Renormalization-group fixed points of general n-vector models

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We make a general study of symmetry and stability of the fixed points of the quartic Hamiltonian of an n-component field (or order parameter) for $n \neq 4$. Simple proofs of known results are given. Among new results, we show that when it exists the stable fixed point is unique; we give some precision on its symmetry and on its attractor basin.

Nearly half a century ago, Landau introduced a model for second-order phase transitions.[1] It is a mean-field theory, and it yields fairly good predictions for the nature of the symmetry changes between the two phases, but it naturally fails to predict the critical exponents. The renormalization-group method of Wilson[2] and its application to quartic Hamiltonian made in Refs. 3—5 can be used to improve Landau theory, as was suggested in Refs. 6—8. For the last seven years many examples of actual second-order phase transitions have been studied by this method. However, for this type of Hamiltonian, a general study of the existence and stability of the renormalization-group fixed points is lacking. The aim of this paper is to explain what is known on this topic and give new results; e.g., when it exists, the stable fixed point is *unique* and all other fixed points are (to the lowest order in the ϵ expansion) on the boundary of its attractor basin. In addition, the "intriguing conjecture" on the critical exponents made at the end of the paper[5] by Brézin, Le Guillou, and Zinn-Justin is proven.

We consider a physical system in thermodynamical equilibrium (e.g., a crystal) with its symmetry group Γ (e.g., one of the 230 three-dimensional crystallographic space groups). At the phase transition the system state is described by a vector $\vec{\phi}$ belonging to an n-dimensional real vector space $\mathscr{E}_n$, which carries an irreducible orthogonal representation of Γ:

$$\Gamma \ni h \to D(h) \in O(n) \tag{1}$$

acting on $\mathscr{E}_n$. The value of the n-dimensional order parameter $\vec{\phi}$ representing the equilibrium state minimizes the free-energy thermodynamic potential, whose most simple form assumed by Landau is

$$P(\vec{\phi}) = \frac{\alpha}{12} u(\vec{\phi}) - \frac{a}{2} \vec{\phi} \cdot \vec{\phi}, \quad \alpha > 0 \tag{2}$$

where $\vec{\phi} \cdot \vec{\phi}$ is an $O(n)$-invariant orthogonal scalar product on $\mathscr{E}_n$ and $u(\phi)$ is a homogeneous, Γ-invariant, positive quartic polynomial

$$u(\underline{D}(h)\vec{\phi}) = u(\vec{\phi}), \quad \forall h \in \Gamma, \tag{3}$$

$$u(\lambda\vec{\phi}) = \lambda^4 u(\vec{\phi}), \quad \vec{\phi} \neq \vec{0} \Rightarrow u(\vec{\phi}) > 0.$$

The coefficient a is temperature dependent and vanishes at the critical temperature.

We recall that the image of D is the set of matrices $\{\underline{D}(h), h \in \Gamma\}$. It is a subgroup of the orthogonal group $O(n)$. We denote by G its topological closure (while $\mathrm{Im}D$ is finite for a transition from crystal to crystal, it is not closed for a transition to an incommensurate phase). The Γ-invariant polynomials on $\mathscr{E}_n$ are determined only by G. We denote by $\mathscr{T}_m$ the vector space of degree in homogeneous polynomials on $\mathscr{E}_n$. Note that

$$\dim \mathscr{T}_m = \begin{bmatrix} m+n-1 \\ m \end{bmatrix}. \tag{4}$$

We denote by $\mathscr{T}_4^G$ the subset of G-invariant quartic polynomials

$$l = \dim \mathscr{T}_4^G = \frac{1}{4!} \left[\frac{d^4}{dt^4} \int_{g \in G} d\mu(g) \det(\mu - tg)^{-1} \right]\bigg|_{t=0}$$

$$= \frac{1}{4!} \int_{g \in G} d\mu(g) [(\mathrm{tr}g)^4 + 6(\mathrm{tr}g)^2 \mathrm{tr}g^2 + 8\,\mathrm{tr}g\,\mathrm{tr}g^3$$

$$+ 3(\mathrm{tr}g^2)^2 + 6\,\mathrm{tr}g^4] \tag{5}$$

(see, e.g., Ref. 9). When G is finite (this is a particular case of compact) the integral over the normalized Haar measure $\int_{g \in G} d\mu(g)$ is simply

$$\frac{1}{|G|} \sum_{g \in G},$$

where $|G|$ is the number of elements of G. So, in the Landau model u is generally taken as an expansion on a basis of $\mathscr{T}_4^G$, i.e.,

$$u = \sum_{\alpha=1}^{l} \lambda_\alpha u^{(\alpha)}.$$

Including the positivity condition, we conclude that the set of polynomials that satisfy (3) forms a convex cone $\mathscr{C}_+(\mathscr{T}_4^G)$ in the l-dimensional vector space $\mathscr{T}_4^G$. Moreover, $l \geq 1$, and $\mathscr{C}_+(\mathscr{T}_4^G)$ is not empty since it always contains

$$s(\phi) = (\vec{\phi} \cdot \vec{\phi})^2. \tag{6}$$

With the introduction of the symmetric operator on $\mathscr{E}_n$,

$$T_u(\phi) = \frac{1}{12} \frac{d^2}{d\vec{\phi}^2} u(\vec{\phi}) ,$$

and so

$$\vec{\phi} \cdot T_u \vec{\phi} = u(\vec{\phi}) \tag{7}$$

(it is represented by the $n \times n$ matrix $\partial^2 u / \partial \phi_i \partial \phi_j$ and it depends quadratically on $\vec{\phi}$), one writes the condition that $\vec{\phi}$ is an extremum of $P(\vec{\phi})$ as

$$\frac{dP}{d\vec{\phi}} \equiv \left[\frac{\alpha}{3} T_u(\vec{\phi}) - aI \right] \vec{\phi} = \vec{0} . \tag{8}$$

This extremum is a minimum when

$$\frac{d^2 P}{d\vec{\phi}^2} \equiv \alpha T_u(\vec{\phi}) - aI \geq 0 . \tag{9}$$

The isotropy group $\Gamma_{\vec{\phi}} = \{ \underline{D}(h)\vec{\phi} = \vec{\phi}, \ h \in \Gamma \}$ of the minimum $\vec{\phi}$ is the symmetry group of the corresponding phase. Its nature depends on the choice of u in $\mathscr{C}_+(\mathscr{T}_4^G)$. This cone is divided into a disjoint union of subdomains corresponding to a phase diagram for the different possibile phases with lower symmetry.

The vector $\vec{\phi} \in \mathscr{C}_n$ can be the value of an n-component field $\vec{\phi}$ defined on our $d = 3$ dimensional space, and the Landau model becomes a dynamical theory with the Hamiltonian density

$$\mathscr{H}(n) = \frac{1}{2} \sum_{\alpha=1}^{3} \frac{\partial}{\partial x^\alpha} \vec{\phi} \cdot \frac{\partial}{\partial x^\alpha} \vec{\phi} + P(\vec{\phi}) . \tag{10}$$

Introducing the scaling variable λ, the renormalization-group equations are of the form

$$\frac{1}{\lambda} \frac{d}{d\lambda} u(\lambda) = \beta(u(\lambda)) , \tag{11}$$

where β is an O(n)-covariant vector field on the vector space $\mathscr{T}_4$. The relevant Hamiltonian of the transition is (10), but with the quartic polynomial $u(\vec{\phi})$ in (2) replaced by an infrared stable fixed point $u^*(\vec{\phi})$ of the renormalization equation. These stable fixed points are the solutions of

$$\beta(u^*) = 0 , \tag{12a}$$

$$\left[\frac{d\beta(u)}{du} \right]_{u=u^*} \Bigg|_{\mathscr{T}_4^G} \geq 0 , \tag{12b}$$

where $|_{\mathscr{T}_4^G}$ means the restriction of the operator on its proper subspace. Indeed, if we consider a $u_0(\lambda,\phi) \in \mathscr{T}_4^G$, because of the O($n$) covariance of β, the trajectory of Eqs. (11) going through $u_0(\lambda,\phi)$ stays in $\mathscr{T}_4^G$. It will contain, in general, several fixed points. A solution u^* of (11) is physically acceptable if $u^*(\phi) \geq 0$ [as we will see, this is implied by (12)], and if u_0 is in the attractor basin of u^*, i.e., there are no other fixed points between u^* and u_0 on the trajectory of u_0. In addition, for every admissible stable fixed point one can compute the critical exponents $\eta(u^*)$ and $\nu(u^*)$ [see Eqs. (40) and (41)].

Our results are obtained from a study of some properties of the vector space $\mathscr{T}_4$ of quartic polynomials on $\mathscr{C}_n$. We refer to Ref. 10 for more details. Since a homogeneous quartic polynomial can be written

$$u(\phi) = \sum_{i,j,k,l=1}^{n} u_{ijkl} \phi_i \phi_j \phi_k \phi_l , \tag{13}$$

$\mathscr{T}_4$ is also the space of completely symmetrical rank-4 tensors on $\mathscr{C}_n$, whose components are the coefficients of $u(\vec{\phi})$. To help readers more accustomed to the latter point of view, we will write the next few equations under the two forms corresponding to the two meanings of the $\mathscr{T}_4$ space. We use Δ uniquely for the Laplacian on $\mathscr{C}_n$,

$$\Delta = \vec{\nabla} \cdot \vec{\nabla} \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial \phi_i^2} . \tag{14}$$

We define a natural O(n)-invariant scalar product on $\mathscr{T}_4$:

$$(u,v) = \sum_{i,j,k,l} u_{ijkl} v_{ijkl}$$
$$= 2^{-10} 3^{-2} [\Delta^4 uv - 6(\Delta^2 u)\Delta^2 v] - \frac{3}{4} \text{tr}[(\Delta T_u)(\Delta T_v)] . \tag{15}$$

We will use also the O(n) equivariant algebra on $\mathscr{T}_4$ (see Ref. 10)

$$u \vee v = \text{tr}(T_u T_v) \tag{16}$$

$$\Longleftrightarrow (u \vee v)_{ijkl} = \frac{1}{6} \sum_{pq} u_{ijpq} v_{pqkl} + u_{klpq} v_{pqij} + u_{ikpq} v_{pqjl} + u_{jlpq} v_{pqik} + u_{ilpq} v_{pqkj} + u_{kjpq} v_{pkil} . \tag{16'}$$

This nonassociative symmetric algebra is similar to algebras already used in physics, e.g., Gell-Mann d algebra for SU(3) (Ref. 11) and the Jordan algebra on the adjoint representation of U(n) (Ref. 12). Radicati and I have studied several similar examples,[13,14] and we have emphasized the role in physics of the idempotent of these algebras [see Eq. (39)]. For a fixed u, we can define a linear operator $\mathscr{D}_u \in \mathscr{L}(\mathscr{T}_4)$, the algebra of linear operators on $\mathscr{T}_4$, by

$$\mathscr{D}_u v = u \vee v , \tag{17}$$

and $\mathscr{D}_u$ depends linearly on u; in fact, (17) defines a O(n)-convariant linear map

$$\overset{\mathscr{D}}{\mathscr{T}_4 \to \mathscr{L}(\mathscr{T}_4)} . \tag{17'}$$

We make the assumption that the set of matrices $\text{Im}\Gamma = G$ is irreducible, i.e., actually G does not leave any (nontrivial) subspace of $\mathscr{C}_n$ invariant. If this were not the case, the Hamiltonian density (10) could be split into a direct sum of G-irreducible Hamiltonians and the problem

would be transformed into a set of similar problems with the smaller n. As a consequence of this hypothesis any quadratic form in $\vec{\phi}$ that can be made from the G-invariant quartic polynomial u is proportional to $\vec{\phi} \cdot \vec{\phi}$. For instance,

$$\frac{1}{12}\Delta u = \gamma(u)\vec{\phi} \cdot \vec{\phi} \iff \sum_i u_{iikl} = \gamma(u)\delta_{kl} , \qquad (18)$$

with the linear form $\gamma(\ )$ on $\mathscr{T}_4$ given by

$$\gamma(u) = \frac{1}{n}(s,u) = \frac{n}{24}\Delta^2 u . \qquad (19)$$

Similarly,

$$\frac{1}{108}\left[\frac{1}{12}\Delta^3 uv - \frac{1}{4}(\Delta u)\Delta^2 v - \frac{1}{4}(\Delta^2 u)\Delta v - (\vec{\nabla}\Delta u)\cdot\vec{\nabla}\Delta v\right] - \frac{3}{2}\mathrm{tr}(T_u\,\Delta T_v + T_v\,\Delta T_u) = \frac{1}{n}(u,v)\vec{\phi}\cdot\vec{\phi} \qquad (20)$$

$$\iff \sum u_{ipqr}v_{jpqr} = \frac{1}{n}(u,v)\delta_{ij} . \qquad (20')$$

In a more precise group-theoretical description we can say that $\mathscr{T}_4$ is the sum of three $O(n)$-irreducible spaces

$$\mathscr{T}_4 = \mathscr{T}_4^{(0)} + \mathscr{T}_4^{(2)} + \mathscr{T}_4^{(4)} \qquad (21)$$

of respective dimensions

$$\begin{bmatrix} n+3 \\ 4 \end{bmatrix} = 1 + \frac{(n+2)(n-1)}{2}$$

$$+ (n+6)(n+1)\frac{n(n-1)}{24} . \qquad (21')$$

This corresponding decomposition for the quartic polynomial u is

$$u = u^{(0)} + u^{(2)} + u^{(4)} ,$$

$$u^{(0)} = \frac{3\gamma(u)}{n+2}s, \qquad (21'')$$

$$u^{(2)} = \frac{1}{2(n+4)}\vec{\phi}\cdot\vec{\phi}q(\vec{\phi}) , \Delta q = 0 , \quad \Delta u^{(4)} = 0 .$$

The quadratic form $q(\vec{\phi})$ is $\Delta u - (12/n^2)\gamma(u)\vec{\phi}\cdot\vec{\phi}$.]

I wish to emphasize another nonrelated group-theoretical remark (independent from the irreducibility of G). Let us denote

$$\widetilde{G} = \mathrm{centralizer}(\mathscr{T}_4^G),$$

$$N_G = \mathrm{normalizer}(\mathscr{T}_4^G) , \qquad (22)$$

i.e., $\widetilde{G}$ is the larger subgroup of $O(n)$, which leaves fixed every element of $\mathscr{T}_4^G$ fixed, and N_G is the largest subgroup of $O(n)$, which transforms $\mathscr{T}_4^G$ into itself. Note that $G \leq \widetilde{G}$, but the equality may not hold. One easily proves that the centralizer is an invariant subgroup of the normalizer. We denote the quotient by

$$Q_G = N_G/\widetilde{G} . \qquad (23)$$

We denote by $[O(n)]_u$ the isotropy group (equal to little group) of u, i.e., the largest subgroup of $O(n)$ that leaves u invariant ($\widetilde{G} \leq [O(n)]_u, u \in \mathscr{T}_4^G$). Finally, it is interesting to consider $N(\widetilde{G})$, the largest $O(n)$ subgroup, which contains $\widetilde{G}$ as an invariant subgroup. Hence,

$$N_G \leq N(\widetilde{G}) . \qquad (24)$$

In Ref. 10 it is proved that the equality holds if and only

if there is a u such that $\widetilde{G} = [O(n)]_u$. For any case, and as was noted in Ref. 15, if one finds a (stable) fixed point u^* by the action of Q_G (since N_G acts on $\mathscr{T}_4^G$ only through this quotient), one can form a Q_G orbit of (stable) fixed points [this is due to the $O(n)$ covariance of β]. Let us illustrate these different remarks by examples. It is traditional to denote by c,

$$c = \sum_{i=1}^{n} \phi_i^4 , \qquad (25)$$

this invariant of the symmetry group B_n of the n-dimensional (hyper) cube. Up to conjugation by $O(n)$, for $n=2$ the only irreducible strict $O(2)$ subgroup is $G = B_2 = C_{4v} = \widetilde{G}$ and $l=2$; indeed any $u \in \mathscr{T}_4^{C_{4v}}$ is of the form $u = \alpha s + \beta c$. For $n=3$, while there are five possible conjugated classes of groups G, namely T, T_h, T_d, O, and O_h, they correspond to a unique $\widetilde{G} = O_h = B_3$ and again $l=2$ and $u = \alpha s + \beta c$. There are many more possible G for $O(n)$, but as shown in Ref. 16 there are only 21 conjugation classes of centralizers $\widetilde{G}$ and only 13 conjugation classes of isotropy groups $[O(4)]_u$. For $n=3$, $N_G = G$, but for $n=2$, $N_G = C_{8v}$, and so Q_G is a two-element group, which naturally leaves s invariant but changes the sign of the orthogonal (harmonic) polynomial

$$w = 4c - 3s = \phi_1^4 - 6\phi_1^2\phi_2^2 + \phi_2^4 . \qquad (26)$$

The renormalization equation cannot be computed exactly; Wilson[2] showed how to compute it as a power expansion in $\epsilon = 4 - d$, so

$$\beta = \sum_{K=0}^{\infty} \epsilon^K {}_K\beta , \qquad (27)$$

where the ${}_K\beta$'s are $O(n)$-covariant polynomial vector fields on $\mathscr{T}_4$. It is convenient to write explicitly the ${}_K\beta$ as a sum of homogeneous polynomials of degree M,

$${}_K\beta = \sum_M {}_K\beta_M , \quad {}_K\beta_M \in \mathscr{T}_M(\mathscr{T}_4) \qquad (28)$$

where $\mathscr{T}_M(\mathscr{T}_4)$ is the vector space of homogeneous polynomials of degree M on $\mathscr{T}_4$. The fixed points defined by (12a) are also functions of ϵ, and physically they must go to zero with ϵ, so they are of the form

$$u^*(\epsilon) = \epsilon^\alpha \widetilde{u}(\epsilon) , \quad \alpha > 0 . \qquad (29)$$

Here, as we will see below, we can take $\alpha = 1$; then the fixed-point equation (12a) becomes

$$\sum_{K,M} \epsilon^{K+M} {}_K\beta_M(\widetilde{u})=0 \Longleftrightarrow \epsilon^{L_0}\sum_{L=0}^{\infty}\epsilon^L B^{(L)}(\widetilde{u})=0 , \quad (30)$$

where we have grouped together the terms with the same power in ϵ, factorized the lowest ϵ power, and used the short notation

$$\beta^{(0)}(\widetilde{u}_0)=0 , \tag{32a}$$

$$\left[\frac{d\beta^{(0)}(u)}{du}\right]_{u=\widetilde{u}_0}\widetilde{u}_1+\beta^{(1)}(\widetilde{u}_0)=0 , \tag{32b}$$

$$\left[\frac{d\beta^{(0)}(u)}{du}\right]_{u=\widetilde{u}_0}\widetilde{u}_2+\left[\frac{d}{du'}\left\{\left[\frac{d\beta^{(0)}(u)}{du}\right]_{u=\widetilde{u}_0}u'\right\}\right]_{u'=\widetilde{u}_1}\widetilde{u}_1+\left[\frac{d\beta^{(1)}(u)}{du}\right]_{u=\widetilde{u}_0}\widetilde{u}_1+\beta^{(0)}(\widetilde{u}_2)=0 , \tag{32c}$$

$$\cdots$$

$$\left[\frac{d\beta^0(u)}{du}\right]_{u=\widetilde{u}_0}\widetilde{u}_S+p(\widetilde{u}_0,\widetilde{u}_1,\ldots,\widetilde{u}_{S-1})=0 . \tag{32d}$$

We recall that

$$\frac{d\beta(u)}{du}v=\lim_{\theta\to 0}\theta^{-1}[\beta(u+\theta v)-\beta(u)] .$$

We have already pointed out that $\widetilde{u}\in\mathscr{T}_4^G$. Therefore for any solution of the nonlinear equation (32a) such that the operator [(restriction of $d\beta^{(0)}/du(\widetilde{u}_0)$ to the eigenspace $\mathscr{T}_4^G)]$

$$\left[\frac{d\beta^{(0)}(u)}{du}\right]_{u=\widetilde{u}_0}\bigg|_{\mathscr{T}_4^G} \text{ is inversible} \tag{33}$$

we can solve the linear equation system (32b) and successively all equations (32) up to the required order in the ϵ expansion (31). The validity and the convergence of the ϵ expansion (31) are a difficult and unsolved problem. We assume here that (31) makes sense and defines $\widetilde{u}$ as an analytic function in ϵ. If the invertible operator (33) is positive, then for some range in ϵ, Eq. (12b) is satisfied and the corresponding fixed point is stable. Hence with the optimist assumptions on the ϵ expansion we have obtained the following result

Theorem 1. Solutions of Eq. (32a) that satisfy (33) yield expansion (31) by solving successively the linear systems of equations (32b)–(32d). These expansions are fixed-point solutions of the renormalization-group equation (12a). We will call them *regular* fixed points. For them, the stability condition (12b) reduces to

$$\left[\frac{d\beta^{(0)}(u)}{du}\right]_{u=\widetilde{u}_0}\bigg|_{\mathscr{T}_4^G}>0 . \tag{34}$$

When condition (33) is not satisfied, one must apply the bifurcation theory (see, e.g., Ref. 17 for a recent review); this will not be done here.

The ϵ expansion of the renormalization equation has been computed for the isotropic $(u=s)$ and cubic $(u=c)$

$$\beta^{(L)}=\sum_{K+M=L+L_0}{}_K\beta_M . \tag{30'}$$

Assuming a power-series expansion,

$$\widetilde{u}=\sum_{S=0}^{\infty}\epsilon^S\widetilde{u}_S . \tag{31}$$

The equation for a fixed point $u^*=\epsilon\widetilde{u}$ reduces to a system of equations

symmetry in Refs. 3 and 4, respectively. For the case of a general quartic polynomial u with irreducible isotropy group $[O(n)]_u$ the computation of β up to second order in ϵ [i.e., $\beta^{(0)},\beta^{(1)},\beta^{(2)}$] has been made by Brézin, Le Guillou, and Zinn-Justin.[5] As was noted by Wallace and Zia,[18] up to this order, β is a gradient. The notations introduced here and Eq. (20) (not used in Ref. 5) yield the following expressions for β and the critical exponents η and ν computed in Ref. 5:

$$\beta=\frac{d\psi}{du} ,$$

$$\psi=-\tfrac{1}{2}\epsilon(u,u)+\tfrac{1}{2}(1+\tfrac{1}{2}\epsilon)(u\vee u,u)-\tfrac{3}{8}(1-\tfrac{1}{2}\epsilon)\Xi(u)$$
$$+\frac{1}{48n}(1+\tfrac{5}{4}\epsilon)(u,u)^2 , \tag{35}$$

where

$$\Xi(u)=\sum_{i,j,k,l,p,q,r,s}u_{ijpq}u_{pqkl}u_{ikrs}u_{rsjl} , \tag{36}$$

$$\eta=\frac{1}{24n}(1+\tfrac{5}{4}\epsilon)(u,u)-\tfrac{1}{32}(u\vee u,u) , \tag{37}$$

$$\frac{1}{\nu}-2=-\tfrac{1}{2}(1+\tfrac{1}{2}\epsilon)\gamma(u)+\tfrac{5}{24}(1-\epsilon)(u,u) . \tag{38}$$

Since the elements of $\mathscr{T}_4$ are rank-4 tensors on $\mathscr{E}_n$, by contraction of indices it is simple to count the number of linearly independent homogeneous polynomials in $\mathscr{T}_m(\mathscr{T}_4)$ that one can form from a given $u\in\mathscr{T}_4$, which are $O(n)$ invariants or components of $O(n)$-covariant vector field on $\mathscr{T}_4$. Using S_m as the number of $O(n)$ invariants and V_m as the number of $O(n)$-covariant vector fields, we find the following for $m\leq 5$:

m	1	2	3	4	5
S_m	0	1	1	4	6
V_m	1	1	4	12	

Since for $m = 1,2,3$ $V_m = S_{m+1}$, this proves the following.

Lemma 1. Any $O(n)$-covariant vector field on $\mathscr{T}_4$ that is an inhomogeneous polynomial of degree ≤ 3 is a gradient vector field.

The remark made in Ref. 18 that β is a gradient was trivial, since Ref. 5 gives β as a third-degree inhomogeneous polynomial, but the conclusion was interesting (e.g., absence of strange attractors). However, we will give stronger results here. It is not known if higher-order terms in the ϵ expansion of β are still gradients, which is not likely, but this is irrelevant for the application of Theorem 1.

The application of the procedure defined in (30) and (31) to Eqs. (34) and (35) yields $L_0 = 2$ and

$$\left[\frac{d\psi^{(0)}(u)}{du} \right]_{u=\tilde{u}_0} \equiv \beta^{(0)}(\tilde{u}_0)$$

$$\equiv -\tilde{u}_0 + \tfrac{3}{2}\tilde{u}_0 \vee \tilde{u}_0 = 0 . \tag{39}$$

We obtain for the lowest order in ϵ of the critical coefficient

$$\eta(\epsilon \tilde{u}_0) = \frac{\epsilon^2}{24n}(u_0, u_0) , \tag{40}$$

$$\nu(\epsilon \tilde{u}_0)^{-1} - 2 = -\tfrac{1}{2}\epsilon\gamma(\tilde{u}_0) , \tag{41}$$

$$\left[\frac{d^2\psi^{(0)}(u)}{du^2} \right]_{u=\tilde{u}_0} \equiv \left[\frac{d\beta^{(0)}(u)}{du} \right]_{u=\tilde{u}_0}$$

$$= 3\mathscr{D}_{\tilde{u}_0} - I \tag{42}$$

[where $\mathscr{D}_{\tilde{u}_0}$ has been defined in (17)].

Theorem 1 and (39) show that the regular fixed points $\epsilon\tilde{u}_0$ are given by idempotents of the $O(n)$-covariant algebra $\vee$ defined in (16). Since (39) is a second-degree equation, either the set of regular fixed points is an algebraic manifold [e.g., orbits of Q_G defined in (23)] or it is a discrete set of at most 2^l points.

The number of discrete regular fixed points is $\leq 2^l$,

$$l = \dim\mathscr{T}_4^G . \tag{43}$$

One fixed point is proportional to s; it is given by $\epsilon\tilde{s}_0$ with

$$\tilde{s}_0 = \frac{6}{n+8}s . \tag{44}$$

For $u \in \mathscr{T}_4^{(0)} \oplus \mathscr{T}_4^{(4)}$, one obtains (see, e.g., Ref. 10)

$$s \vee u = \tfrac{1}{3}[\gamma(u)s + 2u] . \tag{45}$$

This shows that $\mathscr{T}_4^{(0)}$, $\mathscr{T}_4^{(4)}$, and therefore $\mathscr{T}_4^{(2)}$ are eigenspaces of $\mathscr{D}_s$; it also allows computation of the corresponding eigenvalue:

$$\left[\frac{d\beta^{(0)}(u)}{du} \right]_{u=\tilde{s}_0} \Bigg|_{\mathscr{T}_4^{(0)}} = I \tag{46a}$$

$$\left[\frac{d\beta^{(0)}(u)}{du} \right]_{u=\tilde{s}_0} \Bigg|_{\mathscr{T}_4^{(4)}} = \frac{4-n}{n+8}I . \tag{46b}$$

Equation (46a) shows that when $l = \dim\mathscr{T}_4^G = 1$ [e.g., $G = O(n)$] $\tilde{s}$ given by (44) is the only fixed point and is stable for all n. Equation (46b) shows that $\tilde{s}$ is a regular fixed point for $n \neq 4$. Both equations show that $\tilde{s}$ is a stable fixed point for $n < 4$.

Regular stable fixed points correspond to minima of the polynomial

$$\psi^0(\tilde{u}) |_{\mathscr{T}_4^G} \tag{47}$$

with

$$\psi^0 = \tfrac{1}{2}[(\tilde{u} \vee \tilde{u}, \tilde{u}) - (\tilde{u}, \tilde{u})] .$$

At an extremum $\tilde{u}_0$ [solution of (39)] the value of this polynomial is

$$\psi^0(\tilde{u}_0) = -\tfrac{1}{6}(\tilde{u}_0, \tilde{u}_0) . \tag{48}$$

We can now prove the following.

Theorem 2. If it exists, the *stable* regular fixed point is unique. *Proof:* Consider two extrema of ψ^0, labeled $\tilde{u}_0$ and $\tilde{v}_0$. We study the restriction of ψ^0 on the straight line defined by $\tilde{u}_0$ and $\tilde{v}_0$:

$$\psi(\lambda) = \psi^0[(1-\lambda)\tilde{u}_0 + \lambda\tilde{v}_0]$$

$$= \tfrac{1}{6}\{[(\tilde{v}_0, \tilde{v}_0) - (\tilde{u}_0, \tilde{u}_0)]\lambda^2(2\lambda - 3) - (\tilde{u}_0, \tilde{u}_0)\} . \tag{49}$$

When $\tilde{u}_0$ and $\tilde{v}_0$ have different length, $\psi(\lambda)$ has two extrema—one for $\lambda = 0$ (point $\tilde{u}_0$) and one for $\lambda = 1$ (point $\tilde{v}_0$). If the polynomial ψ^0 has a higher value at the shortest extremum, then $\psi(\lambda)$ is concave and this extremum is unstable. Therefore the longest extrema are the only candidates for regular stable fixed points. Assume that we have two of them $\tilde{u}_0$ and $\tilde{v}_0$ of the same length $(\tilde{u}_0, \tilde{u}_0) = (\tilde{v}_0, \tilde{v}_0) = C$; then $\psi(\lambda) = -C/6$. The two Hessians

$$\left[\frac{d^2\psi^0(u)}{du^2} \right]_{u=\tilde{u}} = \left[\frac{d\beta^{(0)}(u)}{du} \right]_{u=\tilde{u}}$$

at $\tilde{u}_0$ and $\tilde{v}_0$ have zero expectation value on the vector $\tilde{u}_0 - \tilde{v}_0$. But this vector $\tilde{u}_0 - \tilde{v}_0$ is not an eigenvector with zero eigenvalue of either Hessian; indeed,

$$\left[\frac{d\beta^{(0)}(u)}{du} \right]_{u=\tilde{u}_0} (\tilde{u}_0 - \tilde{v}_0) = \tilde{u}_0 + \tilde{v}_0 + 3\tilde{u}_0 \vee \tilde{v}_0$$

$$= \left[\frac{d\beta^{(0)}(u)}{du} \right]_{u=\tilde{v}_0} (\tilde{u}_0 - \tilde{v}_0) . \tag{50}$$

If we assume this vector vanishes and if we take its scalar product with $\tilde{u}_0$ and $\tilde{v}_0$, we obtain $(\tilde{u}_0, \tilde{u}_0) = (\tilde{v}_0, \tilde{v}_0) = (\tilde{u}_0, \tilde{v}_0)$, which is impossible since $\tilde{u}_0 \neq \tilde{v}_0$. It is easy to prove that if a symmetric operator, e.g.,

$$\left[\frac{d^2\psi^{(0)}(u)}{d\widetilde{u}^2}\right]_{\widetilde{u}=\widetilde{u}_0},$$

has zero expectation value for a vector, which is not an eigenvector, then it has both positive and negative eigenvalues. This remark shows that $\widetilde{u}_0$ and $\widetilde{v}_0$ are not stable [condition (33) is not satisfied]. This concludes the proof of the theorem. Note that the isotropy group of the stable point is $\geq N_G$, the normalizer of $\mathscr{T}_4^G$.

Is this stable fixed point physically acceptable? We note incidentally that for every fixed point $\widetilde{u}_0 = \frac{3}{2}\widetilde{u}_{0\text{v}}\widetilde{u}_0 = \frac{3}{2}\text{tr}T_{u_0}^2 > 0$ when $\phi \neq \vec{0}$, since it is the trace of a positive matrix (here the square of a real symmetric matrix).

The renormalization-group trajectories are lines of greatest slope [orthogonal to the level lines $\psi^0(u) = \text{const}$]. Those trajectories passing through a local maximum and a saddle point belong to the boundary of an attractor basin. The origin O is the only local maximum of ψ^0, hence the corollary.

Corollary. For the nonstable fixed points $\epsilon\widetilde{u}_0 \neq 0$, the half rays $\{\lambda\widetilde{u}_0\}$, $\lambda > 0$ belong to the boundary of the attractor basin (up to lowest order in ϵ). This boundary is a convex cone of vertex O.

For $n \leq 3$, the isotropic fixed points are stable (46), so it is the only stable point for $n \leq 3$. This was proven in Ref. 5. Since the stable point has greatest length, Eq. (40) shows that it has the largest critical exponent η; this proves the intriguing conjecture of Ref. 5.

Given two fixed points $\widetilde{u}_0, \widetilde{v}_0$ from Eq. (50) we obtain

$$\left[\widetilde{u}_0 - \widetilde{v}_0, \left[\frac{d^2\psi^0(u)}{du^2}\right]_{u=\widetilde{u}_0}(\widetilde{u}_0 - \widetilde{v}_0)\right] = (\widetilde{u}_0, \widetilde{u}_0) - (\widetilde{v}_0, \widetilde{v}_0).$$

This verifies again that at the longest extrema $\widetilde{u}_0$ (which is unique) the expectation value of the Hessian

$$\left[\frac{d^2\psi^0(u)}{du^2}\right]_{u=\widetilde{u}_0}$$

is positive in the (at most 2^l) direction $\widetilde{u}_0 - \widetilde{v}_0$ of the l-dimensional space $\mathscr{T}_4^G$. This does not ensure that

$$\left[\frac{d^2\psi^0(u)}{du^2}\right]_{u=\widetilde{u}_0} > 0$$

and that $\widetilde{u}_0$ is a local minimum. For years no stable fixed points were known when $l \geq 3$. This led Dzyaloshinskii[19] to acquire the conviction that this was due to a general consequence of some kind of topological property of the renormalization group. Counterexamples to this conjecture have been given independently by Grinstein and Mukamel[20] and by the author[10] for arbitrary large n (for phase transitions in crystals n divides 48 and for $n = 48$, $l = 6$ in the given counterexamples).

Finally we give a proof of some results established in Ref. 5 for regular fixed points. The relation $(u_\text{v}v, w) = (u, v_\text{v}w)$ applied to the triple $(s, \widetilde{u}_0, \widetilde{u}_0)$ yields

$$\frac{2}{n}(\widetilde{u}_0, \widetilde{u}_0) = \gamma(\widetilde{u}_0)[2 - \gamma(\widetilde{u}_0)]. \tag{51}$$

If $\widetilde{u}_0 \neq \widetilde{s}$,

$$(\widetilde{u}_0^{(4)}, \widetilde{u}_0^{(4)}) > 0 \Longleftrightarrow 0 < \gamma(\widetilde{u}_0) < \gamma(\widetilde{s}) = \frac{2(n+2)}{n+8}. \tag{52}$$

Equation (51) gives a relation between the lowest approximation of the critical exponents:

$$12\eta(\epsilon\widetilde{u}_0) = [2 - \nu(\epsilon\widetilde{u}_0)^{-1}][\epsilon - 2 + \nu(\epsilon\widetilde{u}_0)^{-1}]. \tag{53}$$

Equations (52) and (41) show that $\epsilon\widetilde{s}_0$ is the fixed point with the lowest value of ν^{-1}.

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