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# Microscopic approach to critical behavior in $^3$He-$^4$He mixtures. Effective Hamiltonian and instability

K. K. Singh and Partha Goswami
*Department of Physics, University of Delhi, Delhi 110007, India*
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A system of weakly interacting bosons and fermions is used as a model to develop a theory of critical behavior of $^3$He-$^4$He mixtures. The fermion amplitudes and the short-wavelength boson amplitudes are eliminated from the problem within the framework of perturbation theory. The resulting effective boson Hamiltonian possesses interesting features. It implies an instability of the mixture and of its $\lambda$ line when certain conditions are fulfilled. The coefficients associated with the quartic and six-operator terms of the effective Hamiltonian have properties characteristic of the classical Landau model often used to discuss tricritical behavior. The condition of stability derived for the mixture agrees with an earlier result of Cohen and Leeuwen for the degenerate phase but is in disagreement with their results for the nondegenerate phase.

## I. INTRODUCTION

The renormalization-group (RG) approach[1] enables one to discuss not only simple critical behavior associated with continuous second-order phase transitions but also multicritical behavior such as that exhibited by systems possessing tricritical points. The tricritical behavior of $^3$He-$^4$He mixtures, in particular, has been discussed by Riedel and Wegner[2] on the basis of a phenomenological single-component classical spin Hamiltonian which may be considered to be a generalization of the expansion of free energy originally introduced by Landau.[3] Justification for using a one-component classical field to describe critical behavior in a two-component quantum system is, however, lacking. In two earlier papers,[4] one of us has shown how critical behavior in a pure Bose system may be studied by performing RG transformations directly on the quantum-mechanical Hamiltonian. The aim of the present work is to develop a theory of the critical behavior of $^3$He-$^4$He mixtures from a microscopic basis.

A system of interacting fermions and bosons is used as a model of $^3$He-$^4$He mixtures. The calculations are performed on the assumption that the interactions among the particles are weak.

It may be pointed out that several years ago Cohen and Leeuwen[5] worked out the phase diagram of a dilute fermion-boson hard-sphere mixture following the methods of Huang and Yang[6] and Lee and Yang.[6] The results obtained by them were physically very interesting but were rendered imperfect by the lack of consistency of the calculations. For example, while they calculated the free energy of the mixture to the first order in the interaction parameters, the condition of stability derived from it involved terms quadratic in the interaction parameters in an essential manner. Later attempts[7] to obtain a Landau-type expansion for the mixture also proved unsuccessful. The need is evident for an approach to the problem that may overcome some of these difficulties.

We aim at developing a comprehensive theory of the fermion-boson mixture which on the one hand will provide a starting point for applying the RG method, and on the other, will lead to a Landau-type description of the phase transitions involved when suitable approximations are introduced. The underlying assumption is that, as in the case of a pure Bose system, the order parameter for the mixture contained in a volume $V$ is the average value of $(b_0/\sqrt{V})$ where $b_0$ is the annihilation operator for bosons in the zero-momentum single-particle state. As in the RG approach, the part of the Hamiltonian of the mixture containing small momentum boson operators is considered to be of special importance. The fermion field amplitudes as well as the short-wavelength boson amplitudes are treated as auxiliary quantities and are eliminated from the problem by taking a partial trace of the density matrix. This results in an effective low-momentum boson Hamiltonian which can be studied in a variety of ways using methods developed for a pure Bose system.[8-11]

For reasons of length and convenience in presentation, the work has been divided into parts. This paper reports the first part of the work. A brief outline of its contents is as follows: After formulating the problem in Sec. II, we carry out in Sec. III a partial trace of the density matrix within the framework of perturbation theory in order to eliminate the fermion field as well as the short-wavelength boson amplitudes. The system is consequently describable in terms of an effective boson chemical potential $\mu_4'$ and an effective boson-boson interaction $u_4'$, both the quantities being functions of the fermion chemical potential and the temperature $T$. Using the well-known result[9,10] that a pure Bose system becomes unstable if the interaction strength becomes negative, we obtain a condition for the stability of the mixture which, somewhat unexpectedly, turns out to be the same as derived by Cohen and Leeuwen[5] for the degenerate phase from their first-order free energy. Since the effective boson interaction contributes a term $(u_4'M^4)$ to the Landau expansion (cf. Sec. V), where $M$ denotes the order parameter, the failure of earlier attempts that used the bare vertex $u_4$ to obtain a Landau-type theory is clarified.

When the effective interaction $u'_4$ becomes zero or negative, the partial trace calculation of Sec. III must be extended to determine the next nonzero term in the effective boson Hamiltonian. The calculation of this term, involving six boson operators, forms the content of Sec. IV. The associated coupling parameter, denoted by $u_6$, is found to be positive. The effective boson Hamiltonian thus possesses all the features of the phenomenological model used by Riedel and Wegner[2] except that the field involved is a quantum field rather than a classical field. Our experience in applying the RG method to a Bose system,[4] however, warrants the expectation that quantum effects may not be important in regard to critical behavior exhibited by the effective Hamiltonian for the mixture.

Section V contains a discussion of a few important points of the paper. Although detailed calculations are planned for later work, we show that, in disagreement with the results of Cohen and Leeuwen, we get the same condition for the stability of the $\lambda$ line whether we approach it from the degenerate phase or the nondegenerate phase.

## II. FORMULATION

The system under consideration is a mixture of spinless bosons of mass $m_4/2$ and fermions of mass $m_3/2$ and spin $\hbar/2$ contained in a box of volume $V=L^3$. The interactions between the particles are assumed to be of the following type: a short-range boson-boson interaction of strength $2u_4$, a short-range fermion-fermion interaction of strength $2u_3$, and a boson-fermion interaction whose Fourier transform is $u_{34}(p)$. The range of $u_{34}$ will be specified later. Using periodic boundary conditions and choosing units such that $\hbar=1$, we find the Hamiltonian of the system in the second quantized formalism can be written in the form

$$H=\sum_q \frac{q^2}{m_4} b_q^\dagger b_q+\sum_{k,\sigma}\frac{k^2}{m_3}a_{k\sigma}^\dagger a_{k\sigma}+\frac{u_4}{V}\sum_{q_1,q_2,q_3,q_4} b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3} b_{q_4}\delta(q_1+q_2-q_3-q_4)$$
$$+\frac{u_3}{2V}\sum_{\substack{k_1,k_2,k_3,k_4\\ \sigma_1,\sigma_2,\sigma_3,\sigma_4}}(\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3}-\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4})a_{k_1\sigma_1}^\dagger a_{k_2\sigma_2}^\dagger a_{k_3\sigma_3}a_{k_4\sigma_4}\delta(k_1+k_2-k_3-k_4)$$
$$+\frac{1}{V}\sum_{k,q,p,\sigma}u_{34}(p)b_q^\dagger b_{q-p}a_{k\sigma}^\dagger a_{k+p,\sigma}+H_s\ , \tag{1}$$

$$H_s=-\frac{h}{2}V\left[\frac{b_0}{\sqrt{V}}+\frac{b_0^\dagger}{\sqrt{V}}\right]. \tag{2}$$

In (1), $b_q$ denotes boson annihilation operator for the single-particle state of momentum $q$, $a_{k\sigma}$, with $\sigma=\pm1$, the fermion annihilation operator for the single-particle momentum-spin state $(k,\sigma)$, and $\delta$ stands for the Kronecker symbol. Following Bogolubov[12] we have introduced a symmetry breaking term $H_s$, $h$ denoting the field conjugate to the real part of the order parameter $(b_0/\sqrt{V})$.

The number operators for the bosons and fermions are (with subscripts 3 and 4 for fermions and bosons, respectively)

$$N_4=\sum_q b_q^\dagger b_q\ , \tag{3}$$

$$N_3=\sum_{k\sigma}a_{k\sigma}^\dagger a_{k\sigma}\ . \tag{4}$$

The quantity of primary interest is the grand partition function $\Xi$ of the mixture, defined by

$$\Xi=\mathrm{Tr}\exp(-\beta H_0)\ , \tag{5}$$

$$H_0=H-\mu_3N_3-\mu_4N_4\ . \tag{6}$$

Here $\mu_3,\mu_4$ denote the partial chemicals, and $\beta$ is the inverse of the product of the Boltzman constant $k_B$ and the absolute temperature $T$.

The thermodynamic potential per unit volume, denoted by $\Omega$ is given by

$$\Omega=-P=-(\beta V)^{-1}\ln\Xi(T,\mu_3,\mu_4,h)\ , \tag{7}$$

where $P$ denotes the pressure of the mixture. The mean number densities $(n_3,n_4)$ of fermions and bosons can be calculated according to

$$n_3=-\frac{\partial\Omega}{\partial\mu_3}\ , \tag{8}$$

$$n_4=-\frac{\partial\Omega}{\partial\mu_4}\ . \tag{9}$$

The thermodynamic average $M$ of the real part of the order parameter is given by

$$M=-\frac{\partial\Omega}{\partial h}\ . \tag{10}$$

We note that while $\Omega$ is a thermodynamic potential of variables $(T,\mu_3,\mu_4,h)$, $\Omega'$ defined by

$$\Omega'(M)=\Omega+hM \tag{11}$$

can be considered as a thermodynamic potential of the variables $(T,\mu_3,\mu_4,M)$ or $(T,\mu_4,\Delta,M)$ where

$$\Delta=\mu_3-\mu_4 \tag{12}$$

is the variable conjugate to the fermion concentration $x$. One may, therefore, talk of the possibility of an expansion of $\Omega'$ in powers of $M$. Another potential function which involves $M$ is the free energy per unit volume $f'$ defined by

$$f'(T,n_3,n_4,M)=\Omega'+n_3\mu_3+n_4\mu_4 \; . \tag{13}$$

If one chooses $(T,P,x)$ or $(T,P,\Delta)$ as independent variables, as is usually done in the discussions of thermodynamics of $^3$He-$^4$He mixtures, thermodynamic potential functions having $(T,P,x,M)$ or $(T,P,\Delta,M)$ as variables cannot be constructed. As a matter of fact, it is easy to check that $\mu_4'$ defined by

$$\mu_4'=\mu_4+hm \; , \tag{14}$$

$$m=M/n \; , \tag{15}$$

$$n=n_3+n_4 \; , \tag{16}$$

is a thermodynamic potential of the variables $(T,P,\Delta,m)$. When these four quantities are chosen as independent variables,

$$M=n(T,P,\Delta,m)m \tag{17}$$

is in general some complicated function of $m$. These remarks have a bearing on the choice of the thermodynamic potentials appropriate for a Landau-type expansion. If the correct order parameter, as we have assumed, is $M$, proper candidates for an expansion in powers of $M$ are $\Omega'\ (T,\mu_3,\mu_4,M)$ and $f'\ (T,n_3,n_4,M)$.

## III. ELIMINATION OF FERMION AND SHORT-WAVELENGTH BOSON AMPLITUDES

In this section we shall try to eliminate approximately the fermion amplitudes $a_{k\sigma}$ and the boson amplitudes $b_p$ with $|p|>p_c$, where $p_c$ is small compared to the boson thermal momentum $\lambda_B^{-1}=(m_4/4\pi\beta)^{1/2}$. Returning to Eq. (6), we split up $H_0$ as follows:

$$H_0=H_F^{(3)}+H_F^{(4)}(|p|>p_c)+H_F^{(4)}(|q|<p_c)+U_I \; , \tag{18}$$

$$H_F^{(3)}=\sum_{k\sigma}(k^2/m_3-\mu_3)a_{k\sigma}^\dagger a_{k\sigma} \; , \tag{19}$$

$$H_F^{(4)}(|p|>p_c)=\sum_{|p|>p_c}\left[\frac{p^2}{m_4}-\mu_4\right]b_p^\dagger b_p \; , \tag{20}$$

$$H_F^{(4)}(|q|<p_c)=\sum_{|q|<p_c}\left[\frac{q^2}{m_4}-\mu_4\right]b_q^\dagger b_q \; , \tag{21}$$

$$U_1=U_4+U_3+U_{34} \; . \tag{22}$$

$U_4$, $U_3$, and $U_{34}$ denote, respectively, the third, fourth and fifth terms on the right-hand side of (1). In (18) we have suppressed the source term since it plays no role in the elimination of momenta $|p|>p_c$. In what follows, boson momenta less than $p_c$ will be denoted by $q$'s and boson momenta greater than $p_c$ by $p$'s. The term $U_4$ in (22) can be split as

$$U_4=U_4(p)+U_4(q)+U_4(p,q) \; , \tag{23}$$

where

$$U_4(q)=\frac{u_4}{V}\sum_{q_1,q_2,q_3,q_4}b_{q_1}^\dagger b_{q_2}^\dagger b_{q_3}b_{q_4}\delta(q_1+q_2-q_3-q_4) \; . \tag{24}$$

$U_4(p)$ is obtained from $U_4(q)$ by replacing each $q$ summation by a $p$ summation, and $U_4(p,q)$ is a term similar to $U(q)$ except for the fact that, in any particular term of the summation, the four momenta can neither be all $q$'s, nor all $p$'s.

Similarly, the term $U_{34}$ in (22) can be broken up as

$$U_{34}=\sum_{q_1,q_2;k,\sigma}\frac{u_{34}(q_1-q_2)}{V}b_{q_1}^\dagger b_{q_2}a_{k\sigma}^\dagger a_{k+q_1-q_2,\sigma} + \sum_{p_1,p_2;k,\sigma}\frac{u_{34}(p_1-p_2)}{V}b_{p_1}^\dagger b_{p_2}a_{k\sigma}^\dagger a_{k+p_1-p_2,\sigma} + \sum_{\substack{q,p;\\k,\sigma}}\left[\frac{u_{34}(q-p)}{V}b_q^\dagger b_p a_{k\sigma}^\dagger a_{k+q-p,\sigma}+\text{c.c.}\right] \; . \tag{25}$$

For the sake of simplicity, we assume

$$u_{34}(q)=\begin{cases}u_{34}, & |q|<p_c\\ 0, & |q|>p_c \; .\end{cases} \tag{26}$$

The last term in (25) exists only in a narrow range $(|q|,|p|)\sim p_c$ and will be ignored.

Treating $H_F^{(3)}+H_F^{(4)}(p)+H_F^{(4)}(q)$ in (18) as the unperturbed Hamiltonian, and $U_I$ as a perturbation, the partition function can be expanded in powers of $U_I$ in the usual manner.[13] The Hilbert space of the system can be written as the direct product

$$h_3\otimes h_4(p)\otimes h_4(q) \tag{27}$$

where $h_3$ denotes the Hilbert space on which the fermion operators act, $h_4(p)$ is the space on which the boson operators $b_p$ act, and $h_4(q)$ is the space on which the boson operators $b_q$ act. The partition function consequently can be written in the form

$$\Xi=\underset{\{h_4(q)\}}{\text{Tr}}\exp[-\beta H_F^{(4)}(q)]\Xi_F\left[1+\sum_{n=1}^{\infty}(-1)^n\int_0^\beta\cdots\int_0^{\tau_{n-1}}d\tau_1\cdots d\tau_n\langle U_I(\tau_1)\cdots U_I(\tau_n)\rangle\right] \; , \tag{28}$$

$$U_I(\tau)=\exp\{\tau[H_F^{(3)}+H_F^{(4)}(p)+H_F^{(4)}(q)]\}U_I\exp\{-\tau[H_F^{(3)}+H_F^{(4)}(p)+H_F^{(4)}(q)]\} \; . \tag{29}$$

$$\Xi_F=\prod_{k,\sigma}\left\{1+\exp\left[-\beta\left[\frac{k^2}{m_3}-\mu_3\right]\right]\right\}\prod_{|p|>p_c}\{1-\exp[-\beta(p^2/m_4-\mu_4)]\}\ , \tag{30}$$

and angular brackets $\langle\ \rangle$ denote thermodynamic average calculated with $H_F^{(3)}+H_F^{(4)}(p)$ over the space $h_3\otimes h_4(p)$. The elementary vertices appearing in $U_I(\tau)$ are represented graphically in Fig. 1. The dashed lines represent low-momentum ($|q|<p_c$) boson operators, the solid lines high-momentum ($|p|>p_c$) boson operators, and the double lines fermion operators.

Our aim is to calculate the term in the large parentheses in (28) in a well-defined approximation scheme, and to combine it with $H_F^{(4)}(q)$ to obtain an effective low-momentum boson Hamiltonian. The approximation scheme will unfold as we proceed.

The graphs corresponding to the first-order term in (28) are depicted in Fig. 2. Their contribution can be collected in the form

$$F_1=-\int_0^\beta d\tau[U_4(q,\tau)+g_1(q,\tau)]\ , \tag{31}$$

$$g_1(q,\tau)=V(2u_4n'^2_4+u_{34}n_3^Fn'_4+\tfrac{1}{2}u_3n_3^{F2})+\sum_q(u_{34}n_3^F+4u_4n'_4)b_q^\dagger(\tau)b_q(\tau)\ , \tag{32}$$

where

$$n_3^F=V^{-1}\sum_k 2\{\exp[\beta\epsilon(k)]+1\}^{-1}\ , \tag{33}$$

$$n'_4=V^{-1}\sum_{|p|>p_c}\{\exp[\beta\epsilon(p)]-1\}^{-1}\ , \tag{34}$$

$$\epsilon(k)=k^2/m_3-\mu_3\ , \tag{35}$$

$$\epsilon(p)=p^2/m_4-\mu_4\ . \tag{36}$$

It is not difficult to see that the second-order term in (28) can be written as

$$\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2[U_4(q,\tau_1)+g_1(q,\tau_1)][U_4(q,\tau_2)+g_1(q,\tau_2)]+\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2\langle U_I(\tau_1)U_I(\tau_2)\rangle_c \tag{37}$$

where the suffix $c$ stands for connected graphs.

We consider first the second-order graphs which give contributions to the four-point vertex $U_4(q,\tau)$. These graphs are shown in Fig. 3. The contribution of Fig. 3(a) can be written as

$$\mathscr{C}_{3(\mathrm{a})}=-\int_0^\beta d\tau\, V^{-1}\sum_{q_1,q_2,q}v_4(q)b_{q_1}^\dagger(\tau)b_{q_1-q}(\tau)b_{q_2}^\dagger(\tau)b_{q_2+q}(\tau)\ , \tag{38}$$

$$v_4=-\frac{u_{34}^2}{V}\sum_k 2n(k)[1-n(k+q)]\beta^{-1}\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2\exp\{[\epsilon(k)-\epsilon(k+q)](\tau_1-\tau_2)\}\ , \tag{39}$$

$$n(k)=\{\exp[\beta\epsilon(k)]+1\}^{-1}\ , \tag{40}$$

$$\epsilon(k)=\frac{k^2}{m_3}-\mu_3\ . \tag{41}$$

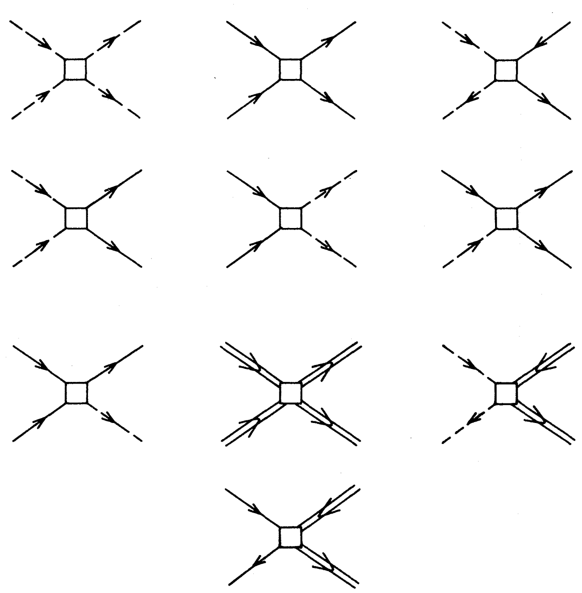

FIG. 1. Vertices in the interaction $U_1$ [Eq. (20)].

In writing (38), use has been made of the fact that, by definition, $q$'s are small in comparison with the thermal momentum so that factors such as $\exp\{\beta[\epsilon_B(q_1)-\epsilon_B(q_1-q)]\}$ can be replaced by unity. Again if the temperature is not high [thermal momentum smaller than the Fermi momentum $(m_3\mu_3)^{1/2}$]

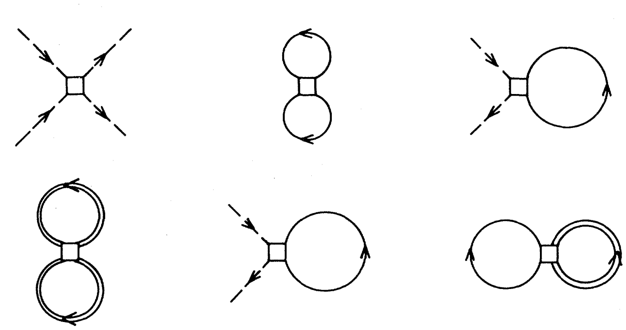

FIG. 2. Graphs corresponding to the first-order term in Eq. (26).

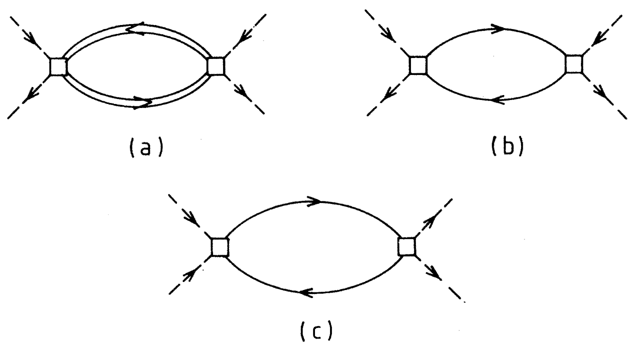

FIG. 3. Second-order graphs contributing to the four-point vertex $u_4(q,\tau)$.

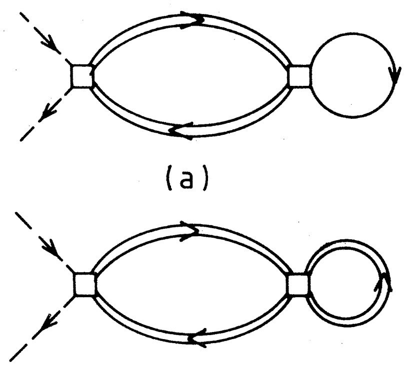

FIG. 4. Second-order graphs contributing to $H_F^{(4)}(q)$.

$$v_4(q)\simeq v_4(0)=-\tfrac{1}{2}u_{34}^2\frac{\partial n_3^F}{\partial\mu_3}\,, \tag{42}$$

where according to (33)

$$\frac{\partial n_3^F}{\partial\mu_3}=\frac{2\beta}{V}\sum_k n(k)[1-n(k)]\,. \tag{43}$$

The contributions of Figs. 3(b) and 3(c) are of order $u_4^2$. We are now in a position to define the scheme of approximation. We regard $u_4$, $u_3$, and $u_{34}$ as weak interactions. If $u_4$ and $u_{34}$ are considered to be of the same order of magnitude, the contributions of the graphs of Fig. 3 to $u_4$ are in the nature of correction terms, and the effective four-point interaction will always have the same sign as $u_4$. On the other hand, if we consider $u_4$ to be of the same order as $u_{34}^2$, the effective interaction $[u_4+v_4(0)]$ can be positive as well as negative, provided $u_4$ is positive. As will become evident later, one needs a four-point interaction of the latter type to describe a tricritical point. Consequently, in what follows we assume $u_4$ to be positive and of the same order as $u_{34}^2$. As regards $u_3$, it will be considered of the same order of magnitude as $u_{34}$. The lowest-order calculation in this scheme is one where all quantities are calculated to the order $u_{34}^2$. Figures 3(b) and 3(c) are now seen to contribute terms of order $u_{34}^4$ and will be ignored.

The second-order graphs which contribute terms similar to $H_F^{(4)}(q)$ in order $u_{34}^2$ are shown in Fig. 4. For contributions from Figs. 4(a) and 4(b), respectively, we find

$$\mathscr{C}_{4(a)}=-\int_0^\beta d\tau\left[-u_{34}^2\frac{\partial n_3^F}{\partial\mu_3}n'_4\sum_q b_q^\dagger(\tau)b_q(\tau)\right]\,, \tag{44}$$

$$\mathscr{C}_{4(b)}=-\int_0^\beta d\tau\left[-u_3u_{34}n_3^F\frac{\partial n_3^F}{\partial\mu_3}\sum_q b_q^\dagger(\tau)b_q(\tau)\right]\,. \tag{45}$$

The second-order graphs shown in Fig. 5 contribute $c$-number terms only. As $u_{34}$ exists only for small momentum transfers, Fig. 5(f) gives only a small correction to Fig. 5(e). Ignoring Fig. 5(f), the contributions of the remaining figures can be written in the form

$$\mathscr{C}_5=-\int_0^\beta Vc_2d\tau\,, \tag{46}$$

$$c_2=-\tfrac{1}{2}u_3^2(n_3^F)^2\frac{\partial n_3^F}{\partial\mu_3}-u_3u_{34}n'_4n_3^F\frac{\partial n_3^F}{\partial\mu_3}-\frac{u_{34}^2}{2}\left[n'^2_4\frac{\partial n_3^F}{\partial\mu_3}+(n_3^F)^2\frac{\partial n'_4}{\partial\mu_4}\right]$$
$$-\frac{4u_3^2}{\beta V^3}\sum_{k_1,k_2,k'}\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2 n(k_1)n(k_2)[1-n(k_1-k')][1-n(k_2+k')]\,. \tag{47}$$

Substituting the contributions of the first- and second-order terms in (28), one can write

$$\Xi=\Xi_F\operatorname*{Tr}_{\{h_4(q)\}}\exp[-\beta H_F^{(4)}(q)]\left[1-\int_0^\beta[u_4(q,\tau)+g_1(q,\tau)-g_2(q,\tau)]d\tau\right.$$
$$\left.+\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2[u_4(q,\tau_1)+g_1(q,\tau_2)][u_4(q,\tau_2)+g_1(q,\tau_2)]+\cdots\right]\,, \tag{48}$$

where $g_1(q,\tau)$ is given by (32) and $g_2(q,\tau)$ denotes the sum of the integrands in (38) and (44)–(46); the ellipsis represents higher-order terms.

The connected graphs arising from the third- and higher-order terms in (28) give contributions to $g_1-g_2$ in the second term in (48) which are of third and higher orders in $u_{34}$. They can be ignored in a calculation up to order $u_{34}^2$. It is, however, important to examine the role of disconnected graphs arising in the various orders. If we had calculated a complete trace of the density matrix, the

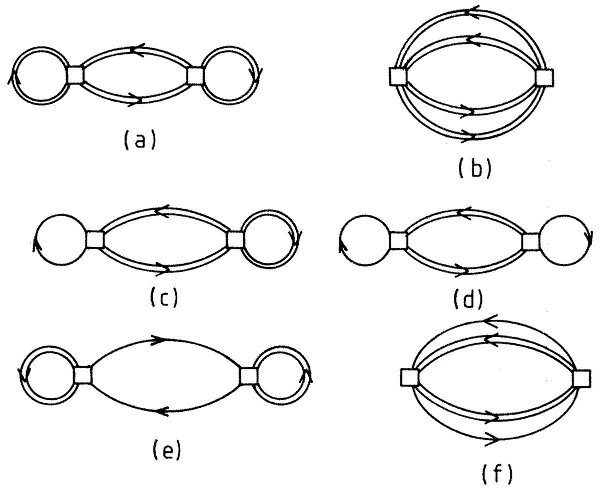


FIG. 5. Second-order graphs contributing $c$-number terms.

connected-graph theorem[13] would ensure that the disconnected graphs lead to exponentiation of the contributions of the connected graphs. When only a partial trace is calculated, the contributions of the connected graphs involve time-ordered products of the uncontracted, small-$q$-boson operators, and the usual method of proving the connected-graph theorem does not work. This problem arises in quantum as well as classical systems,[1] and one needs to verify whether the disconnected graphs lead to exponentiation or not. If they do not, the perturbation method of obtaining a new effective Hamiltonian fails. Consider, e.g., the third term in parentheses in (48). It represents disconnected graphs in the second order but has the form of a time-ordered product of the contributions of the first-order graphs. To establish exponentiation of the connected graph contributions $g_1 - g_2$ up to second order, we need to show that the disconnected graphs in the third and fourth order have the effect of adding a term $-g_2$ to each factor $U_4 + g_1$ in the third term of (48). This is somewhat tedious but easy to check. We have not, however, been able to find a general scheme to show that disconnected graphs arising in higher orders produce terms in (48) which may be identified with the terms in the expansion of $\exp[-\beta(H_F^{(4)} + g)]$, where

$$g = g_1 - g_2 + g_3 - \cdots \tag{49}$$

denotes the contributions of all the connected graphs. An analysis carried out for the case of a pure boson system[14] shows that exponentiation of the connected graphs belonging to the first and second order can be proved to all orders provided the momenta associated with the external legs are small in comparison with the thermal momentum. The analysis is easily extended to the Fermi-Bose mixture under consideration. The conclusion is that, at least for low-order connected graphs, exponentiation is valid provided the external momenta are small. The reason for leaving unintegrated only small boson momenta while carrying out a partial trace in this section has its genesis in this restriction on the validity of exponentiation of connected graphs.

We may now write (48) as

$$\Xi = \Xi_F \operatorname*{Tr}_{\{h_4\}} \exp(-\beta H_e^{(4)}) , \tag{50}$$

where the effective boson Hamiltonian $H_e^{(4)}$ is given by

$$H_e^{(4)} = c_0 + \sum_q \left[ \frac{q^2}{m_4} - \mu_4' \right] b_q^\dagger b_q + \frac{u_4'}{V} \sum_{q_1, q_2, q} b_{q_1}^\dagger b_{q_2}^\dagger b_{q_1 - q} b_{q_2 + q} , \tag{51}$$

$$c_0 = V[c_2 + \tfrac{1}{2} u_3 (n_3^F)^2 + u_{34} n_3^F n_4' + 2u_4 n_4'^2] , \tag{52}$$

$$\mu_4' = \left[ \mu_4 - u_{34} n_3^F - 4u_4 n_4' + (u_{34}^2 n_4' + u_3 u_{34} n_3^F) \frac{\partial n_3^F}{\partial \mu_3} \right] + V^{-1} \sum_{|q| < p_e} v_4(q) , \tag{53}$$

$$u_4' = \left[ u_4 - \tfrac{1}{2} u_{34}^2 \frac{\partial n_3^F}{\partial \mu_3} \right] + O(u_{34}^3) . \tag{54}$$

It is well known[9,10] that a Bose system becomes unstable when the effective interaction $u_4'$ becomes negative. We conclude that the Fermi-Bose mixture is stable as long as

$$\left[ u_4 - \tfrac{1}{2} u_{34}^2 \frac{\partial n_3^F}{\partial \mu_3} \right] \geq 0 . \tag{55}$$

It is convenient to introduce the Born scattering amplitudes $a$ and $b$ associated with the potentials $u_4$ and $u_{34}$. They are given by

$$u_4 = \frac{4\pi a}{m_4} , \tag{56}$$

$$u_{34} = \frac{4\pi b (1 + \nu)}{m_4} , \tag{57}$$

$$\nu = \frac{m_4}{m_3} . \tag{58}$$

The stability condition may then be written in the form

$$(m_4 a)/[2\pi b^2 (1+\nu)^2] \geq \frac{\partial n_3^F}{\partial \mu_3} . \tag{59}$$

In the low-density fermion limit

$$\exp(\beta \mu_3) \simeq \tfrac{1}{2} n_3^F \lambda_B^3(T) \nu^{3/2} \ll 1 , \tag{60}$$

where

$$\lambda_B(T) = \left[ \frac{4\pi\beta}{m_4} \right]^{1/2} , \tag{61}$$

denotes the boson thermal de Broglie wavelength. Consequently, (59) becomes

$$\tfrac{1}{2} n_3^F \lambda_B^3(T) (1+\nu)^2 [b^2 / a \lambda_B(T)] \leq 1 . \tag{62}$$

In view of the inequality (60), we conclude that the stability condition is satisfied in the low-density fermion regime provided $b^2/a\lambda_B(T)$ is not too large compared to 1.

In the high-density fermion limit

$$\beta\mu_3 \simeq (3\pi^2 n_3^F)^{2/3}(m_3 k_B T)^{-1} \gg 1 . \quad (63)$$

The stability condition in this limit reads

$$(3/\pi)^{1/3}\tfrac{49}{12}(n_3^F)^{1/3}\frac{b^2}{a} \le 1 , \quad (64)$$

taking $\nu = \frac{4}{3}$.

As we shall show later, the $\lambda$ line for the mixture at a given pressure $P$ is approximately given by the ideal-gas equation

$$P_F(T,\mu_3) + P_B(T,0) \simeq P \quad (65)$$

where $P_F$ denotes the pressure of the ideal Fermi gas and $P_B$ that of the ideal Bose gas at zero chemical potential. In the high-density fermion limit, this can be written in the form

$$\frac{T(P)}{T_0(P)} = \left[1 + \alpha\left[\frac{x}{1-x}\right]^{5/3}\right]^{-2/5} , \quad (66)$$

$$\alpha = \tfrac{2}{5}(\pi/3)^{1/3}\frac{(2.612)^{5/3}}{1.341} = 1.50 , \quad (67)$$

where $x$ denotes the fermion concentration defined by

$$\frac{x}{1-x} = \frac{n_3}{n_4} , \quad (68)$$

$$n_4 = \frac{2.612}{\lambda_B^3(T)} , \quad (69)$$

and $T_0(P)$ is the Bose-Einstein transition temperature at zero fermion concentration.

On the $\lambda$ line, the stability condition (64) becomes

$$\left[\frac{x}{1-x}\right]^{1/3} \Big/ \left[1 + \alpha\left[\frac{x}{1-x}\right]^{5/3}\right]^{1/5} \le 0.1806\frac{a\lambda_B(T_0)}{b^2} . \quad (70)$$

We infer that the $\lambda$ line is stable for $x \le x_t$, where $x_t$ is determined by the equality sign in (70). The corresponding temperature $T_t$ is obtained from (66). Later (cf. Sec. V) we shall be able to identify $(x_t, T_t)$ as the tricritical point of the mixture. The experimental value for the tricritical concentration for $^3$He-$^4$He mixture is close to $\frac{2}{3}$. One gets this value from (70) for

$$\frac{a\lambda_B(T_0)}{b^2} = 4.915 . \quad (71)$$

The corresponding value of $T/T_0$ is seen to be 0.496 in comparison with the experimental value 0.405. It may be mentioned that for $x = \frac{2}{3}$, the high-density condition (63) is reasonably satisfied.

It is interesting to compare the stability condition (55) with that obtained by Cohen and Leeuwen[5] from a calculation of the free energy to first order combined with thermodynamic stability criteria. In our notation, their result for the condensed phase can be manipulated into the form

$$\left[u_4 - \tfrac{1}{2}u_{34}^2\frac{\partial n_3^F}{\partial\mu_3}\left[1 + u_3\frac{\partial n_3^F}{\partial\mu_3}\right]^{-1}\right] \ge 0 . \quad (72)$$

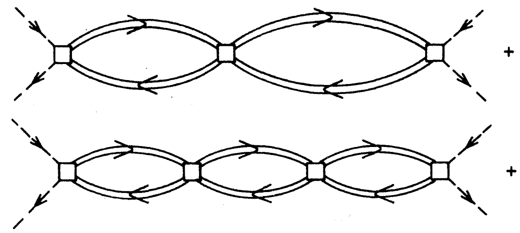

FIG. 6. The series of graphs which lead to the screening of $u_{34}^2$.

Comparison with (55) shows that, notwithstanding the fact that their calculation of free energy is valid only to first order in the $u$'s, (72) is the same as (55) to order $u_{34}^2$. More interestingly, even the factor $[1 + u_3(\partial n_3^F/\partial\mu_3)]^{-1}$ in (72) is not out of place since it seems to represent, in the diagrammatic language, a screening of $u_{34}^2$ by the fermion-fermion interaction. It is easy to check that if one sums up the series of graphs indicated in Fig. 6, one finds for the effective interaction $u_4'$ the expression on the left-hand side of (72). In our approximation scheme, however, the screening of $u_{34}^2$ represents a correction of order $u_{34}^3$. As shown in Sec. V, the stability condition derived by us for the nondegenerate phase disagrees with the result of Cohen and Leeuwen.

## IV. CALCULATION OF SIX-OPERATOR TERM

As pointed out above, when the coefficient $u_4'$ in the effective boson Hamiltonian becomes negative, the system becomes unstable. It is then necessary to take into account the next higher-order contribution to the effective Hamiltonian. Returning to the expansion (28), we find that in the third order, terms involving six boson operators arise from the graphs depicted in Fig. 7. As in the calculation of $u_4'$, if the temperature is not high, Figs. 7(a) and 7(b) contribute to the effective Hamiltonian (51) the term

$$h_6 = \frac{u_6}{V^2}\sum_{\substack{q_1,q_2,q_3,\\ q_1',q_2',q_3'}} b_{q_1}^\dagger b_{q_1-q_1'}\, b_{q_2}^\dagger b_{q_2-q_2'}\, b_{q_3}^\dagger b_{q_3-q_3'} , \quad (73)$$

$$u_6 = \frac{u_{34}^3}{6}y , \quad (74)$$

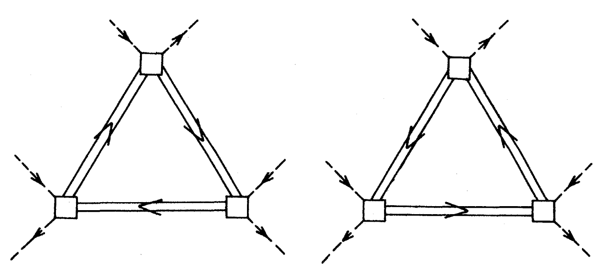

FIG. 7. Third-order graphs contributing six-operator terms.

$$y=\frac{2\beta^2}{V}\sum_k[(1-n_k)^2n_k-n_k^2(1-n_k)] . \qquad (75)$$

It is easy to check that

$$y=\frac{\partial^2 n_3^F}{\partial\mu_3^2} . \qquad (76)$$

In the low-temperature limit $\beta\mu_3\gg 1$

$$u_6\simeq\frac{m_3^2u_{34}^3}{24\pi^2(3\pi^2n_3^F)^{1/3}} . \qquad (77)$$

The coefficient $u_6$ is thus positive. Taking (51) and (73) together, we have derived an effective Hamiltonian for the fermion-boson mixture which may be considered to be the quantum analog of the expansion of free energy introduced by Landau.[3] If one replaces the boson field in the effective Hamiltonian by a classical field, one obtains the phenomenological model used by Riedel and Wegner.[2] It has been shown in earlier works[4] how the quantum nature of the fields becomes irrelevant in regard to critical behavior when one applies the renormalization-group approach to a pure Bose system. One may expect that quantum effects will not be important for the critical behavior exhibited by the effective boson Hamiltonian obtained in this paper.

Evidently, contributions of order $u_{34}^3$ arise to the effective interaction $u_4'$ also calculated in Sec. III. They arise from the first of the polarization graphs in Fig. 6 and the graphs in Fig. 7 when two $q$ lines at any vertex are replaced by two $p$ lines and contracted. These contributions, however, make only small corrections to the stability condition or the tricritical point, and hence will be ignored.

## V. DISCUSSION

By starting with a quantum-mechanical model Hamiltonian for $^3$He-$^4$He mixtures, and carrying out a partial trace with respect to the fermion and short-wavelength boson amplitudes, we have derived an effective boson Hamiltonian given by Eqs. (51) and (73). This effective Hamiltonian involves only small momenta ($|q|\le p_c$), and, as usual in renormalization-group theory,[1,15] can be viewed as a "block Hamiltonian" for $^3$He-$^4$He mixtures. As pointed out in Sec. IV, if one replaces the boson field by a classical field in this Hamiltonian, one gets the phenomenological model used by Riedel and Wegner[2] in their renormalization-group analysis of the tricritical behavior of $^3$He-$^4$He mixtures. The calculations of this paper consequently enable one to understand how a model Hamiltonian of the type used by Riedel and Wegner can arise from a microscopic, quantum-mechanical basis. The earlier approaches[5,7] concerned themselves directly with the calculation of an approximate free energy; the possibility of an equivalent Hamiltonian was not contemplated in these theories.

In keeping with modern ideas on critical phenomena,[15] our effective Hamiltonian is expected to yield a Landau-type theory, provided one ignores fluctuations of the order parameter. For a system of bosons, the order parameter as pointed out in Sec. I is $b_0/\sqrt{V}$ which, following Bogolubov,[12] can be replaced by a $c$ number $M$. The operators $b_k$ with $k\neq 0$ play the role of fluctuations of the order parameter. Upon ignoring the fluctuations completely, the effective Hamiltonian reduces to

$$H_e^{(4)}\simeq c_0-\mu_4'M^2+u_4'M^4+u_6M^6 . \qquad (78)$$

In view of the fact that $u_4'$ can be positive or negative while $u_6$ is always positive, (78) is essentially the Landau expansion corresponding to tricritical behavior. As is planned to be discussed in a future paper, a slightly better and physically more meaningful approximation for the degenerate phase is obtained by treating the kinetic energy $[\sum_q(q^2/m_4)b_q^\dagger b_q]$ as the unperturbed Hamiltonian to take fluctuations into account up to first order. The intersection of the $\lambda$ line ($\mu_4'=0$) and the curve ($u_4'=0$), which we designated as $(x_t,T_t)$ in Sec. III, can now be identified as the tricritical point of the system.

For the nondegenerate phase, the above approximation turns out to be rather poor. We shall consequently use the self-consistent Hartree-Fock approximation for both the phases. As will be shown in a planned paper, this approximation leads to a Ginzburg-type criterion[16] for the validity of the Landau approximation.

In the light of the above discussion, the reason for the inability of earlier attempts[5,7] to obtain a Landau theory becomes clear. In earlier works, corrections to the ideal-gas free energy were calculated to first order using the *bare* vertices ($u_4,u_{34},u_3$). Equation (78), on the other hand, shows that it is the effective or renormalized boson-boson interaction $u_4'$ which governs the stability of the degenerate phase in the absence of the $M^6$ term. The $M^6$ term which was not evaluated in earlier works, plays an important part in restoring stability when $u_4'$ becomes zero or negative.

We have pointed out towards the end of Sec. III that the stability condition obtained by demanding the effective boson interaction to be positive definite agrees with the thermodynamic stability condition derived by Cohen and Leeuwen[5] (abbreviated CL hereafter) for the degenerate phase using first-order expressions for the chemical potentials. The agreement implies that corrections of second order to the chemical potentials somehow do not contribute anything to the thermodynamic stability criterion for the degenerate phase although second-order terms make important contributions to the effective boson-boson interaction $u_4'$. In order to bring this point out in a clear fashion, and also to demonstrate that this is no longer true for the nondegenerate phase, we give below our expression for the chemical potential $\mu_4$. Its derivation will be presented in a planned paper. Up to terms of second order, we find

$$\mu_4=\mu_B^{(0)}(n_4)+\left[u_{34}n_3^F-u_3u_{34}n_3^F\frac{\partial n_3^F}{\partial\mu_3}+2u_4''n_4\right]$$
$$+2u_4n_4+2u_4(n_4-M^2)$$
$$+2u_4''[n_4-M^2-n_4'(p>p_c)] . \qquad (79)$$

In the above equation, $\mu_B^{(0)}$ denotes the chemical potential of the ideal Bose gas, $u_4''$ is the contribution to the bare interaction $u_4$ calculated in Sec. III, viz.,

$$u_4''=-\tfrac{1}{2}u_{34}^2\frac{\partial n_3^F}{\partial\mu_3} \tag{80}$$

and $n_4'(|p|>p_c)$ is the density of bosons with $|p|>p_c$, i.e.,

$$n_4'(|p|>p_c)=\int_{p_c<|p|<\infty}\frac{d^3p}{(2\pi)^3}[\exp(\beta p^2/m_4)-1]^{-1}\ . \tag{81}$$

The second term (within large parentheses) in (79) can be shown to be equal to $n_3u_3$. Comparison with the results of CL shows that (79) differs from their result by the presence of the last term only.

With the expression (79) for $\mu_4$, it is convenient to use the stability condition

$$\left[\frac{\partial\mu_4}{\partial n_4}\right]_{\mu_3,T}>0\ . \tag{82}$$

The same results are obtained if one uses the CL condition

$$\left[\frac{\partial^2 g}{\partial x^2}\right]_{T,p}>0\ .$$

In the degenerate phase $\mu_B^{(0)}=0$ and it is sufficient to use for $n_4-M^2$ appearing in the last two terms of (79), the zero-order result

$$(n_4-M^2)\simeq 2.612/\lambda_B^3\ . \tag{83}$$

One consequently obtains for stability the CL result

$$2(u_4+u_4'')>0 \tag{84}$$

which is identical with the effective interaction condition used in Sec. III.

In the nondegenerate phase, $M$ is zero; the stability condition becomes

$$\frac{\partial\mu_B^{(0)}}{\partial n_4}+4(u_4+u_4'')>0\ . \tag{85}$$

The CL result, on the other hand, is

$$\frac{\partial\mu_B^{(0)}}{\partial n_4}+4u_4+2u_4''>0\ . \tag{86}$$

The disagreement between (85) and (86) is obvious. We note that while (84) and (85) give identical conditions for the stability of the $\lambda$ line, CL, on the basis of (86), were led to the conclusion that the $\lambda$ line when approached from the nondegenerate phase is stable at every point.

Finally it seems appropriate to add a few remarks on the model Hamiltonian we have chosen, particularly the restriction that $u_4$ must be treated as a quantity of the same order of smallness as $u_{34}^2$. This model is not a realistic representation of the interactions in a $^3$He-$^4$He mixture which are neither weak nor disproportionate in strength. It is, however, a useful model in the sense that it enables a theory of critical behavior to be built up from a microscopic basis. As a matter of fact, it is not unusual in many-body theory to consider models obtained by imposing suitable restrictions on the parameters of the Hamiltonian to get useful schemes of calculations. A well-known example is the $\epsilon$-expansion in renormalization-group theory[1] where $4-d$ is treated as a small parameter in order to match second-order terms with the first-order terms to obtain a fixed point. Another example is provided by Huang's work[9] on hard-sphere Bose gas with a long-range attractive interaction between the particles. In order to get physically meaningful results, the strength of the attractive interaction must be regarded as a quantity of second order of smallness to match first-order attractive terms with the second-order repulsive terms.


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