Magnetic susceptibilities of cluster-hierarchical models

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The exact magnetic susceptibilities of hierarchical models are calculated near and away from criticality, in both the ordered and disordered phases. The mechanism and phenomenology are discussed for models with susceptibilities that are physically sensible, e.g., nondivergent away from criticality. Such models are found based upon the Niemeijer—van Leeuwen cluster renormalization. A recursion-matrix method is presented for the renormalization-group evaluation of response functions. Diagonalization of this matrix at fixed points provides simple criteria for well-behaved densities and response functions.

I. INTRODUCTION

A hierarchical model[1] is constructed by repeatedly replacing each single bond between interacting degrees of freedom with a multiply connected graph of many bonds (Fig. 1). As is evident from their construction, hierarchical models are solved exactly by the position-space renormalization-group method. Such models can exhibit phase transitions and nonclassical criticality at finite temperatures.[1,2]

Hierarchical models were originally inspired by position-space renormalization-group approximations for Bravais lattices. In fact, such approximations[3–5] that had been widely and successfully used are the exact solutions of corresponding hierarchical models; this realizability guarantees important physical requirements.[1] Furthermore, an unlimited variety of hierarchical models can now be constructed and used as a testing ground for new concepts. For example, with the introduction of frustration, chaotic renormalization-group trajectories were obtained for the first time, leading to a microscopic picture of a spin-glass phase.[6] Another example is defect structures embedded inside hierarchical models that exhibit novel ordering behavior.[7]

An immediate visual distinction between models on Bravais lattices and hierarchical models is that the latter are not translationally invariant. As described above, different classes of sites have different coordination numbers, smaller and smaller numbers of sites having larger and larger coordination numbers. For example, consider the hierarchical model of Fig. 1 (where the recursion relation of the Migdal-Kadanoff renormalization-group approximation[3] finds an exact application): The mth level of the hierarchy contains $2^{2L-2m+1}$ sites of coordination number 2^{m+1}, where $L\to\infty$ is the total number of levels. Accordingly, the average coordination number of the entire system is $q=3$.

An important thermodynamic consequence of the highly coordinated sites is that the magnetic susceptibility of the disordered phase is infinite. This was noted by Kaufman and Griffiths,[8] who presented an analysis of the infinite-temperature limit under an applied magnetic field, and extended this anomalous susceptibility result to finite temperatures by invoking the Griffiths-Kelly-Sherman (GKS) inequalities. In the present paper, a renormalization-group analysis of the finite-temperature region is given, reaching agreement with the previous work.[8] Then, a family of hierarchical models with more regular coordination is introduced and a criterion for physically reasonable susceptibilities is derived. These new models are inspired by the finite-cluster renormalization-group approximation of Niemeijer and van Leeuwen.[4] Our analysis formulates a compact recursion-matrix method for the renormalization-group evaluation of the response functions.

Our work can also be related to the previous work of Mazenko, Hirsch, Nolan, and Valls,[9] who studied the cumulant approximation of Niemeijer and van Leeuwen[4] and found anomalous algebraic decay of the correlation function in the high-temperature phase. Our criterion satisfies (and is more restrictive than) the criterion given by these workers[9] for normal exponential decay of the correlation function. Thus it appears that the correlation functions of the new models will also be well-behaved. This is

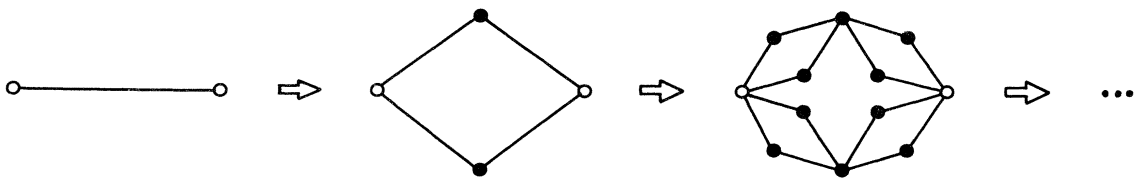

FIG. 1. Construction of the hierarchical model (Ref. 1) that is solved exactly with the Migdal—Kadanoff recursion relation.

important since, with the introduction of frustration, these models will be used to explore the response and correlation functions of the spin-glass phase resulting from chaotic renormalization-group trajectories.[6,10]

The outline of this paper is as follows. In Sec. II the recursion matrix is formulated and the mechanism behind the anomalous response functions is exhibited. In Sec. III the cluster-hierarchical models are introduced and the criterion is established for well-behaved susceptibilities in the disordered phase. The ordered phase is studied in Sec. IV. Thus magnetic susceptibilities are exhibited along the entire temperature range.

II. CALCULATION OF SUSCEPTIBILITIES BY THE RECURSION-MATRIX METHOD

The densities and response functions of two consecutive points along a renormalization-group trajectory can be linked by a recursion matrix, which is constructed from first and second derivatives of the recursion relations. At a fixed point of the transformation, corresponding to a phase transition or a phase sink,[11] the densities and response functions are the components of the left eigenvector of this matrix with eigenvalue equal to the volume rescaling factor B. Unphysical behavior can thus be easily detected by diagonalizing the recursion matrix at a fixed point.

We first give the general formalism. The partition function of a system can be expressed either in terms of the original interactions, $Z(\{K_\alpha\})$, or in terms of the renormalized (primed) interactions, $Z(\{K'_\alpha\})$. Conjugate to each interaction K_α, there is a density M_α (e.g., internal energy, magnetization)

$$M_\alpha=\frac{1}{N_\alpha}\frac{\partial \ln Z}{\partial K_\alpha}\ , \tag{1}$$

where N_α is the total number of α-type interactions in the system. With the use of the two expressions for the partition function and the chain rule, the density recursion is obtained,[12]

$$M_\alpha=B^{-1}\sum_\beta M'_\beta T_{\beta\alpha},\quad T_{\beta\alpha}\equiv\frac{N_\beta}{N_\alpha}\frac{\partial K'_\beta}{\partial K_\alpha}\ . \tag{2}$$

At a fixed point, $M_\alpha=M'_\alpha\equiv M^*_\alpha$ and Eq. (2) becomes[5(b),13] the condition for the left eigenvector of T with eigenvalue B. Note that $\tilde{M}$ is nonzero, since it contains the trivial density $\langle 1\rangle$ associated with the additive constant (G) in the Hamiltonian, which is inevitably generated in a renormalization-group transformation. The densities of an ordinary point are computed by iterating Eq. (2) until a fixed point is effectively reached. The matrix T is evaluated at each successive $\{K_\alpha\}$, so that a numerically different matrix has to be multiplied for each iteration until the fixed point is reached.

Extending this approach to response functions

$$\chi_{\alpha\beta}\equiv\left[\frac{N_\alpha}{N_\beta}\right]^{1/2}\frac{\partial M_\alpha}{\partial K_\beta}=\left[\frac{1}{N_\alpha N_\beta}\right]^{1/2}\frac{\partial^2 \ln Z}{\partial K_\alpha \partial K_\beta} \tag{3}$$

yields

$$\chi_{\alpha\beta}=B^{-1}\left[\sum_{\lambda,\mu}\left[\frac{N_\lambda N_\mu}{N_\alpha N_\beta}\right]^{1/2}\chi'_{\lambda\mu}\frac{\partial K'_\lambda}{\partial K_\alpha}\frac{\partial K'_\mu}{\partial K_\beta}+\sum_\lambda\frac{N_\lambda}{(N_\alpha N_\beta)^{1/2}}M'_\lambda\frac{\partial^2 K'_\lambda}{\partial K_\alpha \partial K_\beta}\right], \tag{4}$$

which, combined with Eq. (2), constitutes a recursion matrix τ operating to the left on a vector with components M_α and $\chi_{\alpha\beta}$. Densities and response functions at a fixed point are the components of the left eigenvector of τ with eigenvalue B, and those at an ordinary point are found by iterative matrix multiplication.[14] The above method can easily be applied with a position-space renormalization-group transformation that is either an approximation for a Bravais lattice, or the exact solution of a hierarchical model. Both situations are readily amenable to the method because the number of interaction types remains finite under renormalization.

In the hierarchical model of Fig. 1, the following Hamiltonian retains its form under renormalization:

$$-\beta\mathcal{H}=\sum_{\langle ij\rangle}[Js_is_j+H_B(s_i+s_j)+G]+H_S\sum_i s_i\ , \tag{5}$$

where $s_i=\pm 1$ is an Ising spin at site i and the sum $\langle ij\rangle$ includes all nearest-neighbor pairs of sites. Magnetic fields that are counted with bonds (H_B) and sites (H_S) must be distinguished due to the nonuniform coordination number. The renormalization-group transformation that solves the model is effected by summing over the two internal spins of the basic graph, thereby inducing direct interactions between the two external spins in order to preserve the free energy of the system. The recursion relations which embody this procedure are

$$\begin{aligned}&J'=\tfrac{1}{4}\ln(R_{++}R_{--}/R^2_{+-})\ ,\\ &H'_B=\tfrac{1}{4}\ln(R_{++}/R_{--}),\quad H'_S=H_S\ ,\\ &G'=BG+\tilde{G},\quad \tilde{G}=\tfrac{1}{4}\ln(R_{++}R_{--}R^2_{+-})\ ,\end{aligned} \tag{6}$$

where

$$\begin{aligned}&R_{++}=xy^2z+2y+x^{-1}z^{-1}\ ,\\ &R_{--}=xy^{-2}z^{-1}+2y^{-1}+x^{-1}z\ ,\\ &R_{+-}=zy+z+y^{-1}z^{-1},\quad x=e^{4J},\quad y=e^{4H_B},\quad z=e^{2H_S}\ .\end{aligned} \tag{7}$$

Under rescaling, a finite site field H_S always induces a finite bond field H'_B, while itself remaining invariant, $H'_S=H_S$. The site field is included in this study because it is the proper generating field for a single-spin expectation value. The up-down symmetric subspace, $H_S=H_B=0$, is conserved under renormalization. Within this subspace, derivatives even in H are in general nonzero, yielding a recursion matrix as given in Table I.

All renormalization-group trajectories initiated in the high-temperature ($J<J_c$) phase flow to the infinite-temperature fixed point ($J^*=0$), which is the sink of this disordered phase. There, for the model of Fig. 1, the recursion matrix reduces to

TABLE I. Recursion matrix τ used in the evaluation of densities and susceptibilities. This matrix operates to the left on the vector $(1;\langle s_i s_j\rangle,\langle(s_i+s_j)\rangle,\langle s_i\rangle;\chi_{BB},\chi_{BS},\chi_{SS})$.

B	$\frac{\partial G'}{\partial J}$	0	0	$\frac{\partial^2 G'}{\partial H_B^2}$	$\left[\frac{q}{2}\right]^{1/2}\frac{\partial^2 G'}{\partial H_B \partial H_S}$	$\frac{q}{2}\frac{\partial^2 G'}{\partial H_S^2}$
0	$\frac{\partial J'}{\partial J}$	0	0	$\frac{\partial^2 J'}{\partial H_B^2}$	$\left[\frac{q}{2}\right]^{1/2}\frac{\partial^2 J'}{\partial H_B \partial H_S}$	$\frac{q}{2}\frac{\partial^2 J'}{\partial H_S^2}$
0	0	$\frac{\partial H_B'}{\partial H_B}$	$\frac{q}{2}\frac{\partial H_B'}{\partial H_S}$	0	0	0
0	0	0	$\frac{\partial H_S'}{\partial H_S}$	0	0	0
0	0	0	0	$\left[\frac{\partial H_B'}{\partial H_B}\right]^2$	$\left[\frac{q}{2}\right]^{1/2}\frac{\partial H_B'}{\partial H_B}\frac{\partial H_B'}{\partial H_S}$	$\frac{q}{2}\left[\frac{\partial H_B'}{\partial H_S}\right]^2$
0	0	0	0	0	$\frac{\partial H_B'}{\partial H_B}\frac{\partial H_S'}{\partial H_S}$	$(2q)^{1/2}\frac{\partial H_B'}{\partial H_S}\frac{\partial H_S'}{\partial H_S}$
0	0	0	0	0	0	$\left[\frac{\partial H_S'}{\partial H_S}\right]^2$

$$\tau=\begin{bmatrix} 4&0&0&0&8&2\sqrt{6}&3\\ 0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&4&0&0\\ 0&0&0&0&0&2&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \tag{8}$$

operating to the left on $(1;\langle s_i s_j\rangle,\langle(s_i+s_j)\rangle,\langle s_i\rangle;\chi_{BB},\chi_{BS},\chi_{SS})$, where the three susceptibilities are second derivatives of the free energy with respect to bond and/or site fields, according to Eq. (3). (Our approach is equally applicable to the second derivative with respect to J, yielding the specific heat.) From the appropriate left eigenvector, it is seen that χ_{BB} is infinite, whereas the other susceptibilities are finite: $\chi_{BS}=\sqrt{6}$ and $\chi_{SS}=1$. It could be argued that χ_{SS} is the important susceptibility, since it corresponds to the physically interesting generating field. However, for any temperature less than the trivial case of infinity, the recursion matrix mixes the three susceptibilities, so they all become infinite. The physical reason behind this is that the highly coordinated sites feel a very large applied field channeled through their many neighbors. This result also holds true for the variant in which the site field is applied only at the lowest level of the hierarchy, so that $H_S'=0$. The susceptibilities for this case are $\chi_{BB}=\infty$, $\chi_{BS}=\sqrt{2}$, and $\chi_{SS}=1$ at $J^*=0$.

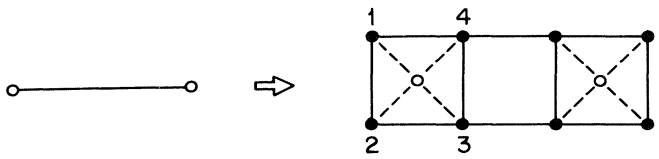

FIG. 2. Construction of the cluster-hierarchical model. In the graph to the right, the dashed lines represent the projection-operator part of the Hamiltonian. The full lines are the interactions Js_is_j.

III. CLUSTER-HIERARCHICAL MODEL

The anomalous susceptibility behavior described in the preceding section need not be characteristic of all hierarchical models. It is eliminated by constructing a model that avoids high coordinations. An example is shown in Fig. 2 and described below. This hierarchical model is related to Niemeijer and van Leeuwen's cluster approximation[4] applied to a square lattice, just as the preceding model is related to the Migdal-Kadanoff bond-moving approximation.[3] In considering this new model, we have also moved from position-space renormalization-group transformations closely related to a decimation to nonlinear transformation, in which a cell spin is introduced as a distinct entity from the site spins, via a projection operator.[15]

The closed circles in Fig. 2 are site spins $s_i=\pm1$, which are replaced by cell spins ($s_i'=\pm1$, represented by open circles) under rescaling. The cell spin is initially connected to its four site spins by a projection operator, which has the following most general form preserving the symmetries of the system:

$$\begin{aligned} P(s',s_{1-4})=\tfrac{1}{2}\{1+s'[&u_1(s_1+s_2)+u_2(s_3+s_4)\\ &+v_1(s_4s_1s_2+s_1s_2s_3)\\ &+v_2(s_2s_3s_4+s_3s_4s_1)]\}\ . \end{aligned} \tag{9}$$

The operator $P(s',s_{1-4})$ is represented by dashed lines in Fig. 2. The interactions Js_is_j between the site spins are represented by full lines, with an average coordination number $q=\frac{5}{2}$. The full Hamiltonian of the hierarchical model is

$$\begin{aligned} -\beta\mathscr{H}=&\sum_{\langle ij\rangle}[Js_i^{(0)}s_j^{(0)}+H_B(s_i^{(0)}+s_j^{(0)})+G]\\ &+H_S\sum_i s_i^{(0)}+\sum_{m,i}\ln P(s_i^{(m)},s_{1-4}^{(m-1)})\ , \end{aligned} \tag{10}$$

TABLE II. Recursion polynomials of the cluster-hierarchical model for the case $u_1=u_2=0$ and $v_1=v_2=1/4$. The polynomial R_{--} is obtained by changing the signs of n_{yp} and n_{zp} in R_{++}.

Polynomials

$$R_{++}=\sum_p (A^{(p)}_{++}/8)(x^{n_{xp}})^{1/2}(y^{n_{yp}})^{1/2}z^{n_{zp}}$$

$$R_{+-}=\sum_p (A^{(p)}_{+-}/8)(x^{n_{xp}})^{1/2}(y^{n_{yp}})^{1/2}z^{n_{zp}}$$

$(A^{(p)}_{++},A^{(p)}_{+-};n_{xp},n_{yp},n_{zp})$		
(8,0;5,10,4)	(28,12;1,2,1)	(36,20;0,−5,−2)
(8,12;3,8,3)	(2,2;1,2,0)	(1,3;−1,4,2)
(8,4;3,6,2)	(10,14;1,0,0)	(12,24;−1,2,1)
(0,8;3,0,0)	(2,2;1,−2,0)	(34,46;−1,0,0)
(0,4;3,−6,−2)	(12,12;1,−2,−1)	(36,24;−1,−2,−1)
(0,12;3,−8,−3)	(12,8;1,−4,−1)	(9,3;−1,−4,−2)
(8,12;2,7,3)	(9,7;1,−4,−2)	(8,16;−2,3,1)
(16,8;2,5,2)	(18,6;1,−6,−2)	(8,8;−2,1,0)
(24,4;2,3,1)	(20,20;0,5,2)	(8,8;−2,−1,0)
(0,4;2,−3,−1)	(12,24;0,3,1)	(24,16;−2,−3,−1)
(0,8;2,−5,−2)	(4,8;0,1,1)	(4,8;−3,2,1)
(0,12;2,−7,−3)	(20,28;0,1,0)	(12,12;−3,0,0)
(2,6;1,6,2)	(20,28;0,−1,0)	(12,8;−3,−2,−1)
(9,7;1,4,2)	(12,8;0,−1,−1)	(4,4;−5,0,0)
(4,8;1,4,1)	(36,24;0,−3,−1)	

where (m) indicates the level of the hierarchy.

The recursion relations are again derived by summing over internal (site) spins within a basic unit to obtain effective interactions involving only the two external (cell) spins. Equations (6) are applicable here with $H'_S=0$ as the only modification, since the site fields are applied at the lowest level of the hierarchy (which in itself does not guarantee well-behaved susceptibilities, as seen in the previous example). The polynomials R are longer than their previous counterparts, but present no essential difficulty. An example is given in Table II. Again the zero-field subspace, closed under the recursion relations, is studied and the recursion matrix is as given in Table I, with the added simplification of $\partial H'_S/\partial H_S=0$. Eigenvector analysis at the disorderd sink $J^*=0$ yields the susceptibilities

$$\chi_{BB}=\frac{26}{5}+\frac{32u^2}{1-10u^2},\quad \chi_{BS}=\sqrt{5}+\frac{16u(u_1+u_2)}{\sqrt{5}(1-10u^2)},$$

$$\chi_{SS}=1+\frac{8(u_1+u_2)^2}{5(1-10u^2)},\quad u\equiv(2u_1+3u_2)/5\ . \tag{11}$$

In order to simulate well a Bravais lattice, we choose $u_1=u_2=0$, giving the susceptibilities of a decoupled spin. Furthermore, to force complete alignment throughout the hierarchy at $J^*=\infty$ (i.e., to preserve the ground state under the renormalization-group transformation), we choose $u_1+u_2+v_1+v_2=\frac{1}{2}$. Within these restrictions, one free parameter is still available, which we set by $v_1=v_2$ in the present study. Now the model is fully specified, and its exact renormalization-group solution yields satisfactory response functions.

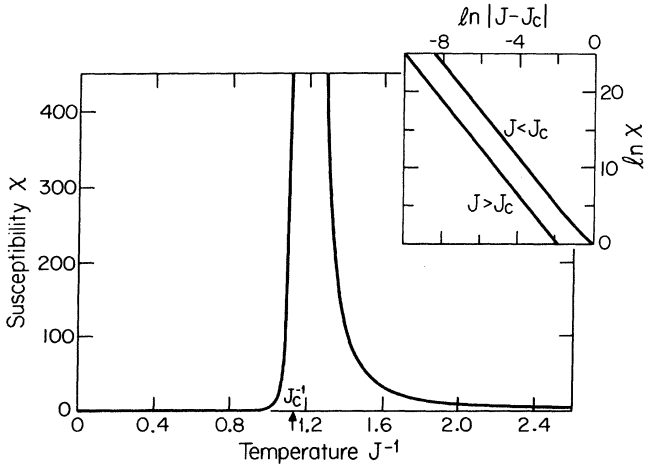

FIG. 3. Calculated susceptibility χ_{SS} of the cluster-hierarchical model.

IV. LOW-TEMPERATURE ORDERED PHASE

The same eigenvector analysis is done with the matrix τ at the sink of the ordered phase, which occurs at zero temperature, $J^*=\infty$. It shows that the susceptibilities are well-behaved at low temperatures ($J>J_c$) in either of the two models discussed. At the sink of the cluster-hierarchical model,

$$\tau=\begin{bmatrix} 10 & 8 & 0 & 0 & B_1 & B_2 & B_3 \\ 0 & 2 & 0 & 0 & -B_1 & -B_2 & -B_3 \\ 0 & 0 & 10 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 20\sqrt{5} & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

with

$$B_1 = 16(25-50u_2+14v_1)/(5-8u_2)\ ,$$
$$B_2 = 24\sqrt{5}(3-6u_2+2v_1)/(5-8u_2)\ ,$$
$$B_3 = 5(13-26u_2+10v_1)/(5-8u_2)\ ,$$

where only the restriction that the ground state be preserved under the transformation $(u_1+u_2+v_1+v_2=1/2)$ was implemented. The recursion matrix in Eq. (12) gives the expected zero-temperature densities and susceptibilities in its eigenvectors

$$(1;\langle s_i s_j\rangle,\langle (s_i+s_j)\rangle,\langle s_i\rangle;\chi_{BB},\chi_{BS},\chi_{SS})$$
$$=(1;1,2M_S,M_S;\chi_{SS}/2,\chi_{SS}/2\sqrt{5},\chi_{SS})\ , \quad (13)$$

where M_S is arbitrary and χ_{SS} is zero or infinity. A single ordered phase corresponds to $M_S=1$, $\chi_{SS}=0$, whereas two-phase coexistence corresponds to $-1<M_S<+1$, $\chi_{SS}=\infty$. Physically sensible behavior of these quantities is obtained throughout the low-temperature phase, using the procedure described in Sec. II.

The susceptibility χ_{SS} is displayed in Fig. 3 along the entire temperature range. In the critical region, the power-law divergence is

$$\chi_{SS}\sim |J-J_c|^{-\gamma},\quad \gamma=3.05\ . \quad (14)$$

Identical values of the critical exponent γ are determined from either of the log-log plots (inset, Fig. 3) obtained numerically by the recursion-matrix method along the flows to either phase-sink fixed point $J^*=0,\infty$, and from

$$\gamma=\left[2\ln\left[\frac{\partial H_B'}{\partial H_B}\right]-\ln B\right]\Big/\ln\left[\frac{\partial J'}{\partial J}\right]\ , \quad (15)$$

reflecting standard eigenvalue analysis[16] at the unstable critical[17] fixed point $J^*=J_c=0.882$. The derivatives in Eq. (15) correspond to the leading magnetic and thermal eigenvalues of the recursion matrix, and $B=10$, in the present model.

At nonzero fields, the odd derivatives do not vanish, and the recursion matrix becomes quite cumbersome. It is therefore more convenient to use the alternate method of directly calculating[18] the free energy per bond,

$$f=\sum_{n=1}^{\infty} B^{-n}\tilde{G}[J^{(n-1)},\ H_B^{(n-1)},\ H_S^{(n-1)}]\ , \quad (16)$$

where a given term is the contribution of the nth renormalization, and then taking two field derivatives numerically. Conversely, the latter method cannot be used directly at zero fields in the ordered phase, since the numerical derivative picks up for susceptibility of the infinite slope corresponding to the magnetization discontinuity. What is desired, of course, is the slope of the $M(H)$ curve just before the discontinuity, which is obtained by doing the calculation at infinitesimal H. Identical values are obtained more readily by the recursion-matrix method at zero fields. Thus the two methods are truly consistent and complementary.

Finally, Mazenko, Hirsch, Nolan, and Valls[9] have given criteria for exponentially decaying correlation functions away from criticality. When applied to our cluster-hierarchical model, these translate into

$$u_2=0$$

and (17)

$$u_1+u_2+v_1+v_2=\tfrac{1}{2}\ ,$$

which are consistent with our choices. Thus, it appears that the correlation function will be physically sensible within both the ordered and disordered phases. Such a study will be presented separately.

ACKNOWLEDGMENTS

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obtained by a one-time bond-moving approximation (Ref. 5) from the square lattice. As such, it could be considered an approximate treatment of the square-lattice Ising model (generalization to hypercubic lattices is obvious). The resulting critical temperature $J_c^{-1}=1.81$ is to be compared with the exact value of $J_c^{-1}=2.27$.

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