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Thermopower and thermal conductivity in two-dimensional systems in a quantizing magnetic field

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We have calculated the nondissipative electrical and thermal currents carried essentially by the edge states in a two-dimensional ideal sample in a quantizing magnetic field. Expressions for the thermopower and thermal conductivity are obtained. The Lorenz number oscillates, as a function of the chemical potential, about the standard value. Effects of disorder are briefly examined.

I. INTRODUCTION

The remarkable discovery[1] of quantized Hall resistance in a two-dimensional electron gas, as realized in inversion layers on semiconductor surfaces in a strong magnetic field, has aroused great interest in the theory of transport in two dimensions. This fascinating transport property of a truly two-dimensional system has been extensively investigated both theoretically and experimentally.

It would be very interesting to measure other transport coefficients, for example, the thermopower, which may shed light on the curious transport properties of the inversion-layer electrons. In a recent theoretical study of the inversion-layer thermopower, Girvin and Jonson[2] stressed the importance of thermopower as a means of estimating the possible thermal voltage error in high-accuracy measurements of the quantized Hall resistance.

This paper contains a calculation of the nondiagonal components of the transport coefficients in the disorder-free limit. We discuss in particular the thermopower and the thermal conductivity.

Following the method of Halperin,[3] we calculate the currents carried by the edge states in a Corbino-type geometry in the presence of a magnetic field and a temperature gradient (Sec. II).

We stress the analogy between the thermopower, as a measure of entropy per particle, with the result of Obraztsov[4] in three dimensions (Sec. III). The significant effect of spin splitting on the thermopower is also noted.

The last section is devoted to a discussion of the thermal conductivity. We find that the Lorenz number, in analogy to the thermopower, is a universal function of the ratios $\hbar\omega_c/k_BT$ and $\mu/\hbar\omega_c$, where ω_c is the cyclotron frequency and μ is the chemical potential. This universal dependency on these ratios, peculiar to two-dimensional ideal systems, originates from the simple form of the thermodynamic potential.

II. EDGE CURRENTS

Consider a disorder-free annular sample in the presence of a magnetic field normal to the surface of the sample as illustrated in Fig. 1. As discussed by Halperin, the edge currents indicated in Fig. 1 cancel out in this case. If, however, the chemical potential at the two edges differs by an amount $\delta\mu$, the edge states carry a net current $I=(-e^2/h)n(\delta\mu/e)$ around the loop, where n is an integer. This current is identical with the total electrical current due to the thermodynamical force $\delta\mu/e$, as shown below.

Let us assume that there is a small temperature difference δT between the two edges of the sample. This implies a difference $\delta\mu$ in the chemical potential at the two edges. The Hamiltonian is given by

$$H=\frac{1}{2m^*}\left[-i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right]^2+V(r)\ , \tag{1}$$

where $V(r)$ is a potential which confines the electrons within the region $r_1<r<r_2$. This potential is zero everywhere inside the sample except at regions close to the sample edges. If we employ the gauge $\vec{A}=(A_r,A_\theta,A_z)$ with $A_r=A_z=0$ and

$$A_\theta=\tfrac{1}{2}Br\ , \tag{2}$$

the eigenfunctions may be written as

$$\psi_{m\nu}(\vec{r})\simeq e^{im\theta}f_\nu(r)\ , \tag{3}$$

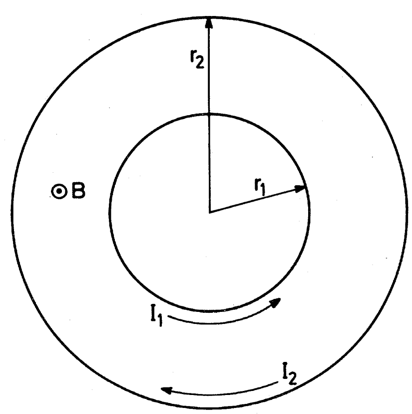

FIG. 1. Annular sample of a two-dimensional electron gas, confined in the region $r_1<r<r_2$, in the presence of a magnetic field B.

where $f_\nu(r)$ satisfies the equation

$$\left[-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial r^2}+\tfrac{1}{2}m^*\omega_c^2(r-r_m)^2+V(r)\right]f_\nu(r)=\epsilon_{m\nu}f_\nu(r)\ . \tag{4}$$

Here r_m is a radius related to the negative integer m by

$$B\pi r_m^2=-m\frac{hc}{e}\ , \tag{5}$$

where $-e$ is the electronic charge. Inside the sample where $V(r)=0$, the eigenenergies $\epsilon_{m\nu}$ are identical with the Landau levels

$$\epsilon_\nu=\hbar\omega_c(\nu+\tfrac{1}{2})\ . \tag{6}$$

In this case, r_m is the center coordinate of the cyclotron motion. It is here assumed that

$$r_2-r_1 \gg R \gg r_c,\quad r_1 \gg r_c\ , \tag{7}$$

where r_c is the cyclotron radius. R is the radial dimension of the regions, at the boundaries of the sample, where the confining potential is different from zero.

The total electrical and thermal currents circulating the sample are given by

$$I=-2e\sum_{m,\nu}f_0(\epsilon_{m\nu})I_{m\nu} \tag{8}$$

and

$$I_q\simeq 2\sum_{m,\nu}(\epsilon_{m\nu}-\mu)f_0(\epsilon_{m\nu})I_{m\nu}\ , \tag{9}$$

where $f_0(\epsilon_{m\nu})$ is the equilibrium Fermi function and

$$I_{m\nu}=\frac{1}{m^*}\int_0^\infty dr\,|\psi_{m\nu}(\vec{r})|^2\left[\frac{m\hbar}{r}+\frac{e}{c}A_\theta\right]$$

$$\simeq\omega_c\int_0^\infty dr\,|\psi_{m\nu}(\vec{r})|^2(r-r_m) \tag{10}$$

is the particle current of the state m,ν. The factor of 2 in Eqs. (8) and (9) account for the spin degeneracy. The effect of spin splitting is not considered at the moment. It is easy to show[5] that

$$I_{m\nu}=\frac{1}{h}\frac{\partial\epsilon_{m\nu}}{\partial m}\ . \tag{11}$$

The current $I_{m\nu}$ vanishes everywhere inside the sample except at the boundaries, as readily seen from Eqs. (10) or (11). For $r_m<r_1+R$ or $r_m>r_2-R$, the radius r_m may no longer be interpreted as the center coordinate of the cyclotron motion and the degeneracy of the Landau levels are lifted as discussed in detail by Halperin.

Let the temperature at the inner edge be T and $T+\delta T$ at the outer edge. We assume that the radius r_1 is sufficiently large so r_m may be considered as a continuous variable. It is readily shown, with the aid of Eq. (11), that the currents become

$$I=\frac{-2e}{h}\sum_\nu\left[-\int_{\epsilon_\nu}^\infty d\epsilon\, f_0(\epsilon,T)+\int_{\epsilon_\nu}^\infty d\epsilon\, f_0(\epsilon,T+\delta T)\right] \tag{12}$$

and

$$I_q=\frac{2}{h}\sum_\nu\left[-\int_{\epsilon_\nu}^\infty d\epsilon(\epsilon-\mu)f_0(\epsilon,T)+\int_{\epsilon_\nu}^\infty d\epsilon(\epsilon-\mu)f_0(\epsilon,T+\delta T)\right]\ . \tag{13}$$

We have here made use of the facts that $\epsilon_{m\nu}\simeq\epsilon_\nu$ for $r_m\simeq r_1+R$ or $r_m\simeq r_2-R$ and that $\epsilon_{m\nu}$ tends to infinity for $r_1-r_m \gg r_c$ or $r_m-r_2 \gg r_c$. Since

$$\frac{\partial f_0(\epsilon)}{\partial T}=-\frac{\partial f_0}{\partial\epsilon}\left[\frac{\epsilon-\mu}{T}+\frac{\partial\mu}{\partial T}\right]\ , \tag{14}$$

the currents may be written as

$$I=\left[\frac{2e^2}{h}\sum_\nu\int_{\epsilon_\nu}^\infty d\epsilon\frac{\partial f_0}{\partial\epsilon}\right]\frac{\delta\mu}{e}-\left[\frac{2e}{h}\sum_\nu\int_{\epsilon_\nu}^\infty d\epsilon\frac{\epsilon-\mu}{T}\left[-\frac{\partial f_0}{\partial\epsilon}\right]\right]\delta T \tag{15}$$

and

$$I_q=\left[\frac{2e}{h}\sum_\nu\int_{\epsilon_\nu}^\infty d\epsilon(\epsilon-\mu)\left[-\frac{\partial f_0}{\partial\epsilon}\right]\right]\frac{\delta\mu}{e}-\left[\frac{2}{h}\sum_\nu\int_{\epsilon_\nu}^\infty d\epsilon\frac{(\epsilon-\mu)^2}{T}\frac{\partial f_0}{\partial\epsilon}\right]\delta T\ . \tag{16}$$

We divide both sides of the above equations with the breadth r_2-r_1 of the sample and express the electrical and thermal current densities in a familiar form:[6]

$$J=\sigma_{\theta r}\frac{\nabla_r\mu}{e}-\beta_{\theta r}\nabla_r T\ , \tag{17}$$

$$J_q=\gamma_{\theta r}\frac{\nabla_r\mu}{e}-\lambda_{\theta r}\nabla_r T\ . \tag{18}$$

Here the off-diagonal transport coefficients $\sigma_{\theta r}$, $\beta_{\theta r}$, $\gamma_{\theta r}$, and $\lambda_{\theta r}$ are given by Eqs. (15) and (16).

The Hall conductivity $\sigma_{\theta r}$ is given by $(-e^2/h)n$, where n is an integer, when the Fermi energy lies between two Landau levels. The thermopower has only a nonvanishing diagonal component, given by

$$Q=\frac{\beta_{\theta r}}{\sigma_{\theta r}}=\frac{1}{e}\left[\sum_\nu\int_{\epsilon_\nu}^\infty d\epsilon\frac{\epsilon-\mu}{T}\frac{\partial f_0}{\partial\epsilon}\right]\Big/\sum_\nu f_0(\epsilon_\nu)\ . \tag{19}$$

The nonvanishing component of the thermal conductivity is the off-diagonal component

$$\kappa_{\theta r}=\lambda_{\theta r}-T\sigma_{\theta r}\left[\frac{\beta_{\theta r}}{\sigma_{\theta r}}\right]^2\ . \tag{20}$$

We note that the above results are also obtained in the case of the usual strip geometry.

III. THERMOPOWER

The thermopower, given by Eq. (19), has been discussed by Girvin and Jonson.[2] They have shown that the thermopower of an ideal two-dimensional electron gas in a quantizing magnetic field is a novel and universal function of the temperature measured in units of the magnetic energy $\hbar\omega_c$.

We shall only stress the analogy with Obraztsov's[4] result in three dimensions, and briefly investigate the effects of spin splitting, disorder, and a periodic potential. In the classical high-field limit where thermal disorder becomes greater than the Landau-level separation, $k_BT > \hbar\omega_c$, the Landau index ν may be assumed continuous and we find, from Eq. (19),

$$Q = -\frac{1}{e}\frac{\pi^2}{3}\frac{k_B^2 T}{\mu} . \tag{21}$$

This is the usual expression for entropy or heat capacity (per particle per unit charge) of a two-dimensional ideal Fermi gas. It is naturally identical with the result obtained from the standard Boltzmann equation in the classical high-field limit.

It is readily confirmed that $\sigma_{\theta r}$ and Q can be rewritten in the form already obtained by Obraztsov[4] as

$$\sigma_{\theta r} = -\frac{Nec}{B} \tag{22}$$

and

$$Q = \frac{\beta_{\theta r}}{\sigma_{\theta r}} = -\frac{S}{eN} , \tag{23}$$

where N and S are the number of electrons per unit area and the entropy per unit area, respectively. These thermodynamic quantities are evaluated from the relationships

$$N = -\left[\frac{\partial \Omega}{\partial \mu}\right]_{T,B} , \quad S = -\left[\frac{\partial \Omega}{\partial T}\right]_{B,\mu} , \tag{24}$$

where the thermodynamic potential (per unit area) is given by

$$\Omega(T,B,\mu) = -k_BT\left[\frac{2eB}{hc}\right]\sum_{\nu}\ln(1+e^{(\mu-\epsilon_\nu)/k_BT}) . \tag{25}$$

We now consider the structure of the thermopower taking spin splitting into account. The effect of spin splitting is included by replacing ϵ_ν in Eq. (19) with

$$\epsilon_{\nu\sigma} = (\nu+\tfrac{1}{2})\hbar\omega_c + \tfrac{1}{2}\sigma g \mu_B B , \tag{26}$$

where σ can take the values ± 1, g is the effective g factor, and μ_B is the Bohr magneton. Figure 2 shows a plot of the thermopower against the chemical potential (in units of $\hbar\omega_c$). The ratio of $\hbar\omega_c$ to k_BT is 53, which is equivalent to a magnetic field strength of 15 T and a temperature of 2 K in the Si(100) inversion layer. An effective g factor of 2 has been used. In the limit of large $\hbar\omega_c/k_BT$, the peak associated with the lowest Landau level tends to $(-k_{B/e})2(\ln 2)/3$, while each of the higher Landau levels are associated with two peaks which tend to the universal values $(-k_B/e)(\ln 2)/(2\nu+1+\sigma/2)$, cf. Fig. 2. At zero temperature, the thermopower or the entropy vanishes as required by the third law of thermodynamics.

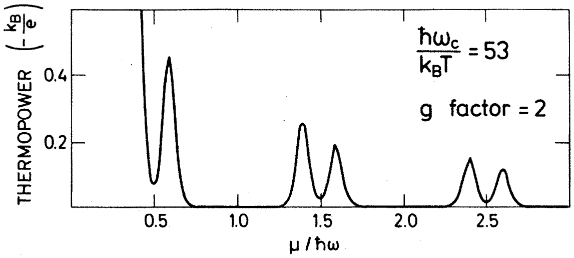

FIG. 2. Thermopower as a function of the chemical potential in units of $\hbar\omega_c$. Splitting of the peaks at the Landau levels is due to the spin of the electrons.

Now consider the effect of disorder. It is interesting to investigate whether the classical high-field result

$$\beta_{ij} = -eL_0T\frac{\partial \sigma_{ij}}{\partial \mu} \tag{27}$$

holds in a quantizing magnetic field. $L_0 = (\pi^2/3)(k_B/e)^2$ is the standard Lorenz number. We find that the above relation does not hold rigorously. Figure 3 (dashed curve) shows the thermopower $Q = \beta_{\theta r}/\sigma_{\theta r}$, where $\beta_{\theta r}$ is evaluated with the aid of Eq. (27) and the exact expression for $\sigma_{\theta r}$, Eq. (22). This approximate result fits fairly well with the exact expression given by Eq. (19); see Fig. 3 (solid line). We assume, on the basis of the above observation,[7] that Eq. (27) holds, approximately, also in the presence of a moderate amount of disorder. Under this assumption we estimate that correction, due to disorder, on the diagonal component of the thermopower is of order $(\sigma_{xx}/\sigma_{xy})^2$. We use henceforth the coordinate notations (x,y) instead of (θ,r). The estimation is valid in the limit of weak disorder, where the width of the Landau levels is of the order of the thermal broadening k_BT. In that case,

$$\frac{\partial \sigma_{xx}}{\partial \mu} \sim \frac{\sigma_{xx}}{k_BT} ,$$

ınd we find, with the aid of Eq. (27), that

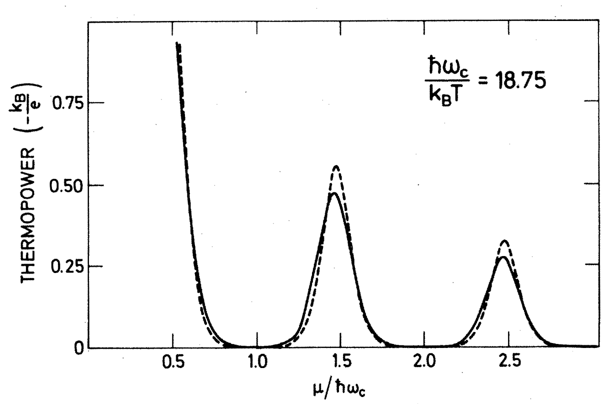

FIG. 3. Thermopower, obtained with the aid of the Mott rule, as a function of the chemical potential in units of $\hbar\omega_c$ (dashed curve). Effect of spin splitting is neglected. The solid curve represents the exact expression for the thermopower.

$$Q_{xx}=\frac{\sigma_{xx}\beta_{xx}+\beta_{xy}\sigma_{xy}}{\sigma_{xx}^2+\sigma_{xy}^2}\simeq\frac{\beta_{xy}}{\sigma_{xy}}+O\left[\frac{\sigma_{xx}^2}{\sigma_{xy}^2}\right].$$

If we carry further the semiclassical treatment, it can be shown, with the aid of Eq. (27), that in the presence of a one-dimensional periodic potential,[8] the thermopower vanishes when the chemical potential lies in a mobility gap.

We next focus on the case where the chemical potential lies in a mobility gap. Streda[9] has shown that the Hall conductivity in this case is given by a quantum correction

$$\sigma=-ec\frac{\partial N}{\partial B}\ . \tag{28}$$

Recently, Widom[10] proposed a formula for the Hall conductivity which happens to give the correct result, Eq. (28), in the limit under consideration. It is interesting to note that the application of Widom's method in the case of a temperature gradient yields a quantum correction on the thermomagnetic coefficient β_{xy}, given by

$$\beta=c\frac{\partial S}{\partial B}\ , \tag{29}$$

where S is the entropy (per unit area) of the conduction electrons. Equations (28) and (29) are analogous. The thermodynamic derivation is accomplished by considering the magnetization current in the presence of only nonuniformities in the chemical potential and temperature. In that case, the edge current due to magnetization is given[10,11] by

$$\begin{aligned}-c\vec{\nabla}\times\vec{\mathrm{M}}&=-c\vec{\nabla}\mu\times\left[\frac{\partial\vec{\mathrm{M}}}{\partial\mu}\right]_{T,\vec{\mathrm{B}}}-c\vec{\nabla}T\times\left[\frac{\partial\vec{\mathrm{M}}}{\partial T}\right]_{\vec{\mathrm{B}},\mu}\\&=\frac{\vec{\nabla}\mu}{e}\times\left[-ec\frac{\partial N}{\partial\vec{\mathrm{B}}}\right]_{T,\mu}-\vec{\nabla}T\times\left[c\frac{\partial S}{\partial\vec{\mathrm{B}}}\right]_{T,\mu}\ ,\end{aligned} \tag{30}$$

where the final expression is obtained with the aid of Maxwell relations. However, it is not obvious from the above derivation that the total conduction current is simply given by the magnetization current when the chemical potential lies in a mobility gap. We are convinced that in the presence of elastic scattering, a more rigorous derivation[12] based on Kubo formalism will confirm that β_{xy} is given by Eq. (29). Thus the nonvanishing component of the thermopower is given by

$$Q=\frac{\beta}{\sigma}=-\frac{1}{e}\frac{\partial S/\partial B}{\partial N/\partial B}\ . \tag{31}$$

We stress that Eq. (31) is expected to be valid only when the chemical potential lies in a mobility gap in a weakly disordered sample and when $\hbar\omega_c \gg k_BT$.

IV. THERMAL CONDUCTIVITY

Figure 4 shows the thermal conductivity, given by Eq. (20), as a function of the chemical potential in units of $\hbar\omega_c$. The effect of spin splitting has been included by replacing ϵ_v with $\epsilon_{v\sigma}$. The stepwise structure (see Fig. 4) indicates that the Weidemann-Franz law is satisfied.

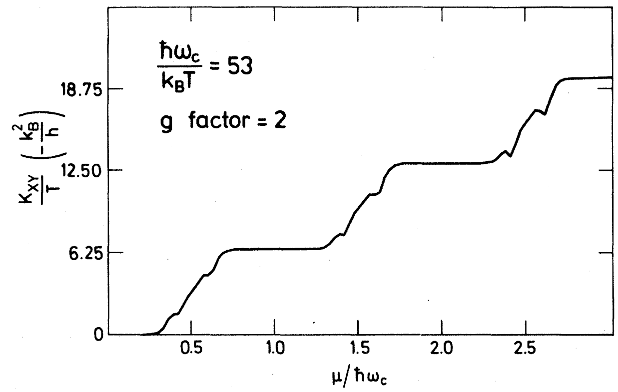

FIG. 4. Thermal conductivity (per unit temperature) as a function of the chemical potential in units of $\hbar\omega_c$. The effect of spin splitting is almost insignificant for the lowest Landau levels.

Figure 5 displays the Lorenz number $\kappa_{xy}/\sigma_{xy}T$ as a function of $\mu/\hbar\omega_c$ for $\hbar\omega_c/k_BT=53$. The Lorenz number is a universal function of $\mu/\hbar\omega_c$ and $\hbar\omega_c/k_BT$. It is interesting to observe, in Fig. 5, the weak oscillatory structure of the Lorenz number about the standard value $L_0=(\pi^2/3)(k_B/e)^2$. When the chemical potential lies between two Landau levels, the Lorenz number is given by L_0. Deviations from this ideal value become more pronounced at intermediate values of the ratio $\hbar\omega_c/k_BT$, for example, $\hbar\omega_c/k_BT=18$. For small values of this ratio, the oscillatory structure naturally disappears and the Lorenz number takes the standard value.

Analogous to the thermopower, we expect correction of order $(\sigma_{xx}/\sigma_{xy})^2$ on the off-diagonal component of the thermal conductivity when the effect of disorder in pure samples is taken into account.

In summary, the off-diagonal components of the transport coefficients in a quantizing magnetic field in ideal two-dimensional systems have been calculated. We have elucidated the structure of the thermopower and revealed

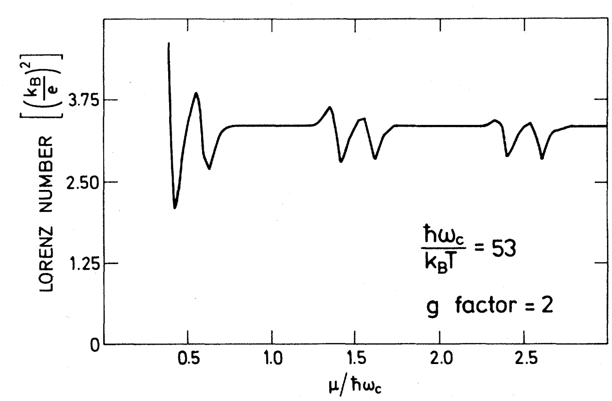

FIG. 5. Lorenz number as a function of the chemical potential in units of $\hbar\omega_c$. As in Fig. 4, spin effect is insignificant for an effective g factor of 2.

novel properties of the Lorenz number in two-dimensions. We have incidentally investigated the validity of the Motts rule in ideal two-dimensional systems and employed this in the estimation of the deviation from ideal transport properties due to the presence of disorder in real samples. There have been reports of measurements of the thermopower in two dimensions in quantizing fields, but as far as we know no firm conclusions have yet been reached.

Note added in proof. Recent theoretical and experimental results for both the diagonal and the off-diagonal components of the thermopower agree quite well [H. Oji, Proceedings of the International Conference on Electronic Properties of Two-Dimensional Systems, Oxford, 1983 (unpublished); H. Oji, Solid State Commun. (to be published)].

[1]K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).

[2]S. M. Girvin and M. Jonson, J. Phys. C 15, L1147 (1982).

[3]B. I. Halperin, Phys. Rev. B 25, 2185 (1982).

[4]Yu. N. Obraztsov, Fiz. Tverd. Tela (Leningrad) 7, 573 (1965) [Sov. Phys.—Solid State 7, 455 (1965)].

[5]This follows, for example, from L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1962), Eq. (72.2). In this case $(1/2\pi)\langle m\nu|\dot{\theta}|m\nu\rangle = \langle m\nu|(1/h)(\partial\hat{H}/\partial m)|m\nu\rangle$, where $\hat{H}$ is given by Eq. (4), except that $m^*\omega_c(r-r_m)$ is replaced with the exact expression $\hbar m/r+(e/c)A_\theta$.

[6]J. M. Ziman, *Electrons and Phonons* (Oxford University, Press, Oxford, 1979), p. 496.

[7]Equation (27) is valid in classical strong fields in the presence of elastic scattering. I. Smrcka and P. Streda [J. Phys. C 10, 2153 (1977)] reached the conclusion that this equation is valid also in a quantizing magnetic field.

[8]Such a potential has been employed in a study of Harper broadening by H. Aoki and T. Ando, Surf. Sci. 113, 27 (1982).

[9]P. Streda, J. Phys. C 15, L717 (1982).

[10]Only a fraction of the Hall current was derived in Widom's paper, A. Widom, Phys. Lett. 90A, 474 (1982), as pointed out by the present author; Phys. Lett. 98A, 127 (1983).

[11]Yu. N. Obraztsov, Fiz. Tverd. Tela (Leningrad) 6, 414 (1964) [Sov. Phys.—Solid State 6, 331 (1964)].

[12]Recently, Eq. (29) has been derived from the Kubo formula: P. Streda and H. Oji (unpublished).