Domain growth in the random-field Ising model

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We study a continuum random-field model of domain growth in quenched nonequilibrium systems. We derive an equation of motion for the interfaces separating domains and find approximate solutions for the growth laws in two and three dimensions. We find what may be a dynamical mechanism for the theoretical prediction that the lower critical dimension of this model is $d_l=2$. Our theoretical predictions can be tested experimentally or by computer simulation.

Recently there has been much interest in systems with random external fields.[1-5] Theories have consisted of equilibrium analyses of Ising-type systems.[1-4] These are systems which, in the absence of a random field, phase separate at low temperatures. When the random field is turned on, there exists a lower critical dimension $d=d_l$, such that, for $d\leq d_l$, there is no phase separation for any temperature. Interfaces wander and become infinitely rough at d_l. Thus a two-phase system effectively becomes one "mixed" phase. Several authors have obtained $d_l=2$ in their equilibrium analyses of random-field models.[2] The value of d_l remains, however, a controversial theoretical issue.[1-3] It has been argued that the random-field model is equivalent to dilute antiferromagnetic systems in a uniform external field.[4] Such systems have been studied experimentally.[5] However, subtle effects have been seen in dilute antiferromagnets which have no obvious origin in the simpler random-field models.[5] For example, hysteresis effects complicate interpretation of the experiments. (Those dynamical effects are not the subject of this paper, however.[6]) Study of the random-field models may nevertheless lead to useful insights into dilute antiferromagnets. In any case, random-field models are of interest in their own right.

So far, the effect of a random external field has been analyzed only in systems at or close to equilibrium. However, the behavior of systems far from equilibrium is a problem of great current interest.[7-12] In this paper we will study the growth of unstable domains far from equilibrium ("spinodal decomposition") in a random-field system.

We consider a simple relaxational model of a system with a (continuous) nonconserved order parameter ψ. The system is quenched from an initially disordered state to a low-temperature nonequilibrium state where phase separation would take place if there were no random field. We consider an intermediate time following the quench, namely, when random interfaces (which separate the domains) have formed but before complicated thermodynamic fluctuations close to equilibrium dominate the dynamics. Therefore, we can discuss the dynamical breaking of long-range order at d_l. For $d\leq d_l$, domains of the ordered phase will eventually stop growing because the system cannot become ordered, even at zero temperature. In two dimensions we find a novel growth mechanism which is consistent with the prediction that $d_l=2$ in this model. Our results for $d=2$ and $d=3$ can be tested experimentally, or by computer simulation of a quenched system.

In the intermediate time regime following the quench, we locate the interface at the nonequilibrium surface $u(\vec{r},t)=0$. Positions on this surface are determined by the $(d-1)$ dimensional vector $\vec{S}$. This can be done if the interface is thin, and gently curved.[8-12] The equation of motion can be derived (i) through analysis of a time-dependent Ginzburg-Landau equation for the nonconserved order parameter ψ (following Allen and Cahn[8] and Kawasaki and Ohta[9]), or (ii) from physical arguments and the requirement of Euclidean invariance (following Bausch, Domb, Janssen, and Zia[10]). In a subsequent paper we will present the detailed derivation of the equation of motion. For now we will only state the result.

We find that the interfacial motion is due to two opposing forces: The surface free energy acts to flatten interfaces through the thermodynamic force K (the curvature),[13] while the random field H makes interfaces wander and become rough. Explicitly, the equation of motion for the velocity normal to the interface is[13]

$$v=(D'/\sigma)\delta F_s/(h\,\delta u)=D'(K-g)\ , \tag{1}$$

at $u=0$, where D' is a diffusion coefficient, σ is the surface tension, h is the differential length in the u direction,[13] and

$$F_s=\sigma\int d^{d-1}S-\int d^dr\,H(\vec{r})\psi(u) \tag{2}$$

is the free energy of the interface in the continuum random-field model.[1,2,14] The first term in Eq. (2) is the "thermodynamic" surface free energy, while the second term gives the interfacial energy due to the random field.[15] We have not included the thermal noise term in Eq. (1). It will have the same form as is given in Refs. 9–12. Thermal noise effects are briefly discussed following Eq. (6b).

If the external field $H(r)$ is a random Gaussian[16] variable with $\langle H(\vec{r})\rangle=0$ and

$$\langle H(\vec{r})H(\vec{r}')\rangle=\nu_0^2\delta^d(r-r')\ ,$$

where ν_0^2 is a constant, then it can be shown that

$$\langle g(S)g(\vec{S}')\rangle=\nu^2(\Lambda/2\pi)^{3-d}\delta^{d-1}(\vec{S}-\vec{S}')\ . \tag{3}$$

In this equation, ν^2 is the dimensionless random-field strength,

$$\nu^2\equiv\nu_0^2(2\pi)^{3-d}(C\Lambda^2\sigma/\Lambda^{d-1})^{-1}\ , \tag{4}$$

where C is the coefficient of the gradient term in the Ginzburg-Landau free energy. It is related to the surface tension through

$$\sigma=C\int h\,du(d\psi/h\,du)^2\ .$$

We have also introduced the ultraviolet cutoff Λ, where $a\Lambda/(2\pi)=O(1)$ where a is the lattice constant.[17]

We have solved Eq. (1) using a physically appealing linearization scheme of Ohta, Jasnow, and Kawasaki.[11,12] The linearization is based on the (approximately) isotropic configuration of interfaces, which exists in the intermediate time regime. Equation (1) becomes tractable in this limit. For example, when there is no random field or thermal fluctuation, Eq. (1) reduces to a linear diffusion equation. The area density $\mathscr{A}$ (which is inversely proportional to the characteristic length scale of the evolving domains, $\bar{R}$) can then be straightforwardly, though tediously, obtained. We now display our results. Details of the derivation will be presented in a later paper.

The growth laws (i.e., the time dependence of the area density $\mathscr{A}$) are expressed in terms of the characteristic size of domains $\bar{R}$.[18] In $d=3$,

$$\bar{R}^2=Dt\{1-3[3/(4\pi)]^{4/3}\nu^2\}\ , \tag{5}$$

where $D=D'(d-1)/d$ and ν^2 is the dimensionless random-field strength given by Eq. (4). In $d=2$,

$$\bar{R}^2=Dt\{1-\pi^{-3/2}\nu^2[\ln(\Lambda^2Dt)-\gamma]\}\ , \tag{6a}$$

or, in dimensionless units ($\bar{r}=\bar{R}\Lambda$, $\tau=D\Lambda^2t$),

$$\bar{r}^2=\tau[1-\pi^{-3/2}\nu^2(\ln\tau-\gamma)]\ , \tag{6b}$$

where γ is approximately 0.809. Thermal noise effects are as given in Ref. 12. They do not change the time dependences of the growth laws, wherein the crucial physics of this problem lies.

We now interpret our results. The time dependence of the $d=3$ growth law, Eq. (5), is unchanged from the zero-field result. However, the diffusion coefficient is renormalized by a field-dependent factor. This renormalization effectively slows down domain growth. This is reasonable. The random field (which roughens interfaces) is competing with the surface free energy (which flattens interfaces).

In two dimensions [Eq. (6)], there are logarithmic corrections to the zero-field linear growth law $\bar{r}^2=\tau$.[19] This new growth law implies that, for any field strength $\nu^2\neq 0$, domains cannot grow larger than a maximum size $\bar{r}_c$, given below. Physically, the interfaces are evolving through their curvature from the initially quenched random configuration to the domain configuration determined by the external field. Unlike $d=3$, where domain growth is only slowed down by a renormalization of D, the random field in $d=2$ is strong enough to stop domain growth. This is consistent with earlier predictions that the lower critical dimension is $d_l=2$.[2] Of course, because of our approximations, this result could be fortuitous. Indeed, as noted above, the value of d_l remains somewhat controversial.[1-6]

Equation (6) implies that two-dimensional domains reach a maximum size $\bar{r}_c$ at a time τ_c, and then begin to shrink.[20] These quantities are given by $\tau_c=\exp[\pi^{3/2}/\nu^2-(1-\gamma)]$ and $\bar{r}_c=\tau_c\nu^2/\pi^{3/2}$. The expression for $\bar{r}_c$ is in accord with that found in Binder's equilibrium analysis.[2] These quantities are very sensitive to the value of the dimensionless field strength. For $\nu^2=0.1$ we obtain $\bar{r}_c\approx 10^{11}$ and $\tau_c\approx 10^{24}$, but for $\nu^2=1$ we obtain $\bar{r}_c\approx 6$ and $\tau_c\approx 170$. This implies that one might be able to "tune" the random field to obtain interesting results in computer simulations of finite-size systems. Note that the scaling of $\bar{r}$ and τ in our continuum theory suggests that they roughly correspond to units of lattice constants and "Monte Carlo steps," respectively. However, a two-dimensional system might have large fluctuation effects, since $\bar{r}_c$ changes by ten orders of magnitude when ν^2 is only changed by one order of magnitude.

Some limitations of our treatment are as follows. Not surprisingly, the hysteresis effects, seen in $d=3$ experiments on dilute antiferromagnets close to equilibrium,[5] are not present in Eq. (5). Our simple relaxational model is probably not sufficiently rich to describe both the domain growth far from equilibrium, and the detailed fluctuational dynamics closer to equilibrium. There may also be some subtle differences between dilute antiferromagnets and the random-field model. The time scale of validity of the $d=3$ growth law is examined below, where we discuss the structure factor.

We stress that the result for $d=2$ [Eq. (6)] may have little physical significance for $\tau>\tau_c$ (after domains have reached their maximum size), since the basic assumptions of the model break down.[21] In our approximate continuum theory, $\tau>\tau_c$ is when the random field, which roughens interfaces, is driving the system. However, the equation of motion [Eq. (1)] requires thin, gently curved interfaces. Further, since the linearization scheme is based on an assumption of isotropy, it will tend to overestimate the effects of randomness. This is certainly what occurs in $d=2$ for $\tau>e\tau_c$, where $\bar{r}^2$ changes sign (and in $d=3$ for large random fields $\nu^2\gtrsim 2\frac{1}{4}$).

We have not calculated the nonequilibrium structure factor $\mathscr{S}(k,t)\sim\langle|\psi(k,t)|^2\rangle$,[22] which is proportional to the scattering intensity, where k is the wave number. However, we can make some comments on its behavior based upon our results for the growth laws, in the intermediate time regime when self-similar growth takes place. From scaling arguments,[7,11] we expect the second moment of $\mathscr{S}(k,t)$, $\sim\int d^dk\,k^2\mathscr{S}(k,t)$, to be roughly proportional to $\bar{R}^{-2}$. Therefore, one can in principle experimentally determine the growth laws.[7] Further, if the zeroth moment of $\mathscr{S}(k,t)$, $\sim\int d^dk\,\mathscr{S}(k,t)$, is approximately independent of the random-field strength, then as the second moment of $\mathscr{S}(k,t)$ increases, the height of the Bragg peak in $\mathscr{S}(k,t)$ will decrease, and vice versa.

Thus, in three dimensions as *time* increases [and so $\bar{R}^{-2}$ decreases from Eq. (6)] the peak in $\mathscr{S}(k,t)$ will sharpen. If the *random field* is increased in a series of experimental quenches, the theory predicts that, at a given time, $\mathscr{S}(k,t)$ will flatten, although as time increases the peak will still sharpen. In the later stages, as the system equilibrates, other processes which we have ignored (e.g., hysteresis effects) will dominate.[23]

The situation in two dimensions is more dramatic. For any nonzero field strength, our approximate theory predicts that the Bragg peak in $\mathscr{S}(k,t)$ will only sharpen until a time τ_c. After this time the peak will decrease. Although the later stages of the equilibration process are not described by our theory,[21] we would expect that, on a macroscopic length scale, the peak will disappear. Thus long-range order will not be established.[24]

To summarize, we have considered domain growth in quenched nonequilibrium systems which have random fields. We have derived a continuum equation of motion[25] for the interfaces separating domains. We have found approximate solutions for the growth laws in two and three

dimensions. In two dimensions, the growth law gives a dynamical mechanism which may be responsible for $d_l=2$ being the lower critical dimension. Our theoretical results could be tested by experiment or by computer simulation. Of course, a detailed quantitative description of experimental systems would require a more realistic dynamical model (for example, one including spin waves) than we have considered.

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[13]Where $h\equiv|\vec{\nabla}u|^{-1}$, $K\equiv-\vec{\nabla}(h\vec{\nabla}u)$, and $v\equiv-h\,\partial u/\partial t$. See Refs. 9 and 12, and G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1970), Chap. 2.

[14]References 1 and 2 have an approximate step-function-like profile for $\psi(u)$.

[15]Note that our treatment is in the continuum limit. Lattice effects could be important, particularly at low temperatures where the surface tension is anisotropic.

[16]Other distributions can have interesting properties: D. Andelman, Phys. Rev. B 27, 3079 (1983), and references therein.

[17]This removes the Λ^d dependence from ν_0^2. There may also be some weak dependence on dimensionality within ν^2 through, for example, the coordination number of a d-dimensional lattice. If temperature effects are important the lattice constant is replaced by the interfacial thickness (Ref. 12).

[18]For convenience, $\bar{R}\equiv(\pi\mathscr{A})^{-1}$ in $d=3$, and $\bar{R}\equiv(4\mathscr{A})^{-1}$ in $d=2$.

[19]The growth law is unchanged for $d=2+\epsilon$, where $\epsilon\ll 1$.

[20]By this we mean the reciprocal of the line length per unit area has such a time dependence.

[21]Metastable states leading to hysteresis effects may be important for $\tau>\tau_c$. The possible existence of metastable states was suggested to us by K. Binder (private communication).

[22]$\mathscr{S}(k,t)$ gives the time-dependent *elastic* scattering in the evolving system. This is in contrast to the inelastic scattering, which is often denoted "$\mathscr{S}(k,\omega)$."

[23]This implies a rough estimate of the time scale over which Eq. (5), for $d=3$, is valid. When the peak has sharpened so that the width is of the order determined by thermodynamic fluctuations, the self-similar regime is over. This is probably an overestimate, however. The valid time scale for Eq. (5) could be smaller because of the onset of metastable, hysteresis effects.

[24]This follows even if domain growth stops at τ_c. Domains of size $\bar{r}_c/\Lambda$ would be too small to create long-range order in the thermodynamic limit.

[25]Our analysis has been in the continuum limit. Lattice models can have different dependencies on the field strength [J. F. Fernandez *et al.,* Phys. Rev. Lett. 51, 203 (1983)], e.g., for a lattice model, it is possible that $\nu^2\rightarrow\nu^\alpha$ ($\alpha<2$) in the formulas given herein.