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Quantized Hall effect and edge currents

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It is shown that the quantized Hall current may always be expressed as the difference between diamagnetic currents flowing at the two edges. It is argued that the high precision of the quantization may be aided by the establishment of a local equilibrium in each edge region. The basic ideas are illustrated by the discussion of a free two-dimensional electron gas in an infinite confining potential. Our derivation establishes the connection between quantum-mechanical and classical thermodynamic explanations for the quantum Hall effect.

I. INTRODUCTION

Since the discovery of the quantized Hall effect by von Klitzing *et al.*,[1] many papers devoted to the theoretical explanation of this interesting phenomenon have appeared (for a review, see, e.g., Ref. 2). In most of these works the influence of surface currents is not made explicit. On the other hand, a recent thermodynamical derivation by Středa and Smrčka,[3] using an approach originally suggested by Widom,[4] leads to the conclusion that the quantized Hall current may be viewed as arising from the response of the edge diamagnetic currents only. In this article we use quantum-mechanical arguments to establish the essential role played by these edge currents in quantizing the Hall conductivity. In so doing we establish the connection between the approach of Laughlin[5] and Halperin[6] and the thermodynamic derivation.

II. IDEAL CASE

We begin by considering a system of noninteracting electrons in the x-y plane in a potential which confines the electrons to a finite range as a function of one of its coordinates. In a magnetic field $\vec{B}=(0,0,B)=\vec{\nabla}\times\vec{A}$ the one-electron Hamiltonian is

$$H=\frac{1}{2m}\left[\vec{p}-\frac{e}{c}\vec{A}\right]^2+V(x)\,, \tag{1}$$

where $V(x)$ is the confining potential. In the Landau gauge $\vec{A}=(0,Bx,0)$, we can separate variables in the usual way and seek solutions of the form

$$\psi(x,y)=\frac{1}{\sqrt{2\pi}}\exp\left[\frac{iXy}{a_L^2}\right]\phi_{n,X}(x)\,, \tag{2}$$

where $\phi_{n,X}(x)$ obeys

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega_c^2(x-X)^2+V(x)\right]\phi_{n,X}(x)=\epsilon_{n,X}\phi_{n,X}(x)\,, \tag{3}$$

and $a_L^2=\hbar c/|e|B$ and $\omega_c=|e|B/mc$. If we impose periodic boundary conditions with length L_y as a function of the y coordinate, which is equivalent to requiring single valuedness in a ring geometry, the allowed values of the quantum number X are separated by $2\pi a_L^2/L_y\equiv\delta X$. The precise form of the confining potential is unimportant for what follows, but for the sake of definiteness we take

$$V(x)=\begin{cases}0\,, & x\in(-L_x/2\,,\,L_x/2) \qquad (4a)\\ +\infty\,, & \text{otherwise}\,. \qquad (4b)\end{cases}$$

Then, if $|X\pm L_x/2|\gg a_L$, we can take $V(x)\equiv0$ so that $\epsilon_{n,X}=\hbar\omega_c(n+\frac{1}{2})$ for integer n and $\phi_{n,X}$ is a one-dimensional harmonic-oscillator eigenstate. Near the boundary, however, we must take account of the confining potential by setting $\phi_{n,X}(\pm L_x/2)=0$. If we write $\epsilon_{n,X}=\hbar\omega_c(\nu_n+\frac{1}{2})$ the general solution to Eq. (3) may be written as

$$\phi_{n,X}(x)=\exp\left[\frac{-Z^2}{2a_L^2}\right]\left[{}_1F_1\left[\frac{-\nu_n}{2};\frac{1}{2};\frac{Z^2}{a_L^2}\right]\alpha+\beta Z\,{}_1F_1\left[\frac{-\nu_n}{2}+\frac{1}{2};\frac{3}{2};\frac{Z^2}{a_L^2}\right]\right]\,, \tag{5}$$

where $Z=x-X$. Near the edge $x=-L_x/2$ we can fix α/β by requiring that $\phi_{n,X}(x)$ vanishes as $x\to+\infty$. This leads to

$$\frac{\beta}{\alpha}=2\tan\left[\frac{\pi\nu_n}{2}\right]\frac{\Gamma(1+\nu_n/2)}{\Gamma(\frac{1}{2}+\nu_n/2)}\,. \tag{6}$$

Then the discrete eigenvalue spectrum for given X can be

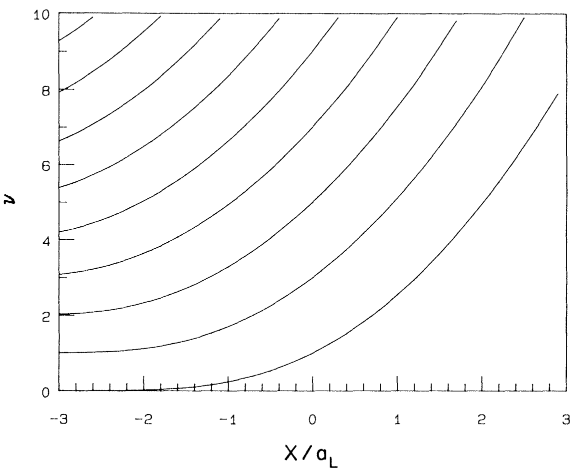

FIG. 1. Eigenvalues $[\epsilon=\hbar\omega_c(\nu+\frac{1}{2})]$ as a function of X, the orbital center in the absence of a confining potential, for an infinite wall confining potential. In this figure $X=0$ is at the edge, $X>0$ is outside the edge, and $X<0$ is inside the edge. Note that at $X=0$ all the odd-integer harmonic-oscillator eigenvalues appear.

determined by finding the zeros of $\phi_{n,X}(-L_x/2)$. The eigenvalues and eigenfunctions at the other edge are simply related. The eigenvalue spectrum near the edge is illustrated in Fig. 1. The important point is that the eigenvalues in the edge region lie between those for the bulk Landau levels. It is clear that this must be the case for any confining potential.

We now turn our attention to the current carried by the electrons in this system. The expectation of the current operator in each eigenstate is given by

$$\langle n,X|j_y|n,X\rangle=\frac{ea_L^2}{h}\frac{\partial\epsilon_{n,X}}{\partial X}\ , \tag{7}$$

so that the total current carried by the nth branch of the energy spectrum, when all the bulk states for that branch are occupied, is

$$\begin{aligned} I_n&=\frac{1}{L_y}\sum_X\langle n,X|j_y|n,X\rangle\Theta(\mu-\epsilon_{n,X})\\ &=\frac{1}{\delta X\,L_y}\int dX\langle n,X|j_y|n,X\rangle\Theta(\mu-\epsilon_{n,X})\\ &=\frac{e}{h}\int dX\frac{\partial\epsilon_{n,X}}{\partial X}\Theta(\mu-\epsilon_{n,X})=\frac{e}{h}(\mu^R-\mu^L)\ , \end{aligned} \tag{8}$$

where μ^R is the chemical potential at the right (positive-x) edge of the sample and μ^L is the chemical potential at the left edge. Note that in the equilibrium case, where μ is constant, no current flows. This remains true even if we include an electric field term in our potential [i.e., $V(x)=-eEx$ in Eq. (4a)]. In that case $\phi_{n,X}(x)\to\phi'_{n,X}(x)=\phi_{n,X'}(x)$, where $X'=X+eE/m\omega_c^2$ and $\epsilon_{n,X}\to\epsilon'_{n,X}=\epsilon_{n,X}-eEX'+(m/2)(cE/B)^2$, but Eqs. (7) and (8) remain valid. In other words, in the equilibrium case, $\mu^R=\mu^L$, the bulk current which flows in the presence of an electric field is exactly canceled by the surface diamagnetic current. In the nonequilibrium case the current flowing in the bulk depends on the internal electric field when there is a chemical potential difference between the two edges. For noninteracting electrons all the current would flow at the edges and the quantized Hall current would just be the difference between the two edge currents. For interacting electrons a chemical potential change at the edges must always be accompanied by a charge redistribution and a consequent internal electric field. (As discussed by MacDonald *et al.*,[7] however, this electric field is unlikely to be constant.) The point we wish to emphasize here is that any bulk current which flows will always be compensated by an additional edge current. As discussed in the next section these statements remain true when we add a random potential term to our Hamiltonian.

III. GENERAL CASE

We consider adding a random potential term to our Hamiltonian which is sufficiently weak that the eigenfunctions $\{\phi_{\alpha,n}(x,y)\}$ and eigenvalues $\{\epsilon_{\alpha,n}(x,y)\}$ can still be labeled by a Landau-level index. (This does not mean that we do not allow any mixing between different Landau levels.) In the equilibrium case the total current from the nth Landau level

$$I_n=\frac{1}{L_y}\sum_\alpha\langle\alpha,n|j_y|\alpha,n\rangle\Theta(\mu-\epsilon_{n,\alpha})=0 \tag{9}$$

must be zero. We will restrict our attention to the Hall current when the chemical potential lies between bulk Landau levels and consider changes in the chemical potential at each edge of the sample. To be precise about this we define a local current density by

$$j_y^n(X)=\frac{1}{\delta X\,L_y}\sum_{m,m'}\sum_\alpha\langle\alpha,n|X,m\rangle\langle X,m'|\alpha,n\rangle \times\langle X,m|j_y|X,m'\rangle\Theta(\mu-\epsilon_{n,\alpha})\ . \tag{10}$$

Here $\{|X,m\rangle\}$ are the eigenstates for the ideal case discussed in Sec. II. Since j_y has no matrix elements between any of these states with different values of X, the total current is

$$I_n=\int dX\,j_y^n(X)\ . \tag{11}$$

Now consider

$$\frac{\partial j_y^n(X)}{\partial\mu}=\frac{1}{\delta X\,L_y}\sum_{m,m'}\sum_\alpha\langle\alpha,n|X,m\rangle\langle X,m'|\alpha,n\rangle \times\langle X,m|j_y|X,m'\rangle\delta(\mu-\epsilon_{n,\alpha})\ . \tag{12}$$

Since the only states which exist at energy μ are localized at the edges of the sample, we shall assume this function is peaked within a distance $W\sim a_L$ of the edge of the sample and drops essentially to zero in the middle. This is certainly true in the ideal case and as disorder is added W

may increase but presumably not drastically. (As we see below, the accuracy of the Hall current quantization may be related to W.) It is worth noting that the existence of states well localized at the edges for energies in the intra-Landau-level gaps which occur in a periodic potential has been established by an explicit calculation.[8] To calculate the Hall current we take the chemical potential to be constant at each edge of the sample, i.e.,

$$I_n=\int_L dX\frac{\partial j_y^n(X)}{\partial\mu}\delta\mu_L+\int_R dX\frac{\partial j_y^n(X)}{\partial\mu}\delta\mu_R \tag{13a}$$

$$=\int_R dX\frac{\partial j_y^n(X)}{\partial\mu}(\delta\mu_R-\delta\mu_L)\,, \tag{13b}$$

where $\int_{L,R}$ denotes an integral over the region near the left or right edge where $\partial j_x^n(X)/\partial\mu$ is nonzero and Eq. (13b) follows from Eq. (13a) by noting that I_n must still be zero if $\delta\mu_L=\delta\mu_R$. Since there are no energy gaps between surface states, electrons lying in the vicinity of the chemical potential at each edge will try to establish a local equilibrium to force the chemical potential to be constant in each edge region. The probability of an electron being scattered from one edge to the other, of course, is very small because of the large momentum difference between the electrons and a chemical potential difference between the two edges would relax very slowly, corresponding to nearly dissipationless current flow. We believe this is why there is little dependence of the Hall conductivity on edge imperfections. For example, the confining potential need not be a function of one coordinate only. (The hypothesis about local equilibrium at edges allows us to neglect the dependence of $\delta\mu_R$ and $\delta\mu_L$ on Landau-level index.)

To express the total Hall current

$$I=\sum_n\int_R dX\frac{\partial j_y^n(X)}{\partial\mu}(\delta\mu_R-\delta\mu_L)$$

in terms of known quantities we note that

$$j_y=c\frac{\partial H}{\partial A_y}\,. \tag{14}$$

Equation (14) is valid for arbitrary H (i.e., with or without a random potential) and, as first pointed out by Laughlin,[5] this relation is essential to the quantum Hall effect. By introducing edge states Halperin[6] was able to improve the argument for the quantization of the Hall current based on Eq. (14). This has been discussed more explicitly by Girvin and Jonson[9] in connection with their treatment of the thermal conductivity and thermopower which we mention briefly below. Here, however, we start from Eq. (14) and take a different approach which relates this quantum-mechanical treatment with the thermodynamic derivation.[3,4] We consider the grand potential at zero temperature

$$\Omega(\mu,B)=\sum_{\alpha,n}(\epsilon_{\alpha n}-\mu)\Theta(\mu-\epsilon_{\alpha n})\,. \tag{15}$$

The Hall current [Eq. (13b)] will eventually be shown to be proportional to $\partial^2\Omega(\mu,B)/\partial\mu\partial B$, and so we begin by looking at the change in Ω with a small change in B. To do so we note that when $B\rightarrow B+\delta B$ then, in the Landau gauge, $A_y\rightarrow A_y+\delta Bx$, so that using a Hellman-Feynman theorem we obtain

$$\begin{aligned}\frac{\partial\Omega(\mu,B)}{\partial B}&=\frac{1}{c}\sum_{\alpha,n}\Theta(\mu-\epsilon_{\alpha,n})\langle\alpha,n\,|\,j_yx\,|\,\alpha,n\rangle\\&=\frac{1}{c}\sum_{\alpha,n}\Theta(\mu-\epsilon_{\alpha,n})\sum_X\sum_{m_1,m_2,m_3}\langle\alpha,n\,|\,X,m_1\rangle\langle X,m_1\,|\,jy\,|\,X,m_2\rangle\langle X,m_2\,|\,x\,|\,X,m_3\rangle\langle X,m_3\,|\,\alpha,n\rangle\,.\end{aligned} \tag{16}$$

Since the thermodynamical definition for the magnetization M is $\partial\Omega(\mu,B)/\partial B=-M$, Eq. (16) may be regarded as providing the quantum-mechanical equivalent of the classical expression relating the current and magnetization densities in the thermodynamic derivation.[3,4] Differentiating Eq. (16) with respect to μ just changes the factor $\Theta(\mu-\epsilon_{\alpha,n})$ to $\delta(\mu-\epsilon_{\alpha,n})$. It follows, making use of Eq. (12), that

$$\frac{\partial^2\Omega(\mu,B)}{\partial\mu\partial B}=\frac{L_y}{c}\sum_n\int dX\frac{\partial j_y^n(X)}{\partial\mu}X \tag{17a}$$

$$=\frac{L_yL_x}{c}\sum_n\int_R dX\frac{\partial j_y^n(X)}{\partial\mu}\,, \tag{17b}$$

so that comparing with Eq. (13b) we have

$$\begin{aligned}I=\sum_n I_n&=\frac{(\delta\mu_R-\delta\mu_L)c}{L_xL_y}\frac{\partial^2\Omega(\mu,B)}{\partial\mu\partial B}\\&=-\frac{(\delta\mu_R-\delta\mu_L)c}{L_xL_y}\frac{\partial N(\mu)}{\partial B}\,,\end{aligned} \tag{17c}$$

with the sum being over occupied Landau levels.

Note that in passing from Eq. (16) to Eq. (17a) we have dropped terms containing factors such as $\langle X,m_2\,|\,x-X\,|\,X,m_3\rangle$, which would give corrections of order a_L/L_x compared to those we have retained. Similarly, to go from Eq. (17a) to Eq. (17b) we replace X by $\pm L_x/2$ at the edges which has corrections $\sim W/L_x$. [Actually, these corrections should cancel to leading order at the two edges to give a correction factor $\sim(a_L/L_x)^2$.] To the same order we may replace $N(\mu)/L_xL_y$ by the number of states per unit area in a bulk Landau level to obtain

$$I=\frac{e}{h}\bar{n}(\delta\mu_R-\delta\mu_L)\,, \tag{18}$$

where $\bar{n}$ is the number of filled Landau levels which is the usual expression for the quantized Hall current.

IV. DISCUSSION AND CONCLUDING REMARKS

Treatments of the quantized Hall effect which begin, as we do, from Eq. (14), normally make use of the so-called gauge-invariance principle which requires the eigenvalue spectrum to be unchanged when $\delta A_y L_y = \Phi_0$. This property is not used explicitly in our derivation; the same information in our case enters through the use of the expression for the number of states per unit area in a Landau level. Equation (17c), however, contains more information since it can, and has been,[10] applied to the case of a periodic potential where gaps appear within a Landau level. The connection between Eq. (17c) and the "gauge-invariance" principle has been discussed recently by Rammal *et al.*[11] Our equation has previously been derived using Widom's[4] thermodynamic argument as elaborated by Středa and Smrčka.[3] The quantum-mechanical derivation of this expression makes the approximation hidden in the classical approach apparent and establishes the connection between this approach and previous quantum-mechanical treatments. It also stresses the fact that the Hall current can always be regarded as the difference of the diamagnetic currents at the two edges. The same expression has also been derived from the Kubo formula,[12] and here, as well, a detailed analysis of the derivation leads to the same conclusion.

In real systems of interacting electrons, at least without a random potential, some current may flow in the bulk, and the surface currents will adjust to cancel this, leaving the quantized value for the total Hall current. (In our derivation, we only look at the difference between the currents in equilibrium and nonequilibrium situations. This is localized at the edges but the total current need not be.) A useful experimental indication of the relative importance of surface and bulk currents may be the thermal conductivity. It can be concluded from the arguments of Girvin and Jonson[9] that plateau values for this quantity are given by

$$\kappa = \frac{\pi^2 k_B^2 T}{3h} \bar{n} \tag{19}$$

if the current flows only at the edges, but it is not certain that this will be the case if some of the heat current flows in the bulk.

It is clear from the quantum-mechanical derivation that while Eq. (17c) has corrections $\sim (W/L_x)^2$, it need not be true that Eq. (18) will have errors of the same order. In fact, if the confining potential can be taken to be a function of one coordinate only and the random potential can be set to zero in the edge regions, Eq. (14) leads directly to Eq. (18) with corrections which decrease exponentially with $(L_x/a_L)^2$. Whether such a situation is ever realized experimentally, however, is an open question.

Finally we note that the derivation of Eq. (17c) may readily be generalized to the case of interacting electrons merely by substituting the language of second quantization for the language of single-particle wave functions which we have employed. It follows that whenever a discontinuity in the chemical potential exists which, as a function of magnetic field, is pinned to a certain filling factor ν of the Landau level,

$$\sigma = -\frac{e^2 \nu}{h} \tag{20}$$

will hold, independent of the origin of the gap.

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