Borel-Padé analysis for the two-dimensional electron in a random potential under a strong magnetic field

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Perturbational calculation of the diffusion constant limited to the lowest Landau level is studied for the two-dimensional electron in the presence of a strong magnetic field and random impurities. This expansion is asymptotic and the conductivity at the band center is estimated as $\sigma_{xx}=1.4e^2/2\pi^2\hbar$ by the Borel-Padé approximation.

A two-dimensional electron in the presence of a strong magnetic field shows a quantized Hall effect. This phenomenon is considered to be connected strongly to the localization of the electron in random impurities and also to the existence of the extended state. Recently Wegner[1] has obtained an exact expression for the density of states in the case of a strong magnetic field and a white-noise distribution of impurities. Further study for the arbitrary short-range random distributions has been worked exactly.[2] However, for the conductivity an exact expression has not been obtained as yet. In this paper, we consider the perturbational expansion about the strength of the random potential. The calculation is simple for the two-particle density-density correlation function due to the fact that only two determinants are necessary for each diagram. This perturbational expansion is asymptotic. The expansion coefficients of the diffusion constant at the band center show alternative signs and the series is Borel summable. By Borel-Padé analysis, the conductivity at the band center is shown to be nonvanishing. For the other energies away from the band center, the series is more complicated. We have not obtained a definite conclusion due to the shortness of the series, but the series shows the tendency of the vanishing diffusion constant energy for the band center.

The lowest Landau level is spanned by the orthogonal set of functions under $eA=B/2(y,-x,0)$ gauge[3] as

$$u_{0,l}(r)=(2^{l+1}\pi l!)^{-1/2}z^l\exp\left(-\frac{eB}{4}|z|^2\right)(eB)^{(l+1)/2}\ ,\quad (1)$$

where B is the magnetic field and $z=x+iy$. The one-particle Green's function becomes

$$\left\langle r\left|\frac{1}{E-H}\right|r'\right\rangle=\frac{eB}{2\pi(E-\hbar/2\omega_c)}\exp\left(-\frac{eB}{4}(|z|^2+|z'|^2)+\frac{eB}{2}z'^*z\right)\ ,\quad (2)$$

with $\omega_c=eB/m_0$, where m_0 is the mass of the electron. The random potential is assumed to take a white-noise distribution,

$$\langle V(r)V(r')\rangle_{\rm av}=w\delta(r-r')\ .\quad (3)$$

The wave function is Gaussian and the impurity scattering is calculated by a Gaussian integral.[1] The nth impurity scattering gives the contribution to the one-particle Green's function as

$$\left\langle\left\langle r\left|\frac{1}{E-H}\right|r\right\rangle_{n\rm th}\right\rangle_{\rm av}=\gamma^{2n+1}\frac{(2\pi w)^n}{(\det M_1)(eB)^n}\ ,\quad (4)$$

where $\gamma=1/2\pi(E-\hbar\omega_c/2)$ and matrix M_1 is related to the diagram, and the determinant of this matrix is equal to the number of Euler trails.[1]

For the two-particle Green's function, we consider a retarded-advanced Green's-function pair which gives diffusion propagators. We denote this quantity as $K(q)$ and $K(q)$, in turn, is expressed by the determinants,

$$K(q)=\int\left\langle r\left|\frac{1}{E-H+i0}\right|r'\right\rangle_{\rm av}\left\langle r'\left|\frac{1}{E-H-i0}\right|r\right\rangle e^{-iq(r-r')}d^2r=\sum\frac{C}{\det M_1}\exp\left(-\frac{\det M_2}{2\det M_1}q^2\right)\ ,\quad (5)$$

where we have used ($\lambda=\det M_2/\det M_1$)

$$\int\frac{d^2p}{2\pi}\exp\left(i\vec{q}\cdot\vec{p}-\frac{p^2}{2\lambda}\right)=\lambda\exp\left(-\frac{\lambda}{2}q^2\right)\ .$$

The constant C in Eq. (5) depends on the one-particle Green's function. Matrix M_2 is obtained from matrix M_1 by putting 1 in the corresponding element due to insertion of r' as in Fig. 1. In the diagram of Fig. 1, for example, matrices M_1 and M_2 become

$$M_1=\begin{pmatrix}2&0&-1&0\\-1&2&0&-1\\0&-1&2&-1\\-1&0&-1&2\end{pmatrix},\quad M_2=\begin{pmatrix}2&0&-1&0\\-1&2&0&-1\\0&-1&2&0\\-1&0&-1&2\end{pmatrix}\ .\quad (6)$$

Instead of the unperturbed propagator

$$\gamma\pm=[2\pi(E-\hbar\omega_c/2\pm i\epsilon/2)]^{-1}\ ,$$

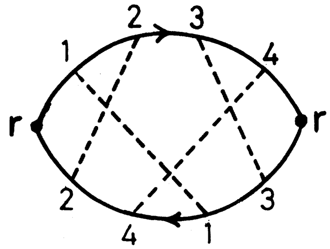

FIG. 1. Diagram of retarded-advanced Green's function with impurity scatterings [Eq. (5)]. r' is inserted between 4 and 3, then matrix M_2 [Eq. (6)] is modified from M_1.

we use the renormalized true propagator $\Gamma_\pm$

$$\Gamma_\pm = 1/2\pi\left[E-\frac{\hbar}{2}\omega_c-\frac{1}{2}\Sigma_\pm \pm \frac{i}{2}\epsilon\right] . \quad (7)$$

The two-particle Green's function $K(q)$ has a diffusion pole for small q and for small ϵ. To obtain an explicit series, we expand as ($\tilde{w}=2\pi w$)

$$\frac{K(q=0)}{K(q)}=\frac{Dq^2+\epsilon}{\epsilon}$$
$$=1+\frac{q^2}{2}\frac{1-\Gamma_+^3\Gamma_-\tilde{w}^2/4-\Gamma_-^3\Gamma_+\tilde{w}^2/4+\cdots}{1-\Gamma_+\Gamma_-\tilde{w}-\Gamma_+^2\Gamma_-^2\tilde{w}^2/2+\cdots} , \quad (8)$$

where D is the diffusion coefficient. The propagator $\Gamma_\pm$ has real and imaginary parts. We write

$$\Gamma_\pm = 1/(A_1 \pm iA_2) = C\exp(\pm i\theta) .$$

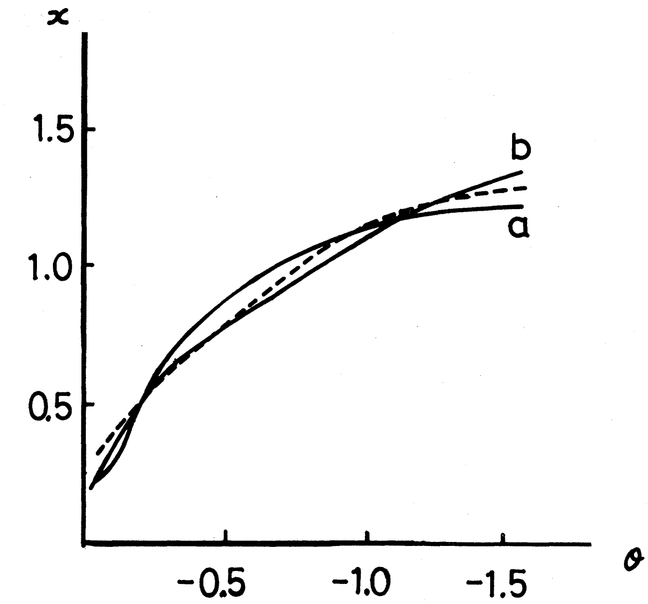

FIG. 2. Lines a and b represent (2,1) and (3,1) complex Borel-Padé approximation, respectively, and the dashed line shows the exact value for x. $\theta=-\sin^{-1}[\rho/(\rho^2+\zeta^2)^{-1/2}]$.

A_1 and A_2 are given as

$$A_1=2\pi\left[E-\frac{\hbar}{2}\omega_c-\frac{1}{2\pi}\mathrm{Re}\Sigma_+\right] , \quad (9a)$$

$$A_2=2\pi\left[\frac{\epsilon}{2}-\frac{1}{2\pi}\mathrm{Im}\Sigma_+\right] . \quad (9b)$$

From Eqs. (8) and (9b), and the fact that $\mathrm{Im}\Sigma_+$ has a factor A_2, the infinitesimally small quantity $\pi\epsilon/A_2$ is written in the following series by the small variable $x=C^2\tilde{w}$,

$$\frac{\pi\epsilon}{A_2}=1-x-\frac{1}{2}x^2\frac{\sin3\theta}{\sin\theta}-\frac{5}{4}x^3\frac{\sin5\theta}{\sin\theta}-\frac{41}{8}x^4\frac{\sin7\theta}{\sin\theta}-\frac{449}{16}x^5\frac{\sin9\theta}{\sin\theta}-\frac{5993}{32}x^6\frac{\sin11\theta}{\sin\theta}-\cdots . \quad (10)$$

In the limit of vanishing ϵ, x is equal to $2\pi^3 w(\rho^2+\zeta^2)$, where ρ and ζ are $-\mathrm{Im}G/\pi$ and $\mathrm{Re}G/\pi$, respectively. By the exact expression for these quantities,[1] ρ and ζ are given as

$$\rho=\frac{\sqrt{2}}{\pi^2\sqrt{w}}\frac{e^{\nu^2}}{1+\frac{4}{\pi}\left[\int_0^\nu e^{t^2}dt\right]^2} , \quad \zeta=\frac{1}{\pi}\left\{\nu\left[\frac{2}{\pi w}\right]^{1/2}-\frac{4}{\pi(2\pi w)^{1/2}}\left[e^{\nu^2}\int_0^\nu e^{t^2}dt\right]\Big/\left[1+\frac{4}{\pi}\left[\int_0^\nu e^{t^2}dt\right]^2\right]\right\} , \quad (11)$$

where $\nu=(\pi/2w)^{1/2}(E-\hbar\omega_c/2)$. The angle θ is expressed as $\sin\theta=-\rho(\rho^2+\zeta^2)^{-1/2}$.

The expansion of Eq. (10) is asymptotic. The asymptotic series $f(x)=\sum a_n x^n$ is written by the series $g(z)=\sum a_n x^n/n!$ as a Borel-Padé approximation,

$$f(x)=\int_0^\infty e^{-z/x}\frac{1}{x}g(z)\,dz , \quad (12)$$

where $g(z)$ is approximated by a ratio of polynomials. We apply this Borel-Padé analysis. Considering $xe^{2i\theta}$ as a complex variable and performing the Borel-Padé approximation, we obtain an accurate value x which satisfies Eq. (10) in the small-ϵ limit. In Fig. 2, the Borel-Padé result is compared with the exact value. At the band center, x becomes $4/\pi$ for $\theta=-\pi/2$. In Table I, the value for x by Borel-Padé approximation for $\theta=-\pi/2$ is presented.

The diffusion constant is obtained from the numerator of Eq. (8),

TABLE I. Borel-Padé analysis for x which gives $\pi\epsilon/A_2=0$ at the band center $\theta=-\pi/2$. The exact value is $x=4\pi=1.273$. (n,m) are degrees of polynomials of Padé:

$$g(z)=\sum_{i=0}^{n} a_i z^i / \sum_{j=0}^{m} b_j z^j \; .$$

n / m	1	2	3	4
1	1.69	1.22	1.35	1.40
2	*	1.31	1.27	1.27

TABLE II. (n,m) Borel-Padé analysis for $2\pi D/A_2$ at the value of $x=4/\pi=1.273$.

n / m	1	2	3	4
1	*	1.288	1.458	1.392
2	*	*	1.411	1.440

$$\frac{2\pi D}{A_2}=1-\tfrac{1}{2}x^2\cos2\theta-\left(\tfrac{41}{36}\cos4\theta+\tfrac{17}{36}\cos2\theta+\tfrac{2}{9}\right)x^3-(4.6839\cos6\theta+1.7161\cos4\theta+1.5838\cos2\theta+0.924\,44)x^4$$
$$-(25.809\cos8\theta+9.3880\cos6\theta+6.3208\cos4\theta+7.3002\cos2\theta+4.0538)x^5$$
$$-(174.19\cos10\theta+63.427\cos8\theta+34.897\cos6\theta+33.964\cos4\theta+38.526\cos2\theta+20.092)x^6-\cdots \; . \qquad (13)$$

Notice that x and θ are functions of a single variable ν. For the band center, this series has alternative signs and is Borel summable,

$$\frac{2\pi D}{A_2}=1+0.5x^2-0.888\,89x^3+3.6272x^4$$
$$-19.545x^5+130.126x^6-\cdots \; . \qquad (14)$$

Borel-Padé analysis gives us $2\pi D/A_2=1.4$ for $x=4/\pi-1.273$ (Table II). The conductivity σ_{xx} is related to the diffusion constant by the Einstein relation as

$$\sigma_{xx}=e^2\rho D \; . \qquad (15)$$

For small w, the density of state becomes $\sqrt{2}/\pi^2 w=1/\pi A_2$ and the diffusion constant D is proportional to $A_2/2\pi$ at $E=\hbar\omega_c/2$. Thus, the conductivity is proportional to $e^2/2\pi^2\hbar$ independent of the strength w. Our estimate gives $\sigma_{xx}=1.4e^2/2\pi^2\hbar$.

For $E\neq\hbar\omega_c/2$, the asymptotic series is more complicated and we have not enough terms to discuss a definite value of D. The constant A_2 in Eq. (13) becomes small away from the band center. We eliminate this A_2 by using Eq. (10) as

$$D=\frac{\epsilon}{2}F(x), \quad F(x)=\sum_{i=0}^{\infty} f_i x^i \; . \qquad (16)$$

If $F(x)$ is not divergent (or at least the power of $\ln\epsilon$), then we have a vanishing diffusion constant since ϵ is an infinitesimally small quantity. The expression for the coefficients f_i are easily obtained from Eqs. (10) and (13). In Fig. 3, f_i $(i=1,\ldots,5)$ are plotted for θ. For $\theta=-\pi/2$, $F(x)$ seems divergent $(x=4/\pi)$ and for other values of θ $(E\neq\hbar\omega_c/2)$, f_i shows oscillation for θ and $F(x)$ seems to have a finite value for x which is determined by Padé analysis or exact expression. Therefore, the conductivity seems to be vanishing except the band center. By the renormalization-group analysis,[4–6] it is shown that there is no extended state in a weak magnetic field. But it is discussed that nonperturbative term gives the extended state near band center due to the instanton effect.[7] The relation of our perturbational treatment to the renormalization-group calculation is not fully understood although maximal crossed diagrams are vanishing for both cases. There is an essential difference in that we have here no small parameter as $1/E_F\tau$ in the weak magnetic field. Numerical simulation[8] and self-consistent approximation[9] also suggest that there is no extended state except the band center. We are planning to make a more detailed analysis using higher-order terms. For comparison with experiments, the value of

$$\sigma_{xx}=1.4e^2/2\pi^2\hbar=1.7\times10^{-5}\ \Omega^{-1}$$

is close to metal-oxide-semiconductor field-effect transistor observation,[10] although we have neglected the mixing with other Landau levels.

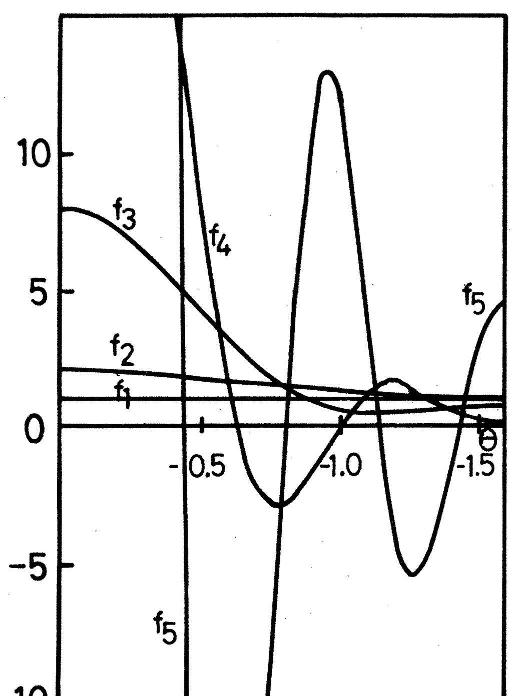

FIG. 3. Coefficients f_i for $D(x)=\sum f_i x^i$ as a function of θ, $\theta=-\sin^{-1}[\rho/(\rho^2+\zeta^2)^{-1/2}]$. $\theta=-\pi/2=-1.57$ corresponds to the band center.

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[1]F. Wegner, Z. Phys. B 51, 279 (1983).
[2]E. Brézin, D. J. Gross, and C. Itzykson, Saclay Report No. spht/83-154 (unpublished).
[3]R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[4]F. Wegner, Z. Phys. B 35, 207 (1979).
[5]E. Brézin, S. Hikami, and J. Zinn-Justin, Nucl. Phys. B 165, 528 (1980).
[6]S. Hikami, Phys. Rev. B 24, 2671 (1981).
[7]H. Levine, S. B. Libby, and A. M. M. Pruisken, Phys. Rev. Lett. 51, 1915 (1983).
[8]T. Ando, J. Phys. Soc. Jpn. 52, 1740 (1983).
[9]Y. Ono, J. Phys. Soc. Jpn. 51, 3544 (1982).
[10]S. Kawaji and J. Wakabayashi, Surf. Sci. 58, 238 (1976).