PHYSICAL REVIEW B VOLUME 29, NUMBER 1 1 JANUARY 1984

Q-state Potts model by Wilson's exact renormalization-group equation

Kathie E. Newman,* Eberhard K. Riedel,† and Shunichi Muto‡
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 27 June 1983; revised manuscript received 31 October 1983)

Critical properties of the Q-state Potts model for dimensions $3 \leq d \leq 6$ are calculated by means of Wilson's exact momentum-space renormalization-group equation. The scaling-field method of Golner and Riedel is used to approximate the functional differential equation by a set of 11 ordinary coupled differential equations. For $d = 4-\epsilon$, lines of critical and tricritical Potts fixed points are found as functions of Q that annihilate as Q approaches a critical value $Q_c = 2+\epsilon^2/a + O(\epsilon^3)$. For $Q > Q_c$, the Potts transition is first order. Along these fixed lines the critical and tricritical exponents (upper and lower sign, respectively) are to leading order: $1/\nu = 2-\frac{1}{6}[\epsilon \pm (\epsilon^2 - a\delta)^{1/2}]$, $\phi/\nu = \mp(\epsilon^2 - a\delta)^{1/2}$, and $\eta = [\epsilon \pm (\epsilon^2 - a\delta)^{1/2}]^2/216 + b\delta$, where $\epsilon = 4-d$, $\delta = Q-2$, and $\delta \leq \delta_c = \epsilon^2/a + O(\epsilon^3)$. While the form of the ϵ and δ dependences is exact, the coefficients a and b cannot be obtained systematically by ϵ expansion, since the upper critical dimensionality of the Potts model is six when $Q \neq 2$. In our truncation, $a = 6.52$ and $b = 0.065$. The results have been extended to dimensions $3.4 \leq d \lesssim 4$ by solving the renormalization-group equations numerically. The percolation limit of the Potts model, $Q = 1$, is also investigated and the critical exponents ν^P, ϕ^P, and η^P determined as functions of dimension for $3 \leq d \leq 6$.

I. INTRODUCTION

The Q-state Potts model has been widely studied because of its intrinsic theoretical interest and its many applications to physical systems.[1] In 1973, Baxter found that the phase transition of the two-dimensional Potts model is first order for $Q > Q_c = 4$ but continuous for $Q \leq Q_c$.[2] An explanation of this changeover by renormalization-group (RG) techniques was given by Nienhuis *et al.*[3] These authors showed by conventional position-space RG methods that the dilute Potts model exhibit lines of critical *and* tricritical fixed points as functions of Q that merge and annihilate as Q approaches Q_c. For $Q \leq Q_c$, the phase transition of the pure Potts model is governed by the line of critical fixed points and, for $Q > Q_c$, by a line of first-order discontinuity fixed points. The interest in the two-dimensional Potts model heightened when conjectures were proposed that relate the critical exponents of the Potts model to the thermal exponent of the eight-vertex model. A conjecture for the thermal critical exponent was first proposed by den Nijs,[4] and later extended by Nienhuis *et al.* to the tricritical thermal[3] and critical and tricritical magnetic exponents.[5,6] Recently, all three exponent relations have been obtained analytically.[7–9]

In contrast, the Potts model at dimensions larger than two is much less explored. This article presents a study of the Q-state Potts model for dimensions $3 \leq d \leq 6$.[10] Since different RG techniques are required than for $d = 2$, here the Potts model is cast into a Landau expansion $H_l[\sigma]$, where σ is a continuous-spin variable with $N = Q-1$ vector components.[11] This model is then investigated using Wilson's exact momentum-space RG equation[12] and the scaling-field method by Golner and Riedel.[13,14] The approach allows one to vary continuously the spatial dimension d and number of states Q and, therefore, to apply *both* numerical and expansion techniques. The scaling-field method has also been used for high-precision calculations of the critical exponents of the isotropic N-vector model in three dimensions[15,16] and for studies of the cubic N-vector and the randomly dilute Ising models in dimensions $2.8 \leq d \leq 4$.[17]

The most striking result of the scaling-field calculation for the Potts model is that the changeover from continuous to first-order behavior with increasing Q occurs by the same mechanism near four dimensions as in two dimensions. Lines of critical and tricritical fixed points exist for small Q and annihilate as Q approaches a critical value Q_c that depends on dimension. (Compare to Fig. 1.) This result is obtained by a novel ϵ expansion near four dimensions and numerically for $3.4 \leq 4 \lesssim dx$. We make two observations.

(i) This topology of the RG phase diagram provides a convenient definition of the critical value $Q_c = Q_c(d)$ in terms of the value of Q for which the lines of critical and tricritical fixed points merge. This determines the largest value of Q for which continuous Potts behavior is possible. By imposing constraints, first-order transitions at $Q < Q_c$ can be induced.

(ii) The result of merging lines of fixed points follows from microscopically derived recursion relations without the use of the Potts lattice-gas concept. In fact, the recursion relations differ strongly from the phenomenological ones proposed by Nauenberg *et al.*[18] to model the results by Nienhuis *et al.*[3] In our calculation near four dimensions, the marginal operator associated with the merging lines of fixed points can be interpreted as a dilution operator.

For a small but interesting range of Q, the Q-state Potts model can be discussed by ϵ expansion near four dimensions, although its upper critical dimensionality is $d_c^* = 6$. The reason is that the model reduces to the Ising model when $Q=2$, for which $d_c^*=4$, and so for $\delta=Q-2\ll 1$ the coupling coefficients between the isotropic and Potts field are of order δ. However, the expansion is special in that for the Potts fixed point only the isotropic fields are of order ϵ, while the Potts fields are of order 1, reflecting the fact that the true upper critical dimensionality is 6. Therefore, even for $d=4-\epsilon$, the solution requires a combination of expansion and numerical techniques. We have obtained the following results.[10] In $d=4-\epsilon$ dimensions, the Potts model exhibits lines of critical and tricritical Potts fixed points for

$$Q\le Q_c(d=4-\epsilon)=2+\epsilon^2/a+O(\epsilon^3) \tag{1.1}$$

that annihilate as Q approaches Q_c. For $Q>Q_c$, the phase transition of the model is first order. An earlier result by Aharony and Pytte,[19] according to which $Q_c=2+\epsilon+O(\epsilon^2)$, is incorrect. To leading order in $\epsilon=4-d$ and $\delta=Q-2$ where $\delta\le\delta_c=\epsilon^2/a+O(\epsilon^3)$, the critical and tricritical exponents (upper and lower signs, respectively) are

$$\frac{1}{\nu}=2-\tfrac{1}{6}[\epsilon\pm(\epsilon^2-a\delta)^{1/2}]\ , \tag{1.2}$$

$$\phi/\nu=\mp(\epsilon^2-a\delta)^{1/2}\ , \tag{1.3}$$

and

$$\eta=\tfrac{1}{216}[\epsilon\pm(\epsilon^2-a\delta)^{1/2}]^2+b\delta\ . \tag{1.4}$$

ϕ denotes both the correction-to-scaling exponent $\phi=-\Delta_c$ (upper sign) and tricritical crossover exponent $\phi=\phi_t$ (lower sign). These results are displayed for $d=3.98$ in Fig. 1 below. For $Q=2$ or $N=1$ the results reduce to those of the Ising model. Also, note the similarity between the above results and those for $d=2$, where, to order δ' with $\delta'=4-Q$,[7–9]

$$1/\nu=\tfrac{3}{2}\mp(3/\pi)(\delta')^{1/2}\ , \tag{1.5}$$

$$\phi/\nu=\mp(8/\pi)(\delta')^{1/2}\ , \tag{1.6}$$

and

$$\eta=\tfrac{1}{4}\pm[(\delta')^{1/2}/2\pi]+(\delta'/\pi^2)\ . \tag{1.7}$$

While the form of the ϵ and δ dependences of the exponents (1.2)—(1.4) is exact, the coefficients a and b cannot be calculated by ϵ expansion since they are determined by the fixed-point coordinates of the scaling fields associated with terms of Potts symmetry, which are of *order 1* near four dimensions. This makes it difficult to determine whether a and b are universal and independent of details of the model. As expected, the values of a and b from the truncated RG equations are not. The truncation of the infinite set of scaling-field equations to 11 coupled differential equations yields $a=6.52$ and $b=0.065$.

The second purpose of this article is to report results for the percolation or one-state Potts problem in dimensions $3\le d\le 6$. The critical exponents of the percolation problem are known exactly for $d=2$ (Refs. 7—9) and $6-\epsilon$.[20] The interpolation as function of dimension between these results has attracted much interest, partly because field-theoretical expansion techniques that worked well in other cases failed to yield the percolation exponents in three dimensions to satisfactory accuracy. Specifically, recent studies by de Alcantara Bonfim *et al.*,[21] Fucito and Marinari,[22] and Reeve *et al.*[23] give no indication for the expected sign change in the exponent η as function of dimension. The scaling-field method is well suited for investigating this question since the dimension can be varied continuously and the calculation performed directly for the dimension of interest. The results for our truncation to 11 scaling-field equations show the expected dependence on dimensionality, but larger truncations must be considered for improved precision.

The outline of this paper is as follows. In Sec. II the Landau Hamiltonian for the Q-state Potts model is defined and the scaling-field formalism described. Results for the Q-state Potts model in dimensions $3.4\le d\lesssim 4$ for general Q and in dimensions $3\le d\le 6$ for $Q=1$ are given in Sec. III. Section IV presents a brief summary. Appendix A provides a guide to the derivation of the scaling-field coupling coefficients and Appendix B presents comments on ϵ expansion near six dimensions.

II. MODEL AND METHOD

To apply Wilson's exact momentum-space RG equation [see Eq. (11.17) of Ref. 12], one must first write the Potts model in terms of a continuous-spin Landau Hamiltonian $H_l[\sigma]$. Here we follow the N-vector-model formulation by Zia and Wallace,[11] adapted to the Wilson equation. The RG investigation then uses the scaling-field method.[16,17] We discuss the method first.

Wilson's exact RG equation has the form of a functional differential equation for the Hamiltonian $H_l[\sigma]$. The idea of the scaling-field approach is to transform this equation into a set of ordinary differential equations, which in turn can be solved by successive approximation. This transformation is carried out in three steps. First, we construct the Gaussian eigenfunctionals $Q_m[\sigma]$ and eigenvalues y_m^G by linearizing the Wilson equation about the Gaussian fixed-point solution $H_G^*[\sigma]$. Then, assuming that the eigenfunctionals $Q_m[\sigma]$ provide a basis in the space of RG Hamiltonians, we expand $H_l[\sigma]$ as follows:

$$H_l[\sigma]=H_G^*[\sigma]+\sum_m \mu_m(l)Q_m[\sigma]\ . \tag{2.1}$$

The expansion coefficients $\mu_m(l)$ in this equation are referred to as scaling fields. Finally, by substituting this expansion into the Wilson equation and projecting all terms onto the basis operators, we obtain the set of scaling-field differential equations,

$$\frac{d\mu_m(l)}{dl}=y_m^G\mu_m(l)+\sum_{j,k}a_{mjk}\mu_j(l)\mu_k(l)+\sum_j a_{mj}\mu_j(l)+a_m\ . \tag{2.2}$$

The coupling coefficients a_{mjk}, a_{mj}, and a_m of Eq. (2.2) are computed using an algorithm that involves an operator-product expansion.[14,16,17] These coefficients are products of combinatorial factors and momentum integrals; for details see also Appendix A. Via these coupling coefficients, Eqs. (2.2) depend on dimension d, number of spin components N (and thus, number of states Q), and a spin-rescaling parameter Δ (in Wilson and Kogut's[12] notation, $\Delta \equiv 1 - d\rho/dl$). The scaling-field equations can be solved in certain limits by ϵ or $1/N$ expansion. Alternatively, the equations can be used to define approximations by truncation. Both methods will be employed in the following discussion.

Information about critical properties follows by studying the recursion relations (2.2) in the usual way.[16,17] The leading thermal eigenvalue equals the inverse of the correlation-length exponent ν and the ratio of the two leading thermal eigenvalues determines the crossover exponent or correction-to-scaling exponent ϕ. The correlation-function exponent η is determined by the fixed-point value Δ^* of the rescaling parameter Δ:

$$\eta = 2\Delta^* \ . \tag{2.3}$$

A condition for calculating Δ^* numerically has been developed. We search for the value of Δ that produces a fixed point with marginal (i.e., zero) eigenvalue.[16,17,24] In our ϵ-expansion approach near four dimensions, we use a different though equivalent condition;[17] however, for ϵ expansion near six dimensions, see Appendix B. The scaling-field method is general and can also be used to study thermodynamic properties of the Potts model.[16]

For the Q-state Potts model, the expansion (2.1) requires in principle all eigenfunctionals of isotropic and Potts symmetry. The latter is the permutation group of Q objects. The calculation proceeds as in Sec. II of Ref. 17, where the scaling-field method is applied to the cubic N-vector model. The Gaussian eigenfunctionals $Q_m[\sigma]$ are defined by Eqs. (2.7)—(2.9) of that reference. They are polynomials in the spin variables $\vec{\sigma}$ and momenta $\underset{\sim}{q}$. For the Potts case, the spin variables are $N = Q-1$ component vectors, $\vec{\sigma} = \{\sigma_\alpha; \alpha = 1, \ldots, Q-1; \ -\infty < \sigma_\alpha < \infty\}$. Two indices, $\overline{m}$ and $\underline{l}$, are required to label the spin dependences of isotropic and Potts symmetry and one principal index, p, is required to characterize the momentum dependence. The latter is parametrized in terms of the set of homogeneous functions in $\underset{\sim}{q}$ to all orders $p, f_p\{\underset{\sim}{q}\}$. We use the index "$m$" to denote the set of indices,

$$m = \{\overline{m}, \underline{l}; p, \ldots\} \ . \tag{2.4}$$

(The notation is chosen so that the cubic eigenfunctionals of Ref. 17 can be easily included in future calculations.) The spin dependence of the Gaussian eigenfunctionals $Q_m[\sigma]$ is defined via the functionals $R_m[\sigma]$ in Eq. (2.9) of Ref. 17. The isotropic eigenfunctionals are defined by

$$R_{\overline{m}0}[\sigma] = \prod_{i=1}^{\overline{m}/2} \vec{\sigma}(\underset{\sim}{q}_{2i-1}) \cdot \vec{\sigma}(\underset{\sim}{q}_{2i}) \ . \tag{2.5}$$

The Potts eigenfunctionals to order σ^6 (written without momentum dependence on the spin variable) are given by

$$R_{0\underline{3}}[\sigma] = N^{3/2} \sum_{\alpha,\beta,\gamma} Q_{\alpha\beta\gamma}\sigma_\alpha\sigma_\beta\sigma_\gamma \ , \tag{2.6}$$

$$R_{2\underline{3}}[\sigma] = R_{20}[\sigma] R_{0\underline{3}}[\sigma] \ , \tag{2.7}$$

$$R_{0\underline{4}}[\sigma] = N \left[N(N+2) \sum_{\alpha,\beta,\gamma,\delta} F_{\alpha\beta\gamma\delta}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta - 3(N+1)R_{40}[\sigma] \right] , \tag{2.8}$$

$$R_{2\underline{4}}[\sigma] = R_{20}[\sigma] R_{0\underline{4}}[\sigma] \ , \tag{2.9}$$

$$R_{0\underline{5}}[\sigma] = N^{5/2} \left[(N+6) \sum_{\substack{\alpha,\beta,\gamma,\\ \delta,\lambda}} G_{\alpha\beta\gamma\delta\lambda}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta\sigma_\lambda - 10R_{2\underline{3}}[\sigma] \right] , \tag{2.10}$$

$$R'_{0\underline{6}}[\sigma] = (N+8)R_{0\underline{6}}[\sigma] - 15(N+4)R_{2\underline{4}}[\sigma] \ , \tag{2.11}$$

and

$$R'_{0\underline{33}}[\sigma] = (N+8)(N^2+N+4)R_{0\underline{33}}[\sigma] - \tfrac{1}{10}N(N+1)(N+2)(N+8)(N^2+17N+12)R_{0\underline{6}}[\sigma] + \tfrac{3}{2}N^2(N+1)(N+4)(N+5)(N+8)R_{2\underline{4}}[\sigma] \ , \tag{2.12}$$

where

$$R_{0\underline{6}}[\sigma] = N^2 \left[N(N+2)(N+4) \sum_{\substack{\alpha,\beta,\gamma,\\ \delta,\lambda,\omega}} H_{\alpha\beta\gamma\delta\lambda\omega}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta\sigma_\lambda\sigma_\omega - 15(N+1)R_{60}[\sigma] \right] \tag{2.13}$$

and

$$R_{0\underline{33}}[\sigma] = N^3(N+2)(N+4)(R_{0\underline{3}}[\sigma])^2 - 6(N+1)^2(N-1)R_{60}[\sigma] \ . \tag{2.14}$$

The $Q_{\alpha\beta\gamma}$, $F_{\alpha\beta\gamma\delta}$, etc. are the Zia and Wallace factors[11] that are defined by

$$A_{\alpha\beta\gamma,\ldots} = \sum_i e^i_\alpha e^i_\beta e^i_\gamma \cdots \ , \tag{2.15}$$

in terms of the $N+1$ vectors e^i_α, $i = 1, \ldots, N+1$, with N components, $\alpha = 1, \ldots, N$, that point to the $N+1$ vertices of a N-dimensional hypertetrahedron. The number of Potts components is

$$Q = N+1 \ . \tag{2.16}$$

We note that our convention for the indices i and α is reversed from that of Ref. 11. Multiple indices $\underline{l}_i$ in $R_{\overline{m},\underline{l}_1,\ldots,\underline{l}_t}[\sigma]$, such as in Eq. (2.14), have a meaning similar to that described in Ref. 17 below Eq. (2.11). The Potts operators of Eqs. (2.6)—(2.12) are traceless.[25] The choice of normalization factors N in Eqs. (2.6)—(2.14) is such that the percolation limit $N \to 0$ can be properly taken. Therefore, in the scaling-field equations the parameter $N=Q-1$ can be varied continuously between zero and large N.

Three general statements concerning the coupling coefficients a_{mjk} can be made. (For details, see Appendix A and Appendix A of Ref. 17.) First, two classes of coupling coefficients vanish because of symmetry, $a_{P,II}=0$ and $a_{I,PI}=0$, where I and P are generic indices denoting the sets of isotropic and Potts operators, respectively. Second, a linked-contraction theorem holds that limits the number of nonzero coupling coefficients a_{mjk}. Similar considerations apply to a_{mj}, for which all $a_{PI}=a_{IP}=0$ and a linked-contraction condition restricts the nonzero a_{II} and a_{PP}. The only nonvanishing coefficient a_m is $a_{20;2}=-2A\Delta$, where A is the normalization of the Gaussian fixed-point Hamiltonian $H_G^*[\sigma]$. Third, the equations for the Potts scaling fields decouple from those for the isotropic scaling fields when $N=1$.[26] The calculation of coupling coefficients involving Potts operators is facilitated by relations summarized in Appendix A. An explicit example of a truncated set of scaling-field equations for the Potts model is given in Sec. III A.

Except in limits, the scaling-field equations must be solved numerically. Approximations are generated by truncation. The present study of the Q-state Potts model is for a truncation that includes the four isotropic scaling fields $\overline{m}\underline{l},p=20,0$, $40,0$, $60,0$, and $20,2$ and the seven Potts scaling fields $0\underline{3},0$, $2\underline{3},0$, $0\underline{4},0$, $2\underline{4},0$, $0\underline{5},0$, $0\underline{6},0$, and $0\underline{33},0$. For simplicity, we delete the index p and refer to the eleven fields as 20, 40, 60, and 2'0, and $0\underline{3}$, $2\underline{3}$, $0\underline{4}$, $2\underline{4}$, $0\underline{5}$, $0\underline{6}$, and $0\underline{33}$. This truncation corresponds to retaining in the expansion (2.1) all operators to order σ^6 with $p=0$ as well as the $p=2$ operator,

$$Q_{2'0}[\sigma] \sim \tfrac{1}{2}\int_{\underline{q}} q^2 \vec{\sigma}(\underline{q})\cdot\vec{\sigma}(-\underline{q}) \ . \tag{2.17}$$

For the Gaussian amplitude factor, the value $A=0.5$ is chosen. Results obtained from truncations depend weakly on the choice of this parameter. With $A=0.5$, the critical exponents for the isotropic N-vector model agree well with the results found by other methods. For details, see Sec. III A of Ref. 17 and Ref. 16.

Eigenvalues are labeled $y_{\overline{m}\underline{l},p}$ or $y_{\overline{m}\underline{l}}$ with superscripts used to distinguish different types of fixed points. Specifically, we denote the thermal eigenvalue y_{20}, the marginal eigenvalue $y_{2'0}$, and the leading irrelevant eigenvalues y_{40}, $y_{0\underline{3}}$, or $y_{0\underline{4}}$, the relative order of the latter depending on I and Q_i. The exponent ϕ equals the leading irrelevant eigenvalue divided by the thermal eigenvalue.

Difficulties with the scaling-field method can typically be traced to the fact that truncations are being used. Two such problems were observed for the Potts model.

(i) Studying a small truncation of only 11 scaling-field equations restricts the range of Q and d for which Potts fixed points can be found. No problems exist for small Q, except when $d<2.8$.[17] However, the critical parameter $Q_c(d)$ can only be determined for $3.4\le d \lesssim 4$. For $d<3.4$, the truncation is insufficient to accurately approximate the physical marginal operator associated with the annihilation of the lines of critical and tricritical fixed points as Q approaches Q_c.[27]

(ii) Near six dimensions, the ϵ-expansion solution of the truncated set of scaling-field equations does not reproduce the exact results for the Potts model.[20] The structure of the equation is such that all scaling fields $\mu_{20,p}$ with $p>2$ and $\mu_{40,p}$, $\mu_{0\underline{4},p}$, $\mu_{2\underline{3},p}$, and $\mu_{0\underline{5},p}$ with $p>0$ assume fixed-point values of orders ϵ and $\epsilon^{3/2}$, respectively, and thus contribute to the critical exponents to leading order. For further comments on ϵ expansion near six dimensions, see Appendix B. The situation is different for the ϵ expansion of the Potts model near *four* dimensions. There the scaling-field method yields exact expressions for the critical exponents and Q_c when $|\delta|=|Q-2|\ll 1$ with, however, the numerical coefficients depending upon the fixed-point values of infinitely many Potts fields. These Potts fields are of order 1 because the upper critical dimensionality of the Potts model is 6.

The strength of the scaling-field method lies in the fact that it allows the numerical calculation of critical properties for the dimension of interest, with no need for subsequent extrapolation or resummation. For the estimation of $Q_c(d=4-\epsilon)$, it is important that analytical ϵ-expansion and numerical methods are available that supplement each other.

III. RESULTS

This section contains three parts: the solution of the Q-state Potts model by ϵ expansion near four dimensions, the numerical analysis of the problem for dimensions $3.4\le d \lesssim 4$, and a discussion of the percolation or one-state Potts problem for dimensions $3\le d\le 6$.

A. Analytical solution for $d=4-\epsilon$

In the Introduction, we motivated the possibility of an analytical solution of the Potts model near four dimensions, although the upper critical dimension is six. The procedure differs from the usual ϵ-expansion approach. We first illustrate the method by considering a simple truncation. Then we show that the form of the expressions obtained for Q_c and the critical and tricritical exponents [Eqs. (1.1)–(1.4)] holds in general, though coefficients a and b must be determined numerically.

Consider the truncation that contains only the set of scaling-field equations for the isotropic fields μ_{20}, μ_{40}, and $\mu_{2'0}$, and the Potts fields $\mu_{0\underline{3}}$ and $\mu_{0\underline{4}}$:

$$\frac{d\mu_{20}}{dl}=2\mu_{20}-\left[2\mu_{20}^2+\frac{2(N+2)}{3}I_1\mu_{20}\mu_{40}+\frac{N+2}{3}I_2\mu_{40}^2+\frac{2(N+2)}{3}K_1\mu_{2'0}\mu_{40}+2(N+1)^2(N-1)I_1\mu_{0\underline{3}}^2\right.$$
$$\left.+(N+1)^3(N+2)(N-1)(N-2)I_2\mu_{0\underline{4}}^2\right]-\Delta\left[2\mu_{20}+\frac{N+2}{3}L_1\mu_{40}\right] , \tag{3.1}$$

$$\frac{d\mu_{40}}{dl}=y_{40}^{G}\mu_{40}-\left[8\mu_{20}\mu_{40}+\frac{2(N+8)}{3}I_1\mu_{40}^2+\frac{12(N+1)^2(N-1)}{(N+2)}\mu_{0\underline{3}}^2+12(N+1)^3(N-1)(N-2)I_1\mu_{0\underline{4}}^2\right]-4\Delta\mu_{40}\,, \tag{3.2}$$

$$\frac{d\mu_{2'0}}{dl}=\left[2[(2-A)-\Delta(1-A)]\mu_{20}^2+\frac{N+2}{24}J_2\mu_{40}^2-4\mu_{20}\mu_{2'0}+\frac{(N+1)^2(N-1)}{4}J_1\mu_{0\underline{3}}^2\right.$$
$$\left.+\frac{(N+1)^3(N+2)(N-1)(N-2)}{8}J_2\mu_{0\underline{4}}^2\right]-\Delta[2A-4A\mu_{20}+2\mu_{2'0}]\,, \tag{3.3}$$

$$\frac{d\mu_{0\underline{3}}}{dl}=y_{0\underline{3}}^{G}\mu_{0\underline{3}}-[6\mu_{20}\mu_{0\underline{3}}+4I_1\mu_{40}\mu_{0\underline{3}}+6(N+1)^2(N-2)I_1\mu_{0\underline{3}}\mu_{0\underline{4}}]-3\Delta\mu_{0\underline{3}}\,, \tag{3.4}$$

$$\frac{d\mu_{0\underline{4}}}{dl}=y_{0\underline{4}}^{G}\mu_{0\underline{4}}-\left[8\mu_{20}\mu_{0\underline{4}}+8I_1\mu_{40}\mu_{0\underline{4}}+\frac{6(N+1)}{(N+2)}\mu_{0\underline{3}}^2+6(N+1)(-2-3N+N^2)I_1\mu_{0\underline{4}}^2\right]-4\Delta\mu_{0\underline{4}}\,. \tag{3.5}$$

Here y_m^G denotes the Gaussian eigenvalues,

$$y_{\overline{m}\underline{l},p}^{G}=d-\tfrac{1}{2}(\overline{m}+\underline{l})(d-2)-p\,, \tag{3.6}$$

where $\overline{m}+\underline{l}$ with $\underline{l}=\underline{l}_1+\cdots+\underline{l}_t$ is the order in σ of $R_{\overline{m},\underline{l}_1,\ldots,\underline{l}_t}[\sigma]$. We remind the reader that the subscript 2′0 stands for $(2,0;p=2)$ and that $p=0$ for all other fields. For $d=4-\epsilon$, both $y_{40}^G=y_{0\underline{4}}^G=\epsilon$ and the spin-rescaling parameter Δ has a fixed-point value of order ϵ^2. The integrals I_n and J_n with $n=1$ and 2 are defined by $I_n\equiv I_n'+\Delta I_n''$ and $J_n\equiv J_n'+\Delta J_n''$. Only the portion of the integrals independent of the spin-rescaling parameter Δ is necessary in calculations to leading order in ϵ: they are $I_1'=\pi/(4A)$ and $J_2'=\pi^2/(3A)$, and, for $A=0.5$, $I_2'=1.305$ and $J_1'=1.909$. The definitions of these and other integrals can be found in Ref. 17.

It is useful to introduce the parameter δ:

$$\delta=Q-2\equiv N-1\,. \tag{3.7}$$

For $|\delta|\ll 1$, the equations for the isotropic and Potts scaling fields, $\mu_I=\{\mu_{20},\mu_{40},\mu_{2'0}\}$ and $\mu_P=\{\mu_{0\underline{3}},\mu_{0\underline{4}}\}$, couple only "weakly" through coupling coefficients $a_{I,PP}$ that are of order δ. It is this property that makes it possible to solve analytically the equations for μ_I when $\epsilon\ll 1$ and $|\delta|\ll 1$. The fixed-point values of the Potts fields μ_P^* are found to be of order 1, which reflects the fact that the upper critical dimension is 6.

Case $\mu_P^*=0$. In this case, Eqs. (3.1)–(3.3) reproduce the ϵ-expansion results for the isotropic N-vector model, i.e., the critical and tricritical fixed points, which we denote by $O_c(N)$ and $O_t(N)$. The fixed-point values of the isotropic scaling fields are

$$\mu_{20,c}^*=\frac{3(N+2)I_2}{8(N+8)^2I_1^2}\epsilon^2+O(\epsilon^3)\,,$$
$$\mu_{40,c}^*=\frac{3\epsilon}{2(N+8)I_1}+O(\epsilon^2)\,, \tag{3.8a}$$

and

$$\mu_{20,t}^*=\mu_{40,t}^*=0\,. \tag{3.8b}$$

[The additional subscripts c and t are used to distinguish between results for $O_c(N)$ and $O_t(N)$.] The corresponding eigenvalues are

$$y_{20,c}=2-\frac{N+2}{N+8}\epsilon+O(\epsilon^2),\quad y_{40,c}=-\epsilon+O(\epsilon^2)\,,$$
$$\eta_c=2\Delta^*=\frac{N+2}{2(N+8)^2}\epsilon^2+O(\epsilon^3)\,, \tag{3.9a}$$

$$y_{20,t}=2,\quad y_{40,t}=\epsilon,\quad \eta_t=0\,. \tag{3.9b}$$

In calculations to this order, one may choose without loss of generality $\mu_{2'0}^*=0$.[16] (See, however, Appendix B for ϵ expansion about six dimensions.) Then Eq. (3.3) yields the exponent η via Eq. (2.3). The fixed points $O_c(N)$ and $O_t(N)$ are unstable under the Potts perturbations, μ_{03}, $\mu_{04}\neq 0$. The respective eigenvalues are

$$y_{0\underline{3},c}=1+\frac{N-4}{2(N+8)}\epsilon+O(\epsilon^2),\quad y_{0\underline{4},c}=\frac{N-4}{N+8}\epsilon+O(\epsilon^2)\,, \tag{3.10a}$$

$$y_{0\underline{3},t}=1+\tfrac{1}{2}\epsilon,\quad y_{0\underline{4},t}=\epsilon\,. \tag{3.10b}$$

Case $\mu_P^*\neq 0$. In the larger space of the Potts model, there exist new branches of critical and tricritical fixed point that we denote by $P_c(Q)$ and $P_t(Q)$.

(i) When $Q=2$, the equations for the isotropic and Potts fields decouple because $a_{I,PP}=0$.[26] For this value of Q, the Potts fields are superfluous and the equations reproduce the eigenvalues (3.9) of the Ising model, $O_c(N=1)$ and $O_t(N=1)$. Equations (3.4) and (3.5) yield at $P_c(Q=2)$ and $P_t(Q=2)$ fixed-point values for the Potts scaling fields that are identical to leading order (order 1),

$$\mu_{0\underline{3},c}^*=\frac{1-\epsilon/12}{(48I_1)^{1/2}}+O(\epsilon^2),\quad \mu_{0\underline{4},c}^*=\frac{-(1-\epsilon/6)}{24I_1}+O(\epsilon^2) \tag{3.11a}$$

and

$$\mu_{0\underline{3},t}^*=\frac{1+\epsilon/4}{(48I_1)^{1/2}}+O(\epsilon^2),\quad \mu_{0\underline{4},t}^*=\frac{-(1+\epsilon/2)}{24I_1}+O(\epsilon^2)\,. \tag{3.11b}$$

The corrections to order ϵ affect only the Potts eigenvalues. For $Q=2$, we find the eigenvalues

$$y_{0\underline{3},c}=-2+\tfrac{1}{3}\epsilon+O(\epsilon^2),\quad y_{0\underline{4},c}=-2+O(\epsilon^2) \tag{3.12a}$$

and

$$y_{0\underline{3},t}=-2,\quad y_{0\underline{4},t}=-2-\epsilon\ . \tag{3.12b}$$

(ii) When $Q=2+\delta$ with $|\delta|\ll 1$, one can substitute expressions (3.11) for $\mu^*_{0\underline{3}}$ and $\mu^*_{0\underline{4}}$ into Eqs. (3.1)–(3.3). Fixed-point expressions for μ^*_{20} and μ^*_{40} are easily found since $a_{I,PP}\propto\delta$. To leading order,

$$\mu^*_{40}=\frac{\epsilon\pm(\epsilon^2-a\delta)^{1/2}}{12I_1}\ , \tag{3.13}$$

with $a=4$ for this truncation. The corresponding fixed points are singly and doubly unstable (upper and lower sign, respectively) with eigenvalues

$$y_{20}=\frac{1}{\nu}=2-\tfrac{1}{6}[\epsilon\pm(\epsilon^2-a\delta)^{1/2}]+O(\epsilon^2)\ , \tag{3.14}$$

$$y_{40}=\frac{\phi}{\nu}=\mp(\epsilon^2-a\delta)^{1/2}+O(\epsilon^2)\ , \tag{3.15}$$

$$\eta=\tfrac{1}{216}[\epsilon\pm(\epsilon^2-a\delta)^{1/2}]^2+\delta b+O(\epsilon^3)\ , \tag{3.16}$$

where $a=4$ and $b=0.051$ for truncation (3.1)–(3.5). Hence, Eq. (3.13) defines the critical and tricritical branches of the Potts fixed points $P_c(Q)$ and $P_t(Q)$ as functions of $\delta=Q-2$ for $d=4-\epsilon$. When $\epsilon\neq 0$, both branches exist for $\delta<\delta_c=\epsilon^2/a+O(\epsilon^3)$. As $\delta\to\delta_c$, the branches annihilate. To leading order, the marginal operator is the dilution operator $Q_{40}[\sigma]$. For $\delta>\delta_c$, "runaway" flow is interpreted as indicating first-order behavior. Therefore, the largest value of Q for which continuous behavior is still possible is given by

$$Q_c=2+\epsilon^2/a+O(\epsilon^3)\ . \tag{3.17}$$

This behavior is summarized in Figs. 1–3.

Now we show that the structure of the ϵ and δ dependence of results (3.17) and (3.14)–(3.16) is correct in general, i.e., not restricted to the truncation defined by Eqs. (3.1)–(3.5). The results are the consequence of the fact that all coupling coefficients $a_{I,PP}$ are of order δ, and the fixed-point coordinates μ^*_P are of order 1 and are identically the same for critical and tricritical solutions to leading order. The untruncated equation for μ_{40} to leading order in ϵ is

$$\frac{d\mu_{40}}{dl}=\epsilon\mu_{40}-2(N+8)I_1\frac{\mu_{40}^2}{3}+\sum_{j,k\in P}a_{40,jk}\mu_j\mu_k+O(\epsilon^3)\ , \tag{3.18}$$

where the sums j and k are over all Potts scaling fields. Since all $a_{I,PP}\propto\delta$, this equation implies a solution of the form (3.13), with a given by

$$a=\lim_{\delta\to 0}24I_1\sum_{j,k\in P}\frac{a_{40,jk}}{\delta}\mu^*_j\mu^*_k\ . \tag{3.19}$$

Similarly, one obtains for b, from the equation for $\mu_{2'0}$,

$$b=\lim_{\delta\to 0}\frac{24I_1}{A}\sum_{j,k\in P}\frac{a_{2'0,jk}}{\delta}\mu^*_j\mu^*_k\ . \tag{3.20}$$

The equations for the Potts scaling fields μ^*_P cannot be solved through systematic expansion near four dimensions. Therefore, we expect the coefficients a and b to depend on all Potts fields in the Landau Hamiltonian (2.1).

The results of Eqs. (3.14)–(3.17) are new. The mechanism of "merging lines" of critical and tricritical fixed points and the changeover in the nature of the Potts transition from continuous to first order has not been obtained before for this range of dimensions, $d<4$.

B. Numerical solution for $3.4\le d\lesssim 4$

A numerical calculation is required to compute the coefficients a and b in Eqs. (3.14)–(3.17) and to extend the results to other d and Q. The results presented here are for the truncation to 11 scaling-field equations defined in Sec. II. Our numerical analysis of the scaling-field equations differs from the ϵ-expansion approach in two ways. (a) The coefficient integrals are calculated for the dimension under consideration and not for the dimension about which one expands. When dimension is changed,[17] the coefficient integrals are recalculated and the fixed-point analysis as function of Q is repeated. (b) The fixed-point value of the RG scaling parameter Δ is determined by searching for a value of Δ^* that produces a fixed point with marginal eigenvalue $y_{2'0}$,

$$y_{2'0}(\Delta^*)=0\ . \tag{3.21}$$

This criterion and the one used in the ϵ-expansion solution are identical to leading order in ϵ for $d=4-\epsilon$. Details are published elsewhere.[17]

The numerical analysis of the Q-state Potts model by the scaling-field method led to the following results.

(i) *d=4*. The values of the Potts fixed-point coordinates μ^*_P for $d=4$ determine the coefficients a and b in Eqs. (3.14)–(3.17). One obtains

$$a=6.52\ \text{ and } b=0.065 \tag{3.22}$$

for the truncation to 11 coupled equations. The corresponding results for $Q_c(d)$, as defined by Eq. (3.17), are shown as the dashed curve in Fig. 2, below.

(ii) *d=3.98*. Figures 1(a)–1(c) exhibit the exponents y_{20}, y_{40}, and η vs Q associated with the critical and tricritical Potts fixed points, $P_c(Q)$ and $P_t(Q)$, for dimension 3.98. The results show the parabolic shape anticipated from Eqs. (3.14)–(3.16). The location of the tip of the curve for the thermal exponent y_{20} offers a convenient definition of $Q_c(d)$. The next-to-leading exponent y_{40} changes sign at or close to $Q_c(d)$, with the degree of agreement measuring the internal consistency of the calculation. That means the critical and tricritical fixed lines form a smooth curve that changes stability at Q_c. For $d=3.98$, the analytical and numerical results for Q_c agree to better than 0.2%.

(iii) *$3.4\le d<4$*. For dimensions in this interval, the graphs for the critical and tricritical exponents are similar to those for $d=3.98$. With decreasing d, the difference

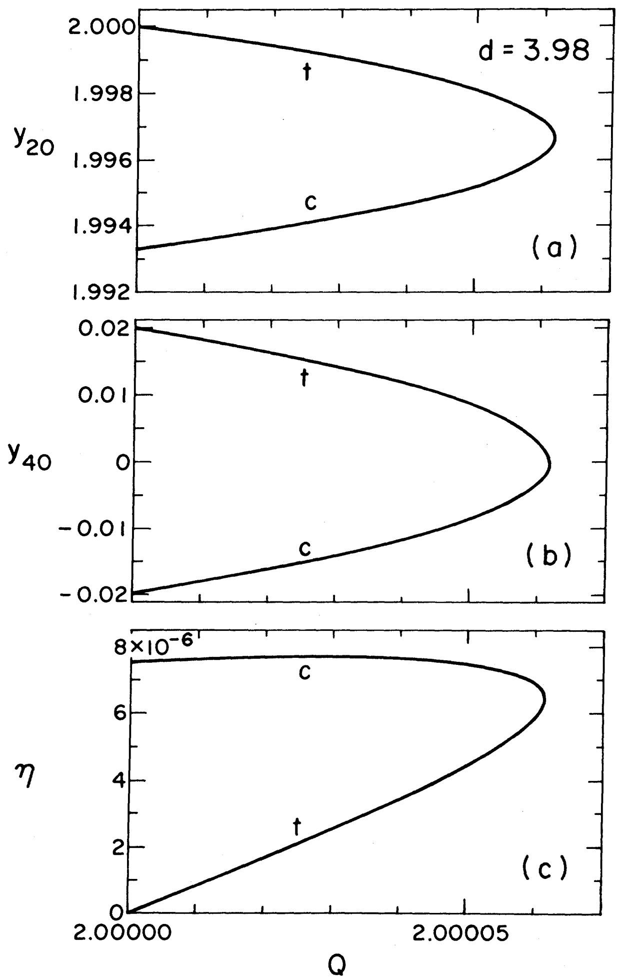

FIG. 1. (a) Thermal exponent y_{20}, (b) next-to-leading thermal exponent y_{40}, and (c) correlation-function exponent η as functions of the number of states Q for the critical (c) and tricritical (t) Potts transitions in $d=3.98$ dimensions as obtained numerically for a set of 11 scaling-field equations.

between the Q at which y_{20} exhibits the tip and y_{40} changes sign increases slightly (to about 0.2% for $d=3.4$). The values of $Q_c=Q_{\text{tip}}$ are recorded as open circles in Fig. 2. The numerical values are $Q_c=2.00637$, 2.0272, and 2.070 for $d=3.8$, 3.6, and 3.4, respectively.

(iv) $3\le d<3.4$. For these dimensions, the calculation yields critical and tricritical fixed points only for values of Q that are smaller than the expected $Q_c(d)$. The difficulty manifests itself in the application of the criterion (3.21) for determining η. We suspect that the truncation does not include sufficient terms in the expansion of $H_I[\sigma]$ to resolve the redundant marginal operator from the physical marginal dilution operator in the region where the critical and tricritical fixed points annihilate. For $d=4-\epsilon$, we found that the physical marginal operator to leading order is the dilution operator $Q_{40}[\sigma]$.

(v) $d>4$. Equations (3.13)–(3.17) apply also when $\epsilon<0$. Figure 3 exhibits numerical results for the leading thermal exponent y_{20} as function of Q for $d=3.98$, 4.0, and 4.02. Similar results are found for dimensions up to $d\simeq4.6$, at which point problems reminiscent to those for

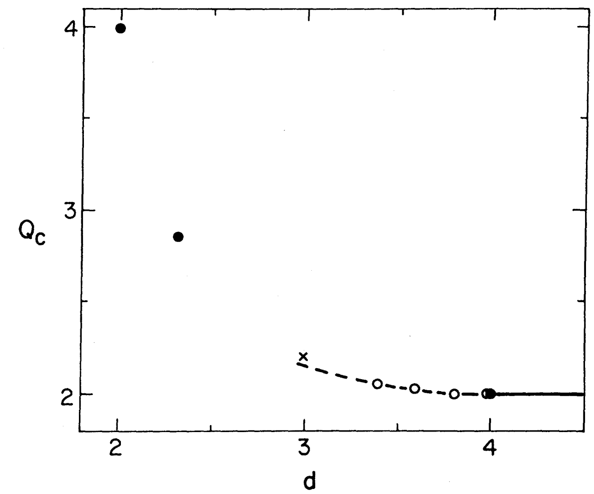

FIG. 2. Critical value Q_c of the Potts model as function of dimension d. Shown are the results by the numerical scaling-field calculation (open circles), extrapolation of scaling-field ϵ expansion (dashed curve), conjecture for $d\ge4$ (solid curve) (Ref. 29), variational RG calculation for $d=2$ and 2.32 (solid circles) (Ref. 28), as well as smooth interpolation of series expansion data for $d=3$ (cross) (Ref. 28). The result Q_c ($d=2$)$=4$ is exact (Ref. 2).

$d\le3.4$ arise. Over the full range $|\epsilon|\le0.6$, one finds $Q_c(d=4+|\epsilon|)\simeq Q_c(d=4-|\epsilon|)$. As $d\to6$, the critical exponents y_{20} and η approach their mean-field values $y_{20}^{\text{MF}}=2$ and $\eta^{\text{MF}}=0$ for all Q. For dimensions $d>3$, the point $Q=2$ with Gaussian eigenvalue $y_{20}^{G}=2$ plays a special role. There the coefficient of the σ^4 term in the fixed-point Hamiltonian $H^*[\sigma]$ changes sign and solutions beyond that point should be discarded as unphysical. The point is located on the tricritical branch for $d<4$ and the critical one for $d>4$. Further remarks concerning the interpretation are deferred until the end of this subsection.

The following comments will place the results into context.

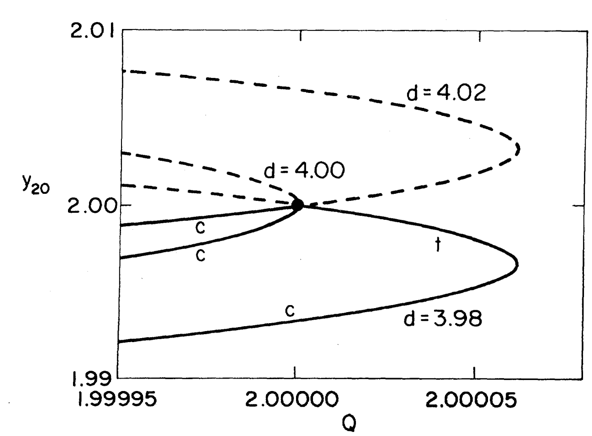

FIG. 3. Thermal exponent y_{20} as function of the number of states Q for the critical (c) and tricritical (t) Potts transitions in dimensions $d=3.98$, 4.0, and 4.02 as obtained numerically from a set of 11 scaling-field equations. Dashed lines indicate nonphysical solutions. As dimension is increased, the point $y_{20}(Q=2)=2$ wanders from the tricritical (t) to the critical (c) branch.

(i) *Exponents.* Most striking is a comparison of the curves in Fig. 1 for $d=3.98$ with the analogous ones for $d=2$ and $d=1.58$ and 2.32 in Fig. 1 and Figs. 2–4, respectively, of Ref. 28. The graphs exhibit similar shapes for $Q \le Q_c(d)$, the difference being a vast change of scale. For the two-dimensional Potts model, the results were obtained by position-space RG methods applied to the Potts lattice gas.[3] This line of attack was motivated by physical considerations. The result—one smooth curve for the critical and tricritical fixed lines—led to the conclusion that these two behaviors are connected. It is interesting that analogous results for a very different range of dimensions have been obtained here by a method that does *not* use the vacancy idea. It is an open question whether this RG mechanism for changeover in phase-transition behavior (here from continuous to first-order behavior) via the annihilation of pairs of lines of fixed points is realized in other systems.

(ii) *Crossover value* $Q_c(d)$. Besides results for Q_c by the scaling-field method, Fig. 2 exhibits results for dimensions $d=2$ and 2.32 from position-space RG calculations as well as an estimate for $d=3$ from a smooth interpolation of series-expansion data.[28] It is not known whether Q_c is universal except for $Q_c=4$ at $d=2$ (Ref. 2) and $Q_c=2$ at $d \ge 4$.[29] The results of our analytical and numerical analysis (dashed curve and open circles, respectively) match smoothly onto that latter result, $Q_c=2$ for $d=4$. This is not the case for the result $Q_c(d=4-\epsilon) = 2+\epsilon+O(\epsilon^2)$ by Aharony and Pytte, which is in error.[19] Our findings are consistent with those by Kogut and Sinclair,[30] who used $1/Q$-expansion techniques to estimate $Q_c(d)$ for $d>2$. Fucito and Parisi[31] mention the possibility that Q_c may be nonuniversal. These three references (Refs. 19, 30, and 31) do not address the question of the RG mechanism responsible for the changeover from continuous to first-order behavior and use definitions for Q_c different from ours. In our definition, $Q_c = Q_{\rm tip}$ gives the largest value of Q for which continuous behavior is possible; however, first-order transitions can be induced for $Q < Q_c$ by appropriate constraints, e.g., dilution.

(iii) *Thermodynamic functions.* Our calculation implies that the three-state Potts model in three dimensions undergoes a first-order phase transition. Thermodynamic quantities may still exhibit large contributions due to fluctuations, so that results by laboratory and computer experiments with insufficient resolution might suggest second-order behavior. Thermodynamic functions can be computed by the scaling-field method[16] but no results have been obtained yet. Calculations by Nauenberg *et al.*[18] were based on phenomenological RG equations that differ considerably from our microscopic ones.

Returning to the interpretation of the evolution as function of d of the pair of Potts fixed points that annihilate as $Q \to Q_c(d)$, we observe the following: For $d>4$, the phase transition of the Potts model is first order when $Q>2$,[29] and continuous when $Q<2$, the latter becoming classical as $d \to 6$.[20] The results in Fig. 3 seem to contradict that picture. However, in analytical RG calculations, lines of fixed points cannot terminate. For example, the line of Potts critical fixed points as function of Q does not terminate at $Q_c(d)$ but changes into and continues as a line of doubly unstable tricritical fixed points. Whether this line has physical significance in a given calculation depends on whether it is connected to a segment of the initial physical parameter space $H_{l=0}[\sigma]$ by RG flow. For $d=4-\epsilon$, the Gaussian eigenfunctional $Q_{40}[\sigma]$ is to leading order the dilution operator of the Potts lattice gas so that tricritical behavior is physical in the limit $Q \le Q_c$. For $Q=2$, the tricritical behavior is classical for all $d \ge 3$ and we discard as nonphysical the portion of the tricritical fixed line with $Q<2$.[32] The observation that lines of fixed points cannot terminate applies also to the evolution as functions of dimension of the pair of Potts fixed lines, $P_c(Q)$ and $P_t(Q)$, and explains why this result changes smoothly as d is varied about $d=4$. (See Fig. 3.) For $d>4$, the situation with regard to the branch of critical fixed points $P_c(Q)$ is similar to the tricritical one for $d<4$. Now the critical behavior at $Q=2$ is classical and the segment $Q>2$ of the line of fixed points is unphysical. In this regime, the classical critical behavior is expected to be preempted by a first-order transition. As $d \to 6$, we observe that the exponents y_{20} and η approach their classical values, but the proof of the change to classical behavior at $d \ge 6$ for $Q<2$ would again require the study of RG flows.

C. Percolation for $3 \le d \le 6$

The limit $Q \to 1$ of the Potts model is related to the percolation problem.[33–35] The latter is representative of physical problems with upper critical dimensionality $d_c=6$. Much effort has been extended to obtain the percolation exponents $y_{20}^P = 1/\nu^P$ and η^P as functions of d for $2 \le d \le 6$. (Exponent relations yield expressions for β^P, γ^P, etc., as well as the cluster exponents σ and τ.[34]) Figures 4(a) and 4(b) exhibit representative results for y_{20}^P and η^P. The only exactly known results are[7–9]

$$1/\nu^P = \tfrac{3}{4}, \quad \eta^P = \tfrac{5}{24} \quad \text{for } d=2\,, \tag{3.23}$$

and ϵ-expansion results near six dimensions,[20]

$$1/\nu^P = 2 - \frac{5\epsilon}{21} + O(\epsilon^2)\ , \eta^P = -\frac{\epsilon}{21} + O(\epsilon^2) \quad \text{for } d=6-\epsilon\,, \tag{3.24}$$

recently extended to order ϵ^3.[21] Interestingly, though the ϵ-expansion results to third order indicate a sign change in η^P with decreasing dimension, the results derived for η^P by a resummation of the series do not.[21] Field-theoretical expansion techniques, which have been applied successfully in other cases, fail to obtain reliable results for the percolation exponents $1/\nu^P$ and η^P in three dimensions.[36] The need for the change in the sign of η^P in the interval $2<d<6$ is reminiscent of the change in the sign of η for the random Ising model for $3<d<4$.[17] (There, ϵ expansion to second order about four dimensions yields the sign change.[37]) In Figs. 4(a) and 4(b) we compare results from the scaling-field method with exact results at $d=2$ and $6-\epsilon$, position-space RG calculations near $d=2$,[28] extrapolation of ϵ-expansion results to second[20] and third order,[21] as well as results from resummation of the ϵ (Ref. 21) or loop expansions,[22,23] and, for $d=3$, from Monte

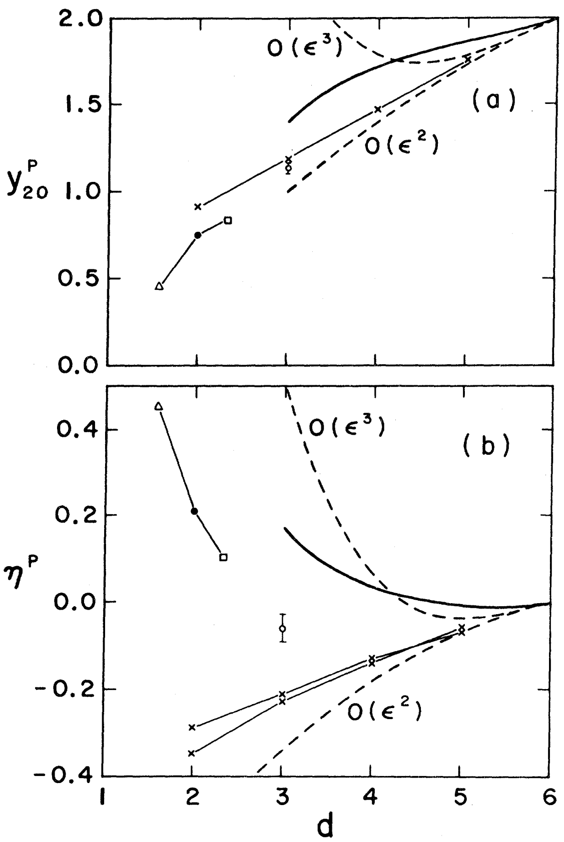

FIG. 4. Exponents of the one-state Potts or percolation problem as functions of dimension d: (a) leading eigenvalue $y_{20}^P = 1/\nu^P$, and (b) correlation-function exponent η^P as functions of dimension d for the percolation problem $Q=1$. Shown are results from scaling-field method (solid curve), extrapolation of ϵ expansion (dashed curve) (Refs. 20 and 21), resummation of the loop (Ref. 22) and ϵ expansions (Ref. 21) to second and third orders, respectively (crosses), Monte Carlo calculation for η^P and series expansion for ν^P (open circles) (Ref. 38), position-space RG for $d=1.58$ by Migdal method (open triangles) and for $d=2.32$ by Kadanoff method (open squares) (Ref. 28), as well as exact results for $d=2$ (solid circles) (Refs. 7–9).

Carlo simulations for $\eta^P = -0.06 \pm 0.03$ and series expansion for $\nu^P = 0.88 \pm 0.02$.[38] The discrepancy between the various results, particularly for η^P, is striking and will be discussed here.

We consider the one-state Potts model in the limit $d=6-\epsilon$. As discussed in Appendix B, the truncated scaling-field equations do not reproduce the exact ϵ-expansion results,[20,21] not even to leading order in ϵ. However, the truncation does preserve the structure of these solutions. Equations (3.1)–(3.5) can serve to demonstrate that; however, since the equations for $\mu_{2\underline{3}}$ and $\mu_{0\underline{5}}$ also contribute to leading order, we consider the truncation to 11 scaling-field equations. Then, with $y_{0\underline{3}}^G = \epsilon/2$ from Eq. (3.6), we find

$$\mu_{0\underline{3}}^* = C\epsilon^{1/2}, \tag{3.25}$$

as well as μ_{20}^*, μ_{40}^*, and $\mu_{0\underline{4}}^* \sim O(\epsilon)$, and $\mu_{2\underline{3}}^*$, $\mu_{0\underline{5}}^* \sim O(\epsilon^{3/2})$, with $\mu_{2'0}^* = \mu_{2'0}^*(\Delta)$ being a parameter of order 1 that is adjusted in the usual way by determining the fixed-point value of Δ such that $y_{2'0}^{(\epsilon)}(\Delta=\Delta^*)=0$. The remaining scaling fields, μ_{60}, $\mu_{2\underline{4}}$, and $\mu_{0\underline{33}}$, have fixed-point coordinates of order ϵ^2 and need not be included in calculations to leading order in ϵ. With $A=0.5$, our truncation yields

$$y_{20}^P = 1/\nu^P \simeq 2 - \tfrac{5}{28}\epsilon, \quad \eta^P \simeq -\frac{\epsilon}{40}, \tag{3.26}$$

and the leading correction-to-scaling eigenvalue $y_{40}^P \simeq -\epsilon$. The results (3.26) for $d=6-\epsilon$ dimensions were obtained numerically with $\epsilon = 10^{-3}$, 10^{-4}, and 10^{-5}. The discrepancy between (3.26) and the exact results (3.24) is due to the fact that the scaling fields $\mu_{20,p}$ with $p>2$ as well as $\mu_{40,p}$ $\mu_{0\underline{4},p}$, $\mu_{2\underline{3},p}$, and $\mu_{0\underline{5},p}$ with $p \geq 2$ are not included in the calculation. These scaling fields all contribute to the coefficient C of the fixed-point coordinate $\mu_{0\underline{3}}^*$ in Eq. (3.25). For further comments, see Appendix B.

The extension of the above results for the one-state Potts model to dimensions $3 \leq d \leq 6$ uses the 11 scaling-field equations. Exponents are determined numerically, as described in Sec. II, for integer and noninteger values of dimension. The results are as follows.

(i) Figure 4(b) exhibits the results for the critical exponent η^P as function of d. Obviously the results for η^P from this truncation are too large in value both in the limit $d \to 6$ and for $d=3$. From the analysis near six dimensions, we suspect as the reason the small size of the truncation. We note, however, that we obtain the correct sign change and curvature in η^P as a function of d necessary to link the results for $d=2$ to $6-\epsilon$.

(ii) Figure 4(a) shows the results for the critical exponent $y_{20}^P = 1/\nu^P$ as function of d. The values of y_{20}^P are larger and exhibit more curvature as function of d than the results by the ϵ and loop expansions as obtained by resummation.[39] We expect that an improved scaling-field calculation will lead to smaller values of y_{20}^P.

(iii) Figure 5 exhibits results for the three leading ir-

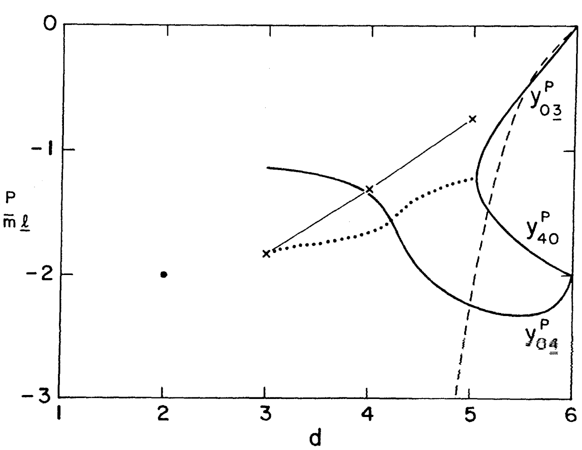

FIG. 5. Leading irrelevant eigenvalues $y_{0\underline{3}}^P$, $y_{0\underline{4}}^P$, and y_{40}^P as functions of dimension d for the one-state Potts or percolation problem from the scaling-field method (solid curve), as well as results from $y_{0\underline{3}}^P$ by resummation of the ϵ expansion (crosses) (Ref. 40), and extrapolation of the $O(\epsilon^3)$ formula (dashed curve) (Ref. 41). For $d=2$, the exact result $y_{40}^P = -2$ is shown (Ref. 8). Dotted part of the curve denotes a conjugate complex pair. pair.

relevant eigenvalues $y^P_{0\underline{3}}$, $y^P_{0\underline{4}}$, and y^P_{40} of the one-state Potts model. (Eigenvalues associated with isotropic eigenfunctionals containing odd powers in the spin σ are not considered here.) The merging of the two solid curves near $d=5$ is an artifact of the short truncation and indicates that the two eigenvalues coalesce forming a conjugate complex pair (dotted curve). Also shown are the results for $y^P_{0\underline{3}}$ obtained from ϵ expansion by a resummation technique[40] (crosses) and by extrapolating the order-ϵ^3 formula (dashed curve).[41] It is of interest that eigenvalues obtained by the scaling-field method cross near $d=4$. Unfortunately, only limited information exists for the correction-to-scaling exponents of the percolation problem at dimension $d=2$. It has been conjectured that $y^P_{40}=-2$,[8] and results by the Kadanoff variational method extrapolated to $Q=1$ yield $y^P_{40}\simeq-2$.[28] Results for dimensions near $d\simeq2$ [such as existed for $y^P_{20}(d)$ and $\eta^P(d)$; see Fig. 4] are not available for $y^P_{40}(d)$. No attempt has been made to link the scaling-field results with the spectrum of correction-to-scaling exponents at $d=2$ since all calculations to date are for only a limited parameter space. Specifically, operators that are expected to have eigenvalues $y\gtrsim-3$, are neglected, e.g., the ones with $y^P_{20,4}$, $y^P_{0\underline{3},2}$, $y^P_{0\underline{4},2}$, and $y^P_{40,2}$.

IV. SUMMARY

The Q-state Potts model has been studied as function of the number of states Q and dimension d. For dimensions $3.4\lesssim d\leq 4$, lines of critical and tricritical Potts fixed points $P_c(Q)$ and $P_t(Q)$ were found for small Q that annihilate as $Q\to Q_c(d)$. This mechanism for the changeover from continuous to first-order behavior as a function of Q is identical to the one discussed by Nienhuis *et al.* for $d=2$,[3,28] but is obtained here for a different range of dimensions by an entirely different technique. By ϵ expansion about $d=4$, we have also obtained the exact dependence of Q_c upon ϵ,

$$Q_c(d=4-\epsilon)=2+\epsilon^2/a+O(\epsilon^3)\ ,$$

with a being a coefficient of order unity. Our results for Q_c as a function of dimension lends full support to the assertion that $2<Q_c<3$ for $d=3$.

The percolation fixed point at $Q=1$ has been studied for dimensions $3\leq d\leq 6$. Unlike calculations that apply resummation techniques to ϵ or loop expansions,[20–23] our results for η^P exhibited a sign change as function of dimension and thus show the curvature necessary to link ϵ-expansion results for $d=6-\epsilon$ to exact results at $d=2$, where $\eta^P(d=6-\epsilon)<0$ and $\eta^P(d=2)=\frac{5}{24}$, respectively.

Finally, the work provides the structure of the scaling-field equations, whose knowledge is necessary for the study of correction-to-scaling exponents, phase diagrams, and thermodynamic functions of the Potts model. Dimension can be varied continuously so that results from this momentum-space RG method at higher dimensions can be linked with those from position-space RG techniques near two dimensions. To do this in detail requires larger truncations of scaling-field equations. This work also opens the possibility to study problems containing Potts symmetry in three spatial dimensions, such as the cubic or random M-vector models.[17]

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation under Grant No. DMR-79-20785. One of us (K.E.N) received support from the Office of Naval Research, Grant No. N00014-77-C-0537 at the University of Illinois, during preparation of part of the manuscript, and wishes to thank John D. Dow for continued encouragement. We also thank J. Adler, A. Aharony, G. Grinstein, B. Nienhuis, M. den Nijs, and M. Schick for interesting discussions.

APPENDIX A: COMPUTATION OF THE COUPLING COEFFICIENTS

The computation of the coupling coefficients a_{mjk} and a_{mj} in the scaling-field equations (2.2) is facilitated by the following rules. (See also Refs. 14–17.)

(1) The following three formulas[11] simplify the combinatorial algebra necessary to derive a_{mjk}:

$$\sum_\alpha e^i_\alpha e^j_\alpha=\frac{N+1}{N}\delta_{ij}-\frac{1}{N}\ , \tag{A1}$$

$$\sum_i e^i_\alpha=0\ , \tag{A2}$$

and

$$\sum_i e^i_\alpha e^i_\beta=\frac{N+1}{N}\delta_{\alpha\beta}\ . \tag{A3}$$

For example, in deriving the coupling coefficient $a_{m,0\underline{3},0\underline{3}}$, one needs to perform the following summation:

$$\sum_\lambda Q_{\alpha\beta\lambda}Q_{\gamma\delta\lambda}=\frac{N+1}{N}F_{\alpha\beta\gamma\delta}-\frac{(N+1)^2}{N^3}\delta_{\alpha\beta}\delta_{\gamma\delta}\ . \tag{A4}$$

Since the result of this summation is of order σ^4, one must project onto the functionals $R_{40}[\sigma]$ and $R_{0\underline{4}}[\sigma]$ of Eqs. (2.5) and (2.8), which yields

$$a_{40,0\underline{3},0\underline{3}}=\frac{12(N+1)^2(N-1)}{N+2} \tag{A5}$$

and

$$a_{0\underline{4},0\underline{3},0\underline{3}}=\frac{6(N+1)}{N+2}\ . \tag{A6}$$

(2) A linked-contraction theorem limits the number of nonzero coupling coefficients a_{mjk}.[14–17] Three rules may be derived from the fact that the combinatorial algebra used in deriving a_{mjk} requires (n + 1) differentiations of both $R_j[\sigma]$ and $R_k[\sigma]$,

$$\sum_m a_{mjk}R_m[\sigma]\propto\frac{\partial^{(n+1)}R_j[\sigma]}{\partial\sigma^{(n+1)}}\frac{\partial^{(n+1)}R_k[\sigma]}{\partial\sigma^{(n+1)}}\ . \tag{A7}$$

The first two rules specify that $a_{mjk}=0$ if the n + 1 derivatives exhaust either $R_j[\sigma]$ or $R_k[\sigma]$. The third rule specifies the relationship between the orders in σ of $R_m[\sigma]$, $R_j[\sigma]$, and $R_k[\sigma]$, which we denote $\widetilde{m}$, $\widetilde{j}$, and $\widetilde{k}$, respectively. Note that Eq. (A7) is schematic in that it is written for $N=1$.

(3) With the exception of the changes in the combinatorial algebra described above, all procedures follow Ref. 17. We have tabulated all coefficients a_{mjk} with m, j or $k=20$, 40, 60, $0\underline{3}$, $2\underline{3}$, $0\underline{4}$, $2\underline{4}$, $0\underline{5}$, $0\underline{6}$, and $0\underline{33}$.[25] Coefficients with m, j, or $k=2'0$ are treated as special cases, as described in Ref. 17. For example, $a_{2'0,jk}$ for $j,k \in P$ is nonzero when the triangular conditions described under (A7) are satisfied and $|\tilde{j}-\tilde{k}| \neq 2$. This coefficient is then obtained by replacing the integral I_n in $a_{20,PP}$ by $-J_n/(2d)$, where n is the number of contractions.

APPENDIX B: ϵ EXPANSION NEAR SIX DIMENSIONS

In calculations by the scaling-field method, we distinguish between analytical and numerical ϵ expansion. This refers to two schemes for the determination of the fixed-point value of Δ which in turn yields the critical exponent η via Eq. (2.3) and defines the physical fixed point for which the spectrum of eigenvalues is determined by linearization.

The two methods were introduced in ϵ-expansion studies near four dimensions.[16,17] For analytical calculations, it is most convenient to choose "the gauge"

$$\mu^*_{20,2}=0 , \tag{B1}$$

and to determine Δ^* from the equation $d\mu^*_{20,2}/dl=0$. This gauge choice yields exact ϵ-expansion solutions for those fixed points for which the μ^*_m are smaller or of $O(\epsilon)$ for all m.[16] Exceptions include the random Ising fixed point for $d=4-\epsilon$ (Ref. 17) and the Q-state Potts fixed point for $d=6-\epsilon$. For numerical studies employing truncations, the physical fixed-point value Δ^* is determined by the condition

$$y_{20,2}(\Delta^*)=0 , \tag{B2}$$

and then $\mu_{20,2}$ assumes in general a nonzero fixed-point value $\mu^*_{20,2}=\mu^*_{20,2}(\Delta^*)$. (See Sec. II for details.) Applying the latter scheme *at fixed dimension* $d=4-\epsilon$, with $\epsilon \ll 1$, yields results identical with those obtained from the gauge choice (B1) for all cases but the Potts fixed point near six dimensions.[16,17] [For the random Ising fixed point, consistent results are obtained using Eqs. (B1) and (B2), but they agree only approximately with the exact ϵ-expansion solutions.[17]] The reason for the discrepancy between methods (B1) and (B2) for the Potts model near six dimensions is that the analytical scheme using gauge (B1) is justified only if, to leading order, $\mu^*_{20,2}$ is the field conjugate to the marginal redundant operator. This is not the case for the Potts model near six dimensions: Because $\mu^*_{0\underline{3}} \propto O(\epsilon^{1/2})$, we find that the marginal redundant eigenvector has a strong $O(\epsilon^{1/2})$ component in the $\mu_{0\underline{3}}$ direction.[16] This results in a value for $\mu^*_{20,2}$ of order 1 if method (B2) is applied.[42] Since we cannot justify the gauge choice (B1) for this fixed point, we accept as the only applicable method the numerical ϵ-expansion scheme (B2).

Parenthetically we note that if criterion (B1) is applied to the case of the one-state Potts model over, e.g., the range of dimensions $5.8 \leq d \leq 6$, then one obtains as a function of dimension a much larger drop in η^P and a somewhat larger one in y^P_{20} than with criterion (B2).

---

*Present address: Department of Physics, University of Notre Dame, Notre Dame, IN 46556.
†Corresponding author.
‡Present address: Fujitsu Limited, Tokyo, Japan.
1For a review, see F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).
2R. J. Baxter, J. Phys. C 6, L445 (1973).
3B. Nienhuis, A. N. Berker, E. K. Riedel, and M. Schick, Phys. Rev. Lett. 43, 737 (1979); B. Nienhuis, E. K. Riedel, and M. Schick, J. Phys. A 13, L31 (1980).
4M. P. M. den Nijs, J. Phys. A 12, 1857 (1979).
5B. Nienhuis, E. K. Riedel, and M. Schick, J. Phys. A 13, L189 (1980). In this reference, on the left-hand side of Eq. (7) the factor $(y_T^{8v}-2)$ is missing.
6R. B. Pearson, Phys. Rev. B 22, 2579 (1980).
7J. L. Black and V. J. Emery, Phys. Rev. B 23, 429 (1981).
8B. Nienhuis, J. Phys. A 15, 199 (1982).
9M. P. M. den Nijs, Phys. Rev. B 27, 1674 (1983).
10Preliminary reports of this work were presented by the present author at The Ninth Midwest Solid State Theory Symposium at Argonne National Laboratory, 1981 (unpublished), and at The 46th Statistical Mechanics Meeting, Rutgers University, New Brunswick, New Jersey, 1981 (unpublished). See also Bull. Am. Phys. Soc. 28, 304 (1983).
11R. K. P. Zia and D. J. Wallace, J. Phys. A 8, 1495 (1975).
12K. G. Wilson and J. Kogut, Phys. Rept. 12C, 75 (1974). The Wilson equation is introduced and discussed in Sec. 11 and Appendix A of this reference.
13G. R. Golner and E. K. Riedel, Phys. Rev. Lett. 34, 856 (1975); Phys. Lett. 58A, 11 (1976).
14F. J. Wegner, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6, p. 7. This article contains reviews of the Wilson RG equation and the scaling-field method.
15K. E. Newman, Ph. D. dissertation, University of Washington, 1982 (unpublished).
16E. K. Riedel, G. R. Golner, and K. E. Newman (unpublished); K. E. Newman and E. K. Riedel (unpublished).
17K. E. Newman and E. K. Riedel, Phys. Rev. B 25, 264 (1982).
18M. Nauenberg and D. J. Scalapino, Phys. Rev. Lett. 44, 837 (1980); J. L. Cardy, M. Nauenberg, and D. J. Scalapino, Phys. Rev. B 22, 2560 (1980).
19A. Aharony and E. Pytte, Phys. Rev. B 23, 362 (1981). Using the argument that the Potts coupling coefficients ω, etc., are small, these authors do not include terms of order $\delta=Q-2$ in their Eq. (13) and elsewhere. This argument is incorrect near four dimensions.
20R. G. Priest and T. C. Lubensky, Phys. Rev. B 13, 4159 (1976); 14, 5125(E) (1976); D. J. Amit, J. Phys. A 9, 1441 (1976).
21O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane,

J. Phys. A 14, 2391 (1981).

[22]F. Fucito and E. Marinari, J. Phys. A 14, L91 (1981).

[23]J. S. Reeve, J. Phys. A 15, L521 (1982); see also J. S. Reeve, A. J. Guttmann, and B. Keck, Phys. Rev. B 26, 3923 (1982).

[24]T. L. Bell and K. G. Wilson, Phys. Rev. B 11, 3431 (1975).

[25]Numerical calculations were performed using the basis operators (2.5)—(2.10) and (2.13)—(2.14) instead of (2.5)—(2.12). Tabulations of coefficients are available upon request.

[26]All coupling coefficients $a_{I,PP}$ are proportional to $N-1$. Certain subsets of Potts operators decouple at other positive integer values of N. For example, all coupling coefficients $a_{I,jk}$ with j or k equal to $0\underline{4}$, $2\underline{4}$, or $0\underline{5}$ contain a factor $(N-1)(N-2)$, and all coupling coefficients $a_{I,jk}$ with j or $k=0\underline{5}$ contain a factor $(N-1)(N-2)(N-3)$. For a general proof, one would use basis operators that are all traceless, which is not the case for our basis. See Ref. 25.

[27]It is probably a coincidence that the value of $d=3.4$ is close to $\frac{10}{3}$, which is the dimension at which Gaussian eigenvalues $y^G_{2\underline{3}}$ and $y^G_{0\underline{5}}$ become positive. See also comments in Sec. III B on $d>4$.

[28]B. Nienhuis, E. K. Riedel, and M. Schick, Phys. Rev. B 23, 6055 (1981).

[29]E. Pytte, Phys. Rev. B 22, 4450 (1980).

[30]J. B. Kogut and D. K. Sinclair, Phys. Lett. 86A, 38 (1981).

[31]F. Fucito and G. Parisi, J. Phys. A 14, L499 (1981).

[32]See also the discussion on Gaussian and classical behaviors in subsection III(i) of Ref. 28, and Ref. 8 in F. J. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973).

[33]P. W. Kasterleyn and C. M. Fortuin, J. Phys. Soc. Jpn. (Suppl.) 16, 11 (1969).

[34]D. Stauffer, Phys. Rept. 54, 1 (1979).

[35]J. W. Essam, Rep. Prog. Phys. 43, 833 (1980).

[36]Using loop expansions to orders 3 and 4, Reeve (Ref. 23) has recently obtained $\eta^P=-0.15$ and -0.13, respectively, in comparison to the second-order result $\eta^P=-0.23$ of Ref. 22.

[37]C. Jayaprakash and H. J. Katz, Phys. Rev. B 16, 3987 (1977).

[38]A consistent set of critical exponents for the percolation problem for $d=3$ are $\nu^P=0.88\pm0.02$ [D. S. Gaunt and M. F. Sykes, J. Phys. A 16, 783 (1983)], $\eta^P=-0.06\pm0.03$ [A. Margolina, H. J. Herrmann, and D. Stauffer, Phys. Lett. 93A, 73 (1982)] and $\beta^P=0.420\pm0.008$ [J. Adler (private communication)].

[39]The results of Ref. 21 shown as crosses in Figs. 4 and 5 are those obtained by the conformal mapping technique. For y^P_{20}, these and the results of Ref. 22 agree to within error bars, while for η^P, the upper and lower crosses at $d=3$ refer to Refs. 21 and 22, respectively.

[40]J. Adler, M. Moshe, and V. Privman, in Percolation Structures and Processes, edited by G. Deutscher, R. Zallen, and J. Adler [Annals of the Israel Physical Society 5, 397 (1983)].

[41]J. Green (née Kirkham) (private communication) has found

$$y^P_{0\underline{3}}=-\epsilon+671\epsilon^2/[7^2\times18] -\epsilon^3[1\,097\,253/(7^4\times81\times8)+93\times4\zeta(3)/7^3]+O(\epsilon^4)\ .$$

Results from the analysis of this series are quoted in Ref. 40.

[42]This is *not* the case for the random Ising model. The difference between the random Ising fixed point at $d=4-\epsilon$ and the Potts fixed point at $d=6-\epsilon$ is that operators of different order in the spin are conjugate to the scaling fields of $O(\epsilon^{1/2})$. For the random Ising model near four dimensions, it is the operator $Q_{04}[\sigma]\sim\sigma^4$, while for the Potts model near six dimensions, it is the operator $Q_{0\underline{3}}[\sigma]\sim\sigma^3$. This leads to different couplings of μ_{04} to $\mu_{20,2}$ than $\mu_{0\underline{3}}$ to $\mu_{20,2}$ in Eq. (2.2), for the two cases. For the former, method (B2) results in $\mu^*_{20,2}\sim O(\epsilon)$.