Universality of correlation-length amplitudes and their relation with critical exponents for quantum systems

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Recently it has been found that for several two-dimensional classical models the amplitude of the correlation length of a finite system at a critical point is universal and related to the correlation-function exponent η. Here we propose that for quantum systems the ratios of two such amplitudes are universal and are related to the ratios of the corresponding exponents. This proposition is confirmed numerically for the transverse Ising model for spin $S=\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. It is used to calculate η in a simplified way.

For classical as well as quantum systems the thermodynamic properties near a continuous phase transition can be described in terms of scaling and universality. If the system is finite, of length L, L^{-1} can be added to the parameters used in the scaling analysis.[1] Its role is special, as one knows from dimensional analysis how it behaves under a change of scale, in contrast with the other parameters. This property has been widely used to estimate critical behavior of such systems.[2,3] For several models, universality of the correlation-length amplitudes have been established[4–6] as a consequence of the scaling behavior of L^{-1}. These results appear particularly transparent in a renormalization-group (RG) framework.[1,7] Furthermore, a simple relation connects this amplitude with the correlation-function exponent η. In the same spirit, η has been related to spectral properties for quantum systems.[8] Finally, finite-size effects have been studied for lattice-gauge theory.[9] Here we investigate the difference between classical and quantum systems. We conclude that for the quantum case, in general, amplitude-exponent relations have to be replaced by relations between amplitude ratios (not to be confused with the universal amplitude ratios above and below the critical point for infinite systems) and exponent ratios.

For a number of two-dimensional (2D) classical, isotropic systems, the following equation has been found to hold

$$\pi\eta = A \quad , \tag{1}$$

where η is the order-parameter correlation-function exponent $[C(r)\sim r^{-\eta},\ r\rightarrow\infty]$, A is the amplitude of the inverse correlation length ($\xi^{-1}=A/L,\ L\rightarrow\infty$). The system is finite (length L) in one direction but otherwise critical. We note that here the correlation length must be defined in a precise way, unlike the usual situation where an all-over factor is immaterial in a scaling analysis. Equation (1) implies immediately that A is universal. For the 2D Ising model, this relation holds, as inspection of the exact solution for finite L shows.[10] Other models which have been investigated include Anderson localization[4] and the XY model.[5] Numerically, Eq. (1) has been verified for percolation, lattice animals, and the Potts model.[6] The Gaussian, symmetric eight-vertex, and N-component cubic models also satisfy the conjecture, as Nightingale and Blöte report,[7] combining analytical and numerical arguments. In addition, these authors considerably generalize Eq. (1). They propose a form that replaces it for anisotropic systems and they suggest that Eq. (1) is valid for other types of operators as well. The correlation-length amplitudes for the two special cases of the $S=1$ anisotropic Heisenberg model have been compared with η.[11] Preceding all these calculations establishing Eq. (1), Haldane[8] has derived analogous relations for Luttinger liquids (1D) which include the spin $S=\frac{1}{2}$ Heisenberg-Ising model.

Let us now consider the validity of Eq. (1) for quantum systems. Two situations can occur. If the model is the anisotropic limit of an isotropic classical one, the appropriately chosen limiting quantum model then satisfies Eq. (1), as the $S=\frac{1}{2}$ Ising model in a transverse field, for example. In general, when no such analog is known, there is an ambiguity in the choice of the Hamiltonian. Two Hamiltonians differing by an all-over factor certainly do describe the same critical behavior, but their gap (or correlation length) amplitudes differ by this factor. Equation (1) then is modified to $A=f\eta$, where f is undetermined. We now suggest that f depends only on the choice of the all-over factor of the Hamiltonian. So, amplitude-exponent relations for different operators $\hat{O}_x,\hat{O}_y$ have the same factor f. From this it follows that

$$\frac{\eta_x}{\eta_y}=\frac{A_x}{A_y} \quad , \tag{2}$$

where the exponents η and the amplitudes A refer to their respective operators. In practice, this allows the determination of all correlation-function exponents from the amplitude ratios, if any one of them is known.

Now let us indicate how to implement and test Eq. (2). Suppose that a D-dimensional quantum system with the Hamiltonian $\hat{H}(h)$ has a second-order phase transition in the ground state at $h=h_c$. With the aid of the phenomenological renormalization group (PRG),[2,3] h_c and the correlation-length exponent can be estimated from $Lg_L(h)=Mg_M(h')$, with $L>M$, where g_L and g_M are the lowest energy gaps of a finite system of size L and M, respectively. The ground-state correlation function of the operator $\hat{O}_x(r)$ is defined by

$$C_x(r)=\langle 0|\hat{O}_x(r)\hat{O}_x(0)|0\rangle-[\langle 0|\hat{O}_x(0)|0\rangle]^2 \quad , \tag{3}$$

and at h_c for $r\rightarrow\infty$, $C_x(r)\sim r^{-(D+z-2+\eta_x)}$, where z is the dynamical critical exponent. Numerically $C_x(r)$ can be estimated at $r=L/2$, the maximum possible r for a periodic

TABLE I. Finite chain PRG results for the spin-S transverse Ising model. The chains of length L are compared with $L+1$ and the results are linearly extrapolated in L^{-1} to $L\to\infty$. The correlation exponents η_s and η_E are calculated with Eq. (3). The error bars increase with growing S and typical values are shown for $S=2$. Extensive calculations by the present authors support universality with S (Ref.14).

Spin	$S=\frac{1}{2}$	$S=1$	$S=\frac{3}{2}$	$S=2$	Exact
$(h/J)_c$	1.000	1.325	1.476	1.655(005)	1.0 $(S=\frac{1}{2})$
ν	0.997	1.008	1.015	1.010(05)	1.0
η_s	0.249	0.255	0.258	0.26(03)	0.25
η_E	1.96	1.95	1.99	1.75(40)	2.0

system of size L (L even). There are usually two independent critical exponents, one of which (ν) can be determined by the PRG. The other, related to the order parameter, can be estimated with Eq. (3). The universal ratios, Eq. (2), provide an alternative way to calculate this exponent. If $\hat{O}_E$ is the energy density operator, scaling law relates ν to η_E: $D+z+2-2/\nu=\eta_E$. From $\eta_s/\eta_E=A_s/A_E$ one then obtains

$$\eta_s=\frac{A_s}{A_E}\left[D+z+2-\frac{2}{\nu}\right] , \tag{4}$$

where A_s and A_E are the amplitudes for the spin-spin and energy-density–energy-density correlation lengths, respectively. Equation (4) is to be evaluated at the fixed point of the PRG, $h=h'=h^*(L,M)$.

As an example, the spin-S transverse Ising chain is considered for $S=\frac{1}{2}$, 1, $\frac{3}{2}$, and 2. The Hamiltonian is

$$H_L=-\frac{J}{2S^2}\sum_{i=1}^{L} S_i^x\cdot S_{i+1}^x-\frac{h}{2S}\sum_{i=1}^{L} S_i^z , \tag{5}$$

where $\vec{S}_i$ are spin-S operators and $\vec{S}_{L+1}=\vec{S}_1$. The Hamiltonian has been chosen such that for $S=\frac{1}{2}$ it corresponds to the correct anisotropic limit of the 2D classical Ising model. For $S\to\infty$, $L\to\infty$, $(h/J)_c$ approaches its mean-field value $(h/J)_{\rm MF}=2$.[12] For $S>\frac{1}{2}$ this model is not exactly soluble and its classical analog does not have a simple form as for $S=\frac{1}{2}$. However, universality arguments[13] and high-temperature expansions[12] support the belief that the critical behavior is unchanged for all finite S. Finite lattice calculations[14] clearly confirm this hypothesis. In Table I some relevant data are collected. The dynamic exponent z is equal to 1 for all S considered.[15] For Eq. (2) to hold it is necessary that $A_s(S)/A_E(S)=\eta_s/\eta_E=\frac{1}{8}$ independently of S. In Fig. 1 we show $A_s(S,L)/A_E(S,L)$ vs $1/L$. For all S considered it tends towards the universal value $\frac{1}{8}$ ($\eta_s=0.25$, $\eta_E=2$).

Let us now turn to the calculation of η_s. We have calculated this exponent both directly [Eq. (3)] and through the amplitude ratios [Eq. (4)]. Figures 2 and 3 show the results. The direct determination has strong even-odd oscillations with L. In contrast, the method using the amplitude ratios gives results which are quite smooth. For spin $S=2$, for example, where only four points are available, the extrapolation with the new method is much more reliable. Furthermore, the numerical effort is considerably reduced, as no eigenvectors have to be calculated. Also, higher gaps can be used to determine additional exponents. It is straightforward to test and use the present relation in other systems.[16] Most of the models studied with the finite-size methods lend themselves to a similar analysis.[2,3]

Conceptually it would be interesting to investigate whether there are restrictions to the validity of the amplitude-exponent relations, classically or quantum mechanically. Higher-D pose additional problems, as the finite geometry, important for the validity of these relations, can be chosen in different ways.

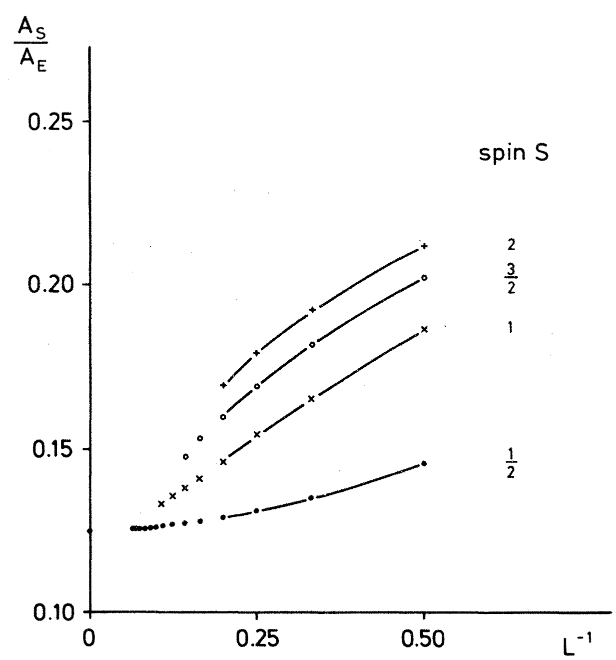

FIG. 1. The amplitude ratio $A_s/A_E=g_s/g_E$ for the spin-S transverse Ising model. The energy gaps g_s and g_E ($g\sim\xi^{-1}$) are energy differences between the first excited states in the subspace of spin- and energy-density, respectively, and the ground state. For all values of S the values of the ratios extrapolate to the universal value $\frac{1}{8}$ for $L\to\infty$. The ratios were obtained at the fixed points of the PRG comparing sizes L with $L+1$.

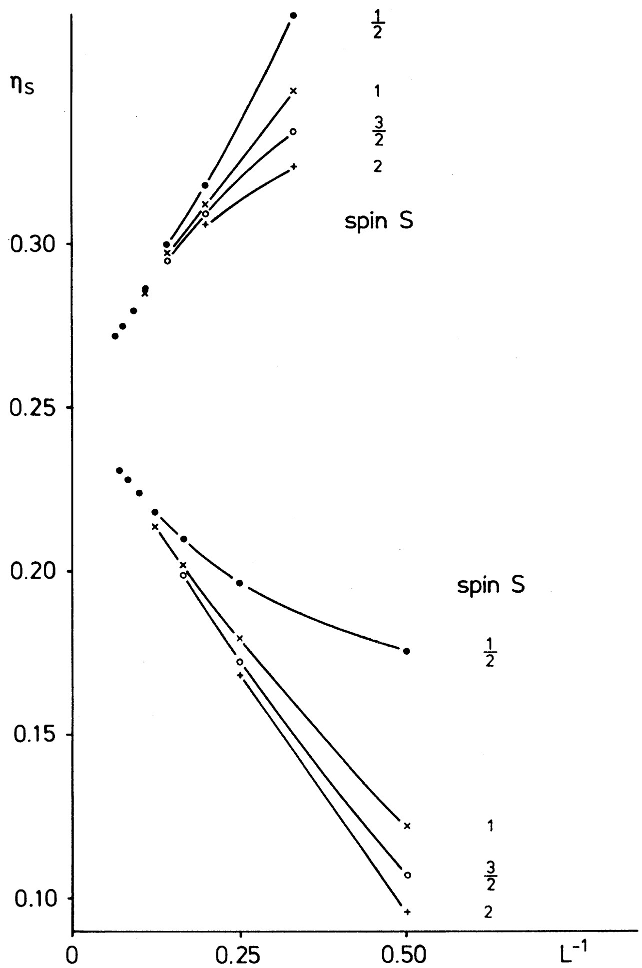

FIG. 2. Estimates of the critical exponent η_s from the correlation function [Eq. (3)] of a finite chain of length L. η_s is defined by

$$\eta_s = \ln[C_x(L/2)/C_x(L'/2)]/\ln(L'/L)$$

(with $L' = L + 2$) taken at the PRG fixed point. For full account of these calculations see Ref. 14.

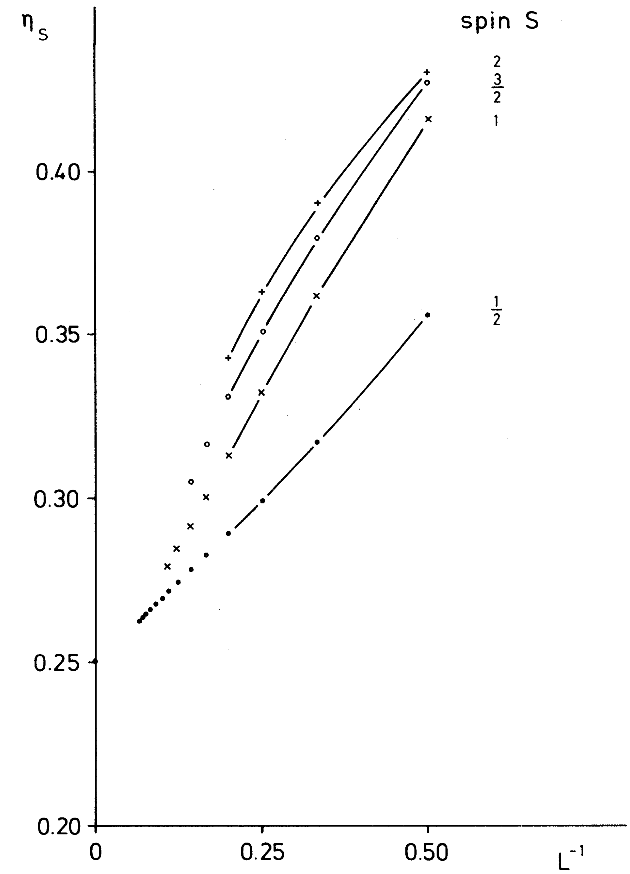

FIG. 3. Estimates of η_s using Eq. (4) and the amplitude ratios A_s/A_E, Fig. 1. A much better determination is possible than from Fig. 2, especially for higher S.

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[15]It must be pointed out that the convergence of the PRG implies $z=1$, thus no separated analysis is necessary if the PRG is credible.
[16]One of us (M.K.) has looked at the $S=1$ anisotropic Heisenberg chain [as defined by R. Botet, R. Jullien, and M. Kolb, Phys. Rev. B 29, 3914 (1983)] and estimated the exponent η from amplitude ratios. The result holds for all values of the anisotropy, in contrast with Ref. 11, where the conclusion depends on the choice of the Hamiltonian (the correct all-over factor needs to be chosen).