Mechanical detectors of second sound

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The detection sensitivity of porous microphones is investigated theoretically. The superfluid enhancement of the membrane's spring constant is found to be the effect that limits the sensitivity. Ways to minimize or circumvent it are proposed and compared with each other. Under favorable circumstances, an increase in the signal by three orders of magnitude is possible.

Second sound exists in superfluids as a direct consequence of the spontaneously broken gauge symmetry. It takes the form of a propagating temeprature wave[1] in ^{4}He II, of a concentration wave in ^{3}He-^{4}He mixtures[2] in the mK regime, and of a spin wave[3,4] in ^{3}He-A_1. It can be detected by either measuring the respective propagating density, using for instance a thermometer in the first and a pickup coil[4] in the last case, or more universally, by measuring the fluctuating counterflow velocity which is always present in second sound. Two appropriate mechanical devices have been designed to date, the widely employed oscillating superleak transducer[5] (OST), which is a microphone with a porous diaphragm, and the Peshkov transducer[6] (PT), which is an ordinary microphone with a massive diaphragm, but positioned behind a rigid superleak. The latter seems out of fashion nowadays, but nevertheless provides a useful alternative both as a generator[7] and as a detector. The detecting sensitivity of the OST has been studied recently by Saslow,[8] with special attention paid to the membrane's mass enhanced by the motion of the superfluid. Because detectors usually work far below their resonance frequency, it is really the superfluid enhancement of the membrane's stiffness constant (and, if it is lacking, that of the damping constant) which determines the detector's response. In this work, these effects are calculated both for the OST and PT, and ways to maximize their sensitivity discussed. In addition, the possibility of detecting second sound in the A and B phases of superfluid ^{3}He is investigated.

Throughout this paper, the propagating density has been indentified with $\delta\sigma/\sigma$, where σ denotes entropy per mass. This is correct for ^{4}He II, and ^{3}He-A and -B at low fields.[1,9] The formulas, however, can be easily rendered more generally valid. As was shown in Ref. 10, substituting δx_ρ for $\delta\sigma/\sigma$ yields the corresponding equations for the ^{3}He-A_1 phase. Similarly, $\delta\sigma/\sigma$ becomes

$$(\chi_c^{-1}\delta c/c+\chi_\sigma^{-1}\delta\sigma/\sigma+\chi_\xi^{-1}\delta\xi/\xi)/(\chi_c^{-1}+\chi_\sigma^{-1}+\chi_\xi^{-1})$$

in the ^{3}He-^{4}He mixture, where c and ξ denote the concentration and the relative magnetization (polarization), respectively. The three susceptibilities are $\chi_\kappa^{-1}=\kappa^2\partial^2\epsilon/\partial\kappa^2$, where ϵ is the energy per mass and κ stands for c, σ, and ξ. The second-sound velocity assumes the appropriate expression for the system under consideration. It is, e.g., $c_2^2=\rho_s/\chi_\sigma\rho_n$ in ^{4}He II and ^{3}He-B, and $c_2^2=\rho_s(\chi_c^{-1}+\chi_\sigma^{-1}+\chi_\xi^{-1})/\rho_n$ in the mixture. At low temperatures and small magnetic fields, the latter reduces to $c_2^2=\rho_s/\chi_c\rho_n$.

We start by examining the response of the OST in its detecting mode. The force $A(\Delta P-\delta P)$ on the membrane, exerted by the incoming wave, is given by its area A and the difference between the pressure on its internal side (facing the back plate) ΔP and the external side (facing the incoming wave) δP. The pressure difference is to be evaluated under three conditions:

(i) The chemical potential is equal[10,11] on both sides of the membrane, or $\Delta\mu=\delta\mu$.

(ii) The hydrodynamic variables of the internal cavity are as given by[12]

$$\Delta\sigma/\sigma=w_0/i\omega L,\quad \Delta\rho/\rho=v_0/i\omega L\ , \tag{1}$$

where v_0 and w_0, respectively, denote the barycentric velocity $(\rho_s v_s+\rho_n v_n)/\rho$ and the counterflow velocity $\rho_s(v_n-v_s)/\rho$, both at the membrane. If the volume V of the internal cavity does not display one-dimensional geometry, the distance L between the backplate and the membrane has to be generalized to V/A. The first of Eqs. (1) can be proven by integrating the continuity equation $\dot\sigma/\sigma+\vec\nabla\cdot\vec w=0$ over the volume V, yielding

$$\int(\dot\sigma/\sigma)\,dV=-\oint\vec w\cdot d\vec s\ .$$

With q_2^{-1} being much larger than any logitudinal dimension of the internal cavity, $\dot\sigma$ and σ are spatially uniform quantities; in addition, we have $w=w_0$ at the membrane and $w=0$ elsewhere at the surface of the volume V. Therefore $-i\omega(\Delta\sigma/\sigma)V=-w_0A$, or the first of Eqs. (1). The second equation can be similarly derived by starting from $\dot\rho/\rho+\vec\nabla\cdot\vec v=0$.

(iii) The hydrodynamic variables right outside the membrane are as given by

$$\delta\sigma/\sigma=(w_0-2w_{\rm in})/c_2,\quad \delta\rho/\rho=(v_0-2v_{\rm in})/c_1\ . \tag{2}$$

The subscript "in" refers to the incoming waves, and "re" below refers to the reflected ones. With $\delta x=\delta x_{\rm in}+\delta x_{\rm re}$, where x stands for ρ, σ, w, and v, and with the eigenvectors of first and second sound being $\delta\rho_{\rm in}/\rho=-v_{\rm in}/c_1$, $\delta\rho_{\rm re}/\rho=v_{\rm re}/c_1$, and $\delta\sigma_{\rm in}/\sigma=-w_{\rm in}/c_2$, $\delta\sigma_{\rm re}/\sigma=w_{\rm re}/c_2$, respectively, Eqs. (2) are easily derived.

The evaluated pressure difference is

$$A(\Delta P-\delta P)=\eta(-\delta L/L+2w_{\rm in}/c_2+2v_{\rm in}/c_1)\ , \tag{3}$$

where $\eta=A\rho/(c_1^{-2}+\rho_s c_2^{-2}/\rho_n)$. For small enough displacements, the dynamics of the oscillating porous membrane is described by a pendulum equation,

$$M\ddot L+\gamma\dot L+K\delta L=f\ , \tag{4}$$

for which Eq. (3) yields the force f and the effective spring constant $K=K_0+\Delta K$, with K_0 being the vacuum one:

$$f=2\eta(w_{\text{in}}/c_2+v_{\text{in}}/c_1), \quad \Delta K=\eta/L \quad . \tag{5}$$

This is a plausible result. Following Sherlock and Edwards,[5] one can obtain its low-temperature limit by resorting to the phonon gas model: For a massive microphone detecting first sound, the corresponding formula is $\Delta K=A\rho c_1^2/L$. In ^{4}He, second sound can be interpreted as first sound in the phonon gas, of density ρ_n; hence one may take $\Delta K=A\rho_n c_2^2/L$, in good agreement with the above equation for $\rho_s\to\rho$.

The enhancement of the mass M is discussed in detail by Saslow[8] and need not be repeated here. It modifies the working range of the detector, $\omega^2<<K/M$, and can be effectively limited by employing a more porous membrane. Note that the diffusive currents across the pores that have such devastating effects in the generating mode and severely restrict the porocity there[10] are quite harmless here. For instance, the difference in the fountain pressure on both sides of the membrane is identically zero in the limit of vanishing K_0 and at most $(\rho\rho_n/\rho_s)c_2^2\,\delta\sigma/\sigma$ for infinite K_0. This is smaller by a factor q_2L_G than the pressure difference at the generator G. Because the OST has an enhanced stiffness constant, the effect of damping, $\gamma=\gamma_0+\Delta\gamma$, with a usually small[5] intrinsic γ_0 and the tiny enhancement of order $\Delta\gamma=O((q_2L)^2,(q_1L)^2)$ will be negligible. In addition, depending on the experimental situation, there will be electromagnetic corrections to the enhancement of M, γ, and K, not considered here. So we may take the signal, $\delta L/L$ for the working range $\omega^2<<K/M$, as

$$\delta L/L=2(w_{\text{in}}/c_2+v_{\text{in}}/c_1)(1+K_0/\Delta K) \quad . \tag{6}$$

It is maximal for maximal ΔK and hence minimal L. Equation (6) changes the "reciprocity relation" of Saslow[8] to $w_{\text{in}}/v_{\text{in}}=c_2/c_1$. The liquid's correction to the stiffness constant, ΔK, varies over a wide range of values. With the upper limit of $A=10$ cm^2 and the lower limit of $L=5\mu$, it goes up to $\Delta K=10^{10}$, 4×10^7, and 10^5 dyn/cm for ^{4}He II and ^{3}He-A_1, ^{3}He-^{4}He mixture at low temperatures, and ^{3}He-A and -B phases, respectively. The vacuum stiffness constant K_0 is typically 10^7 dyn/cm for the usual way of mounting.[5] If measures are taken to avoid tightness at room temperatures, one should be able to reduce K_0 significantly, though probably not much below 10^4 dyn/cm, where the sagging due to gravitation is of the order of a micrometer. Nevertheless, $K_0/\Delta K<<1$ can be achieved for any of the above systems, yielding an optimized signal strength of second sound,

$$\delta L/L=2w_{\text{in}}/c_2 \quad , \tag{7}$$

characteristic of nearly rigid wall reflection, where the incoming fountain pressure $\sim\delta\sigma/\sigma\cong2w_{\text{in}}/c_2$ is compensated by the internal pressure $\sim\Delta\sigma/\sigma=-\delta L/L$.

If one increases L by moving the backplate, and inserts a wire net between the backplate and the diaphragm at the distance d from the latter, to serve as the second electrode, the signal strength, now $\delta L/d$, is amplified by a factor L/d. Maximal signal strength is achieved for maximal L (which of course is still subject to the constraint $q_2L<<1$). For ^{4}He at $\omega=10^2$ sec^{-1}, one can take $L=1$ cm and $d=5$ μm, yielding an amplification of 2×10^3. The transparency of the meshed electrode must be such that v_n can flow freely; hence, the linear dimensions of the holes must be larger than the viscous penetration depth.

Another variant of the OST is given by moving the backplate still further away, to the limit $q_2L>>1$. (The intermediate range, $Lq_2\cong1$, leads to undesirable resonances.) The mesh is left where it is. If K_0 is sufficiently small, the membrane can freely follow the counterflow velocity, behaving essentially as a free surface, not reflecting any portion of the incoming wave. This yields the biggest possible signal. The pressure difference has to be reevaluated for this case. The conditions (i) and (iii) remain unchanged, while Eqs. (1) of (ii) become

$$\Delta\sigma/\sigma=-w_0/c_2, \quad \Delta\rho/\rho=-v_0/c_1 \tag{8}$$

for Lq_2, $Lq_1>>1$. If instead $q_2^{-1}<<L<<q_1^{-1}$ is satisfied, the second of Eqs. (8) remains $\Delta\rho/\rho=v_0/i\omega L$. The resulting expression is

$$A(\Delta P-\delta P)=\Delta\gamma(w_{\text{in}}+xv_{\text{in}}-v_{\text{M}}) \quad , \tag{9}$$

where $\Delta\gamma=2A\rho(c_1^{-1}+\rho_s c_2^{-1}/\rho_n)^{-1}$, $x=1$ for the first and $\Delta\gamma=2\rho_n c_2/\rho_s$, $x=-iq_1L$ for the second case. With $v_{\text{M}}=\dot{L}$, this expression yields an enhancement, $\Delta\gamma$, of the damping rather than the stiffness coefficient, cf. Eq. (4). Detecting second sound, the signal is

$$\delta L/d=(w_{\text{in}}/d)(K_0/\Delta\gamma-i\omega)^{-1} \quad . \tag{10}$$

For $K_0<<\omega\Delta\gamma$, we have $v_{\text{M}}=w_{\text{in}}$, and the amplification, as compared to Eq. (7), is $(2q_2d)^{-1}$. This is larger still than in the previous case, where we had L/d with $q_2L<<1$. The price to be paid here is the waves reflected from the backplate. One can, however, partly fill the space behind the membrane with sinter, and connect it to the cooling source. If the heat exchange can be made fast enough, such that the temperature of the liquid in this cavity is approximately constant, we can take $\Delta\sigma/\sigma=0$ instead of the first of Eq. (8). Then Eqs. (9) and (10) (for the second case) are correct even for $q_2L<<1$, and there will be no reflections.

In ^{4}He II, taking $K_0=10^4$ dyn/cm and $\omega=10^3$ sec^{-1}, we have $500K_0\lesssim\omega\Delta\gamma$, and the amplification $(2q_2d)^{-1}\cong10^4/5$ is large. Note, however, that the value of $q_2^{-1}\cong2$ cm must be much smaller than L if the rate of heat exchange cannot be sufficiently increased. In the A and B phases of superfluid ^{3}He, circumstances are not as favorable. Because of the tiny second-sound velocity, one is confined to very low frequencies to minimize damping. In this limit, K_0 usually dominates whatever superfluid enhancement there is. Taking $K_0=10^4$ dyn/cm and $\omega=20$ sec^{-1}, we have

$$\delta L/d=(\Delta\gamma c_2/K_0d)(w_{\text{in}}/c_2) \quad , \tag{11}$$

yielding a tenfold amplification over Eq. (7).

Depending on the system and the temperature range, the PT can provide a useful alternative. In addition, the fact that one is employing a massive membrane here means there is a wider choice of material with possibly more compliant ones among them. Instead of the pressure difference, one now has to calculate ΔP, the pressure change in the cavity enclosed by the rigid superleak and the membrane. It alone drives the membrane. (Assuming there is no liquid or vapor behind the membrane, I have taken the pressure there to be constant. Its variation is of course easily included if the assumption is incorrect.) The pressure ΔP has to be evaluated, again under condition (i) and (iii), where v_0 and w_0 now denote the respective velocities at the superleak, with $v_0+w_0=0$. In addition, we have[13] instead of condition (ii), or Eqs. (1),

$$\Delta\sigma/\sigma = w_0/i\omega L, \quad \Delta\rho/\rho = (v_0 - \beta v_M)/i\omega L \tag{12}$$

for $L = V/A_{SL} << q_1^{-1}, q_2^{-1}$; A_{SL} and A_M are the areas of the superleak and the membrane, respectively, while $\beta = A_M/A_{SL}$. Taking $\eta = \rho A_{SL}\rho_n c_2^2/\rho_s$, $\zeta = \rho_n c_2^2/\rho_s c_1^2$, and d as the distance between the membrane and the other electrode plate, the signal, now proportional to $\delta V/dA_M = \delta L/d\beta$ is given by

$$\frac{\delta L}{d\beta} = \frac{2L(v_{in}/c_1\zeta - w_{in}/c_2)/d\beta}{1 + K_0 L/\eta\beta^2 - iq_1 L/\zeta} \ . \tag{13}$$

If the PT is bigger, such that $q_2^{-1} << L << q_1^{-1}$, we have to substitute $\Delta\sigma/\sigma = -w_0/c_2$ for the first of Eqs. (12). Or if the cavity is partly filled with sinter, as discussed for the OST, the constraint becomes $\Delta\sigma/\sigma = 0$. Then we have

$$\frac{\delta L}{d\beta} = \frac{2(v_{in}/c_1\zeta - w_{in}/c_2)/d\beta}{(1 - L\nu)K_0/\eta\beta^2 - \nu/\zeta} \ , \tag{14}$$

where $\nu = iq_2\zeta + iq_1$ for the first, and $\nu = iq_1$ for the second case. The signal strengths, Eqs. (13) and (14), are comparable to that of the OST. But for special parameter ranges, an additional amplification of the order of β^{-1} may be achieved.

Now we turn to the problem of estimating the possibility of detecting second sound in ^{3}He-*A* or -*B*. First we need to reinvestigate the generation of second sound and shall, in the following, examine the case where the chamber in which the wave propagates is of length D, with $q_1^{-1} >> D >> q_2^{-1}$. In contrast, the geometry assumed previously[10,11] was $D >> q_1^{-1} >> q_2^{-1}$. The reason for this modification lies in the strong damping of second sound, which restricts the distance that a temperature pulse can propagate to a length a few times larger than the second-sound wavelength, i.e., $D \gtrsim q_2^{-1}$. (With the strong damping, however, the system displays the same behavior as in the limit $D >> q_2^{-1}$. This condition therefore need not be changed.) The dimension of D and the vast difference between q_2 and q_1 necessarily implies $D << q_1^{-1}$. In other words, instead of a propagating first-sound pulse, there is instantaneous relaxation of the density fluctuation within D. We therefore need to calculate the driving efficiency under the altered condition: With $\Delta\mu = \partial\mu$, Eqs. (1), and $\delta\sigma/\sigma = w_0/c_2$ and $\delta\rho/\rho = -v_0/i\omega D$ we obtain $v_0/w_0 = \rho_n c_2^2/\rho_s c_1^2$, valid also for the strictly semi-infinite geometry, but the ratio

$$(\delta\rho/\rho)/(\delta\sigma/\sigma) = (\rho_n c_2^2/\rho_s c_1^2)\, i/q_2 D$$

(with $q_2 D$ or order unity) is now much larger than in the previous case.[11] Nevertheless, the formula for the second-sound driving efficiency[10] $\delta\sigma/\sigma = v_M/c_2$ remains, especially in the two superfluid phases of ^{3}He under consideration, an extremely good approximation. This has two consequences: First, the discussion of the nuisance effects in Ref. 5 remains valid. Second, with Eq. (11) in mind, we find that an optimally designed OST would, in the absence of sound damping, have its membrane displaced by $\delta L_D = \delta L_G(-i\omega\Delta\gamma/K_0)$, where D and G refer to detector and generator, respectively. The reduction factor is $\frac{1}{30}$ for $K_0 = 10^4$ dyn/cm, $\omega = 20$ sec^{-1}. And the remaining question is how strong a signal can be generated: The superfluid velocity in the pores of the generator's membrane has to be subcritical, $v_s^p < v_c$. In addition, we have the continuity equation of the superfluid in the vicinity of the membrane of porocity α, $v_s - v_M = \alpha(v_s^p - v_M)$. Together, these lead to the final formula for the signal strength

$$\delta L_D/d < v_c(\rho/\rho_s\alpha - 1)^{-1}(\Delta\gamma/K_0 d)\, e^{-r/\xi} \ , \tag{15}$$

where ξ describes the bulk damping in between,[4] $\xi = 1$ cm for 3 Hz, and $q_2^{-1} = 0.5$ cm. Although with $\rho_s \rightarrow \rho$, $\alpha \rightarrow 1$ the signal could in theory be made arbitrarily strong, the value of α is restricted by nuisance effects, especially thermal diffusion across the pores, in complete analogy to the discussion about the A_1 phase in Ref. 10.

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[13] For derivation see Eqs. (13) of Ref. 7. The sign mistakes have been corrected.