D}v^2+W(\eta)\right], \tag{36}$$

which can be written in the integral form

$$z(\eta)=\frac{2}{D}\int_0^\eta d\eta\eta\left[\frac{1}{2D}v^2+W(\eta)\right]. \tag{37}$$

By substituting the definition of W from Eq. (23) and carrying out a termwise integration over η we obtain

$$z(\eta)=c_1\eta^2+c_2\eta^{n+2}+c_3\eta^{2n+2}, \tag{38}$$

where the constants are

$$c_1=\frac{1}{D}\left[2A_2+\frac{v^2}{2D}\right], \tag{39}$$

$$c_2=\frac{2A_{n+2}}{D}, \tag{40}$$

$$c_3=\frac{2A_{2n+2}}{D} . \tag{41}$$

We now employ the definition of z to solve for η

$$\begin{aligned}\pm(x-vt)&=\int_{\eta_0}^{\eta}d\eta'z(\eta')^{-1/2}\\ &=\int_{\eta_0}^{\eta}d\eta'[\eta'(c_1+c_2\eta'^n+c_3\eta'^{2n})^{1/2}]^{-1},\end{aligned} \tag{42}$$

where $\eta_0=\eta(x=vt)$. Changing the integration variable to $\zeta=\eta^n$, Eq. (42) becomes

$$\pm(x-vt)=\frac{1}{n}\int_{\eta_0^n}^{\eta^n}d\zeta[\zeta(c_1+c_2\zeta+c_3\zeta)^{1/2}]^{-1}, \tag{43}$$

which upon another change of variable, $\tau=1/\zeta$ assumes the form of a standard elliptic integral,

$$\pm(x-vt)=-\frac{1}{n}\int_{\eta_0^{-n}}^{\eta^{-n}}d\tau(C_3+C_2\tau+c_1\tau^2)^{-1/2}, \tag{44}$$

whose value depends on the constants c_1, c_2, and c_3, and another constant defined as

$$\Delta=4c_3c_1-c_2^2=\frac{4}{D^2}\left[2A_{2n+2}\left[2A_2+\frac{v^2}{2D}\right]-A_{n+2}^2\right]. \tag{45}$$

Equation (44) is obtained, after a series of transformations, as a result of using the Hamiltonian, Eq. (15), to derive the equation of motion of η, Eq. (22). The latter equation has been integrated out by assuming the form of η as in Eq. (18) and keeping the velocity v constant. The variable τ is simply $\tau=\eta^{-n}$. The analytical expressions for the integral Eq. (44) are given in Appendix A. Substituting those expressions into Eq. (44) and solving for η we obtain the following possibilities.

(i) If $c_1 > 0$ and $\Delta > 0$, then

$$\eta(x,t)=\{(1/2c_1)\sqrt{\Delta}\sinh[x_1 \mp n\sqrt{c_1}(x-vt)] -c_2/2c_1\}^{-1/n}, \quad (46)$$

where

$$x_1=\sinh^{-1}[(2c_1\eta_0^{-n}+c_2)/\sqrt{\Delta}] . \quad (47)$$

(ii) If $c_1 < 0$ and $\Delta < 0$, then

$$\eta(x,t)=\{(1/2c_1)\sqrt{-\Delta}\sin[x_2 \pm n\sqrt{-c_1}(x-vt)] -c_2/2c_1\}^{-1/n}, \quad (48)$$

where

$$X_2=\sin^{-1}[(2c_1\eta_0^{-n}+c_2)/\sqrt{-\Delta}] . \quad (49)$$

(iii) If $c_1 > 0$ and $\Delta = 0$, then

$$\eta(x,t)=\{(1/2c_1)\exp[X_3 \pm n\sqrt{c_1}(x-vt)]-c_2/2c_1\}^{-1/n}, \quad (50)$$

where

$$X_3=\ln(2c_1\eta_0^{-n}+c_2) . \quad (51)$$

(iv) If $c_1 > 0$ and arbitrary Δ, then

$$\eta(x,t)=\left[\frac{1+\Delta}{4c_1}\sinh[X_4 \pm n\sqrt{c_1}(x-vt)] + \frac{1-\Delta}{4c_1}\cosh[X_4 \pm n\sqrt{c_1}(x-vt)]-\frac{c_2}{2c_1}\right]^{-1/n}, \quad (52)$$

where

$$X_4=\ln\{2[c_1(c_1\eta_0^{-2n}+c_2\eta_0^{-n}+c_3)]^{1/2}+2c_1\eta_0^{-n}+c_2\} . \quad (53)$$

Note that Eq. (52) becomes identical to Eqs. (50) and (46) if $\Delta=0$ and 1, respectively. A special use of this solution with $\Delta=-1$ and $n=4$ is listed in Table 1 of Hasse.[10]

All the solutions given in Eqs. (46), (48), (50), and (52) are essentially waves propagating through the system with a constant velocity v and a permanent profile. These formulas determine the real part of $\bar{\eta}$ [see Eq. (18)] which is related to the envelope wave,[9] whereas the carrier wave is given by $e^{i\phi}$ where ϕ is displayed in Eq. (34). Therefore, each of these time-dependent solutions can, in general, be written as

$$\eta(x,t)=\eta_0 f[\delta_e(\Delta X_e - V_e t)]\exp[i\delta_c(\Delta X_c - v_c t)] , \quad (54)$$

where the index e refers to the envelope and c refers to the carrier properties, η_0 is the amplitude, and f is the function of the profile expressed in (i)–(iv). It is readily seen that

$$\delta_c=\frac{v}{2},\quad v_c=\frac{v}{2},\quad \delta_e=n\sqrt{c_1},\quad v_e=v . \quad (55)$$

It is perhaps worth mentioning that apart from the time-dependent solutions (i)–(iv), Eq. (22) possesses a number of trivial, time- and/or space-independent solutions which have been presented in conjunction with the earlier models. These are the following.

(v) The solution corresponding to the disordered phase (stable for $T \geq T_c^*$),

$$\eta(x,t)=0 . \quad (56)$$

Here

$$\delta_c=v_c=\delta_e=v_e=\eta_0=0 .$$

(iv) The solution corresponding to the homogeneously ordered phase

$$\eta(x,t)=\pm\left[\frac{-(n+2)A_{n+2}+[(n+2)^2A_{n+2}^2-8(2n+2)A_2A_{2n+2}]^{1/2}}{2A_{2n+2}}\right]^{1/n} . \quad (57)$$

This solution exists only when $T < T_c^*$. Here $\delta_c=v_c=v_e=0$; $\delta_e=\infty$ and $\eta_0 \neq 0$.

(vii) Provided $T < T_c^*$ and $D > 0$, there exist solutions describing a stationary boundary wall between the solutions (vi) with opposite signs,

$$\eta(x,t)=\eta_0\tanh\frac{x}{\sqrt{2}\xi(T)} . \quad (58)$$

Here $\delta_c=v_c=v_e=0$; $\delta_e=[\sqrt{2}\xi(T)]^{-1}$. In the next section we proceed to analyze in detail the properties and existence ranges of the time-dependent solutions (i)–(iv).

IV. DISCUSSION OF THE SOLUTIONS

We now discuss the constraints imposed on the parameters c_1, c_2, and c_3 and through them on the coefficients A_2, A_4, A_6, and D in order for the time-dependent solutions to exist. From the requirements on the signs of c_1 and Δ stated for each of the solutions we easily deduce the following.

(1) Solution (i) requires that $D > 0$ (dissipative systems) and the temperature range must be restrained to

$$T > T_c\left[1-\frac{v^2}{4aD}+\frac{A_{n+2}^2}{2A_{n+2}^2}\right]\equiv T_{c_1} . \quad (59)$$

However, T_{c_1} may be made arbitrarily small by the appropriate choice of parameters.

(2) Solution (ii) requires that $D < 0$ and $T > T_{c_1}$.

(3) Solution (iii) exists only if $D > 0$.

(4) Solution (iv) may exist both if $D < 0$ and if $D > 0$. In the former case one needs

$$T < T_c\left[1-\frac{v^2}{4aD}\right]\equiv T_{c_2} , \quad (60)$$

and in the latter case one needs $T > T_{c_2}$. From the form of these solutions we note that f contained in Eq. (54) is comprised of a function raised to the $(-1/n)$th power. Therefore, additional restrictions arise if n is even ($n=2$ is the most common case). In this case we require that in the entire range of $x-vt$, the function whose root is taken should be positive, otherwise unphysical divergences appear.

Upon analyzing this condition in all the four cases we find the following additional restrictions that have to be imposed in order for the solutions to attain physical meaning.

(1) Solution (i) requires that $c_2 < 0$ or else there will be at least one singular point. This combined with the previous requirement that $D > 0$ results in $A_{n+2} < 0$ which means that only systems exhibiting first-order transitions may possess this solution.

(2) Similarly, solution (ii) may only exist if $c_2\sqrt{-\Delta} > 1$ which implies that $c_2 > 0$ and $(c_2 - 4c_3c_1)^{1/2}$. However, c_3 is positive and c_1 is negative, and therefore the latter conditions can never be realized and consequently solution (ii) is unphysical.

(3) Solution (iii) exists only if $c_2 < 0$ which is equivalent to $A_{n+2} < 0$. The latter condition is satisfied only by systems undergoing first-order transitions.

(4) Solution (iv) can only be admitted if $c_2 < -2\sqrt{c_1c_3}$, which in terms of the Hamiltonian constants, is written as

$$\frac{A_{n+2}}{D} < -\frac{1}{|D|}\left[2A_{2n+2}\left[2A_2 + \frac{v^2}{2D}\right]\right]^{1/2} . \quad (61)$$

This inequality is satisfied irrespective of the sign of A_{n+2} and therefore solution (iv) may exist in system exhibiting both first- and second-order transitions. The assumption that $D > 0$ is followed by $A_{n+2} < 0$, and conversely if $D < 0$; then it is required that $A_{n+2} > 0$. We conclude that solution (iv) may exist either in systems exhibiting first-order transitions with $D > 0$ or in systems exhibiting second-order transitions with $D < 0$. The final analysis we make is with respect to the form of the traveling solutions. Using Eq. (54) and Eqs. (46), (48), (50), and (52), we notice the following common features.

(1) The envelope velocity $v_e = v$ is given in Eq. (33) where we adopt

$$\langle H \rangle = A_2\bar{\eta}^2 + A_{n+2}\bar{\eta}^{n+2} + A_{2n+2}\bar{\eta}^{2n+2} \quad (62)$$

along with Eq. (57) for $\bar{\eta}$. Therefore we find that $\bar{\eta} \to 0$ as $T \to T_c^*$ and $\bar{\eta} = 0$ when $T > T_c^*$, $v \to 0$ as $T \to T_c^*$, and $v = 0$ when $T > T_c^*$. The envelope velocity v decreases continuously to zero with temperature if the transition is of second order and has an abrupt discontinuity at T_c^* in the case of a first-order transition. We conclude that all time-dependent solutions become stationary (or vanish altogether) above the transition temperature T_c^*.

(2) In all four cases the envelope width is $\delta_e = n\sqrt{|c_2|}$ and thus $\delta_e \to 0$ as $T \to T_c^*$.

(3) The profile of the time-dependent solutions is preserved throughout their propagation and can be desired as follows. Solution (i) is a kink-antikink couple, solution (ii) is a periodic function, solution (iii) is a kink, and solution (iv) is of an intermediate form between (i) and (iii). Solution (iv) becomes (i) when $\Delta = 1$ and (iii) when $\Delta = 0$. We have depicted these four profiles in Figs. 1–4.

(4) The amplitude for each solution can be written as

$$\eta_0 = (1/2c_1)^{-1/n} \text{ and } \eta_0 \to 0 \text{ as} T \to T_c^* .$$

The properties of solutions (i)–(iv) are summarized in Table I.

Finally, by comparison with the work of Parliński and Zieliński[7] we note that solution (iii) corresponds to their kinklike solution which has been interpreted as an interface between two allowed phases and solution (ii) corresponds to their kink-antikink couple which has been identified with the nucleation center. We emphasize here that we have found that these solutions exist only for first-order phase transitions in a limited temperature range and they require that $D > 0$ which characterizes a dissipative medium. The other two solutions which we include in Table I have not been demonstrated previously. The periodic solution, however, may only exist if n is odd (the order parameters must be invariant under time reversals

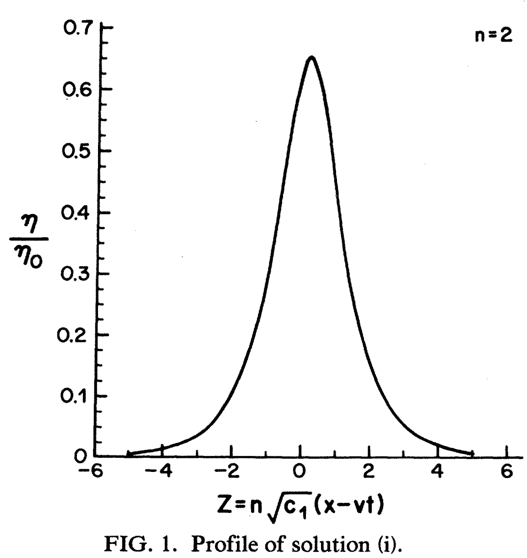

FIG. 1. Profile of solution (i).

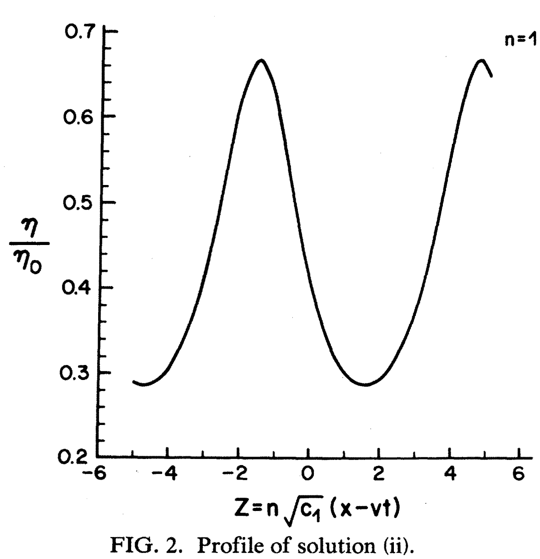

FIG. 2. Profile of solution (ii).

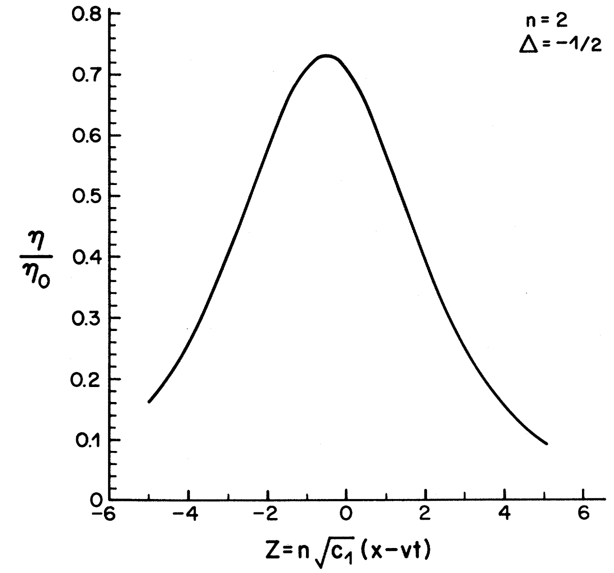

FIG. 3. Profile of solution (iii).

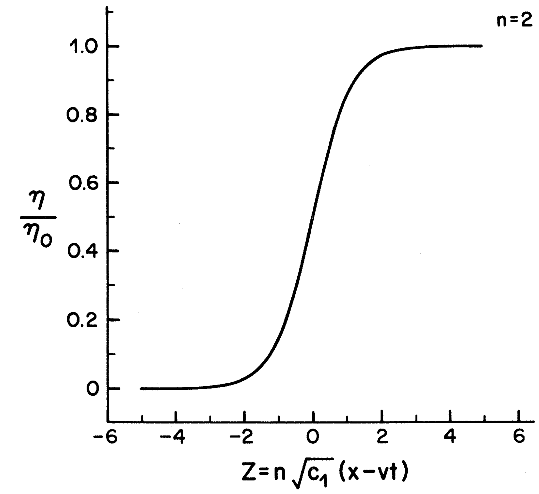

FIG. 4. Profile of solution (iv).

or space inversions). Furthermore, solution (iv) exists both in systems undergoing first- and second-order transitions. Both these new solutions appear if only the appropriate temperature ranges are ensured to coincide with the prescribed sign of the dissipation constant D. We may identify solution (ii) with elementary excitations[11] around the extremal value $\bar{\eta}$ and solution (iv) with local regions of order within the correlation length. Also notice that all the time-dependent solutions can propagate in both directions along the x axis and can have both positive and negative values of the amplitude as long as no external field breaking this symmetry is applied.

The physical meaning of the solutions with $D<0$ is, however, uncertain to the authors, as for the order parameter to be stable against fluctuations it is required that $D>0$. The opposite case can only be taken if a higher-order term proportional to $|\nabla^2\eta|$ is present to ascertain the stability. We intend to examine this question and also investigate the role of external fields in a future paper.

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APPENDIX

From the table of integrals[12] we find that the value of the following elliptic integral,

$$I=\int dx(c_3+c_2x+c_1x^2)^{-1/2}\ , \tag{A1}$$

depends on the sign of c_1 and $\Delta=4c_1c_3-c_2^2$, namely the following four possibilities exist.

(i) If $c_1>0$, then

$$I=(1/\sqrt{c_1})\ln\{2[c_1(c_3+c_2x+c_1x^2)]^{1/2}+2c_1x+c_2\}\ . \tag{A2}$$

TABLE I. Properties of solutions (i)—(iv).

Solution	Profile	Dissipation constant	Temperature	If $n=2$
(i)	Kink-antikink couple	$D>0$	$T_c^*\geq T>T_{c_1}$	First order
(ii)	Periodic function	$D<0$	$T_c^*\geq T>T_{c_1}$	Disappears
(iii)	Kink	$D>0$	$T_c^*\geq T$	First order
(iv)	Kink-antikink couple or kink depending on Δ	$D<0$	$T<T_{c_2},\ T\leq T_c^*$	Second order
		$D>0$	$T_c^*\geq T>T_{c_2}$	First order

(ii) If $c_1 > 0$ and $\Delta > 0$, then

$$I = (1/\sqrt{c_1})\sinh^{-1}[(2c_1 x + c_2)/\sqrt{\Delta}] . \quad \text{(A3)}$$

(iii) If $c_1 < 0$ and $\Delta < 0$, then

$$I = (-1/\sqrt{-c_1})\sin^{-1}[(2c_1 x + c_2)/\sqrt{-\Delta}] . \quad \text{(A4)}$$

(iv) If $c_1 > 0$ and $\Delta = 0$, then

$$I = (1/\sqrt{c_1})\ln(2c_1 x + c_2) . \quad \text{(A5)}$$

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