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Comment on "Specific sine-Gordon soliton dynamics in the presence of external driving forces"

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It is shown that the results presented by Reinisch and Fernandez have an alternative interpretation where the soliton does behave as a Newtonian particle. The key features required for this alternative interpretation are (i) properly defining where the center of the soliton is, and (ii) expanding the solution so as to avoid any secular terms. When these two objectives are achieved, then the center of the soliton is found to satisfy Newton's equation of motion for a point particle.

Recently there has been some controversy about the validity of soliton perturbation theories and the interpretation of a soliton as a particle. This was first noted in Ko and Kuehl's[1] study of the $K\,dV$ equation with time-dependent coefficients. Their result for the position of a soliton x_m, when transformed into the notation of Kaup and Newell,[2] gave

$$\frac{dx_m}{dt}=4\eta^2-\frac{\Gamma}{3\eta} \quad ,$$

where $2\eta^2$ is the amplitude of the soliton and Γ is the damping. On the other hand, a soliton perturbation theory found[2]

$$\frac{d\bar{x}}{dt}=4\eta^2+\frac{\Gamma}{3\eta} \quad ,$$

where $\bar{x}$ is Kaup and Newell's position for the soliton. These sign differences are real, and numerical results did support Ko and Kuehl's result.[3]

More recently Reinisch and Fernandez have numerically studied[4] the sine-Gordon kink under the influence of a constant torque. They also found their numerical results at variance with the predictions of soliton perturbation theories[5–10] and have proposed to explain this by declaring the soliton to be a *non*-Newtonian particle. What I propose is that one does not have to be that drastic and also that the Ko and Kuehl observation and the Reinisch and Fernandez observation may have a common explanation.

First, let me state some facts, then I shall give my interpretation of these results.

(1) The soliton or kink is *not* rigid and is *not* a "point particle."[11] (Therefore one must qualify to what extent one is referring to it as a "Newtonian" or a "non-Newtonian" particle. Should one look at the short-time or long-time scales to see this?)

(2) Any "extended particle" will respond with a time delay to an externally applied force.[12] (This is also verified by Reinisch and Fernandez's numerical results. To the extent that the soliton is not a point particle, one could say that the soliton was non-Newtonian. In this respect, Reinisch and Fernandez were correct. What they observed were the combined transient effects of a soliton reshaping itself[11] and experiencing a time delay.)

(3) The expansion used by Reinisch and Fernandez contained secular terms. (The presence of secular terms limits the validity of their expansion to short-time scales.)

(4) The concept of a "soliton" comes from considering the solution for $t\to+\infty$, whereby the general solution separates into "a collection of solitons in a sea of radiation." (Thus to identify or locate a soliton, one should use a solution valid for *large*-time scales, not short-time scales.)

(5) The definition of the center of a soliton used by Reinisch and Fernandez is different from the definition used in soliton perturbation theories.

Now what I want to do here is to present an alternative interpretation of the Reinisch and Fernandez result.[4] As they did, I start with the perturbed sine-Gordon equation

$$u_{tt}-u_{xx}+\sin u=\epsilon R(x,t) \quad . \tag{1}$$

I now difffer from their procedure and instead expand u as

$$u(x,t)=U_0(\chi)+\epsilon u^{(1)}(\chi,t)+\dots \quad , \tag{2}$$

where χ is to be determined and $U_0(\chi)$ is exactly the one-soliton soliton given by

$$U_0(\chi)=4\tan^{-1}(e^{\pm\chi}) \quad . \tag{3}$$

χ shall be defined such that no secular terms will appear in (2). I shall define the center of the soliton to be at the center of U_0, which is where $\chi=0$. This also differs from the definition of the center according to Reinisch and Fernandez, who took it to be where $u_x(x,t)$ was a maximum. To avoid relativistic effects and to maintain simplicity, I shall take $\chi_t\leq O(\epsilon)$, and require that

$$\chi_x^2=1+\chi_t^2 \quad . \tag{4}$$

Then from (1), (2), and (4), the first-order result is

$$U_{0\chi}\chi_{tt}+\epsilon u_{tt}^{(1)}+\epsilon L u^{(1)}=\epsilon R \quad , \tag{5}$$

where

$$L=-\partial_\chi^2+\cos U_0(\chi) \quad . \tag{6}$$

The operator L has one zero eigenvalue, which is a bound state whose eigenfunction is proportional to $U_{0\chi}$.[7] If I now demand that $u^{(1)}$ must not contain any secular terms in this (first) order, then $u^{(11)}$ must be orthogonal to this bound-state eigenfunction. Thus I take

$$u^{(1)}(\chi,t)=\int_{-\infty}^{\infty}dk\,a_k(t)f_k(\chi) \quad , \tag{7}$$

whence both χ and a_k are uniquely determined by

$$\chi_{tt}=\epsilon\frac{\int_{-\infty}^{\infty}f_b(\chi)R(x,t)\,d\chi}{\int_{-\infty}^{\infty}f_b(\chi)U_{0\chi}(\chi)\,d\chi} \quad , \tag{8}$$

and

$$a_{ktt}+\omega_k^2 a_k=\int_{-\infty}^{\infty} f_k(\chi)R(x,t)\,d\chi \quad . \tag{9}$$

In the above, f_b and f_k are the eigenfunctions of L (Ref. 7) and $\omega_k^2=1+k^2$. Equations (8) and (9) are the results for a general forcing term $R(x,t)$. As one may see from (8), the acceleration χ_{tt} of the soliton is directly proportional to the bound-state component of $R(x,t)$, while from (9), the amplitude of the continuous spectrum is driven by the kth component of $R(x,t)$.

Now, as in Ref. 4, let us take R independent of x, so that we have a constant torque being applied to the sine-Gordon field. Also, take $R=0$ if $t<0$ and R constant for $t>0$. Then for $t>0$, (8) yields

$$\chi_{tt}=\pm\epsilon\frac{\pi}{4}R \quad , \tag{10}$$

from which we obtain

$$\chi=x\pm\frac{\pi}{4}\epsilon R\frac{t^2}{2} \quad . \tag{11}$$

Since the center of the soliton is at $\chi=0$, it then follows that the soliton (our definition of the center at least) does behave as a Newtonian particle.[2,5–11]

However, as was indeed pointed out in Ref. 4, *such is not observed.* And to understand what has occurred in these numerical experiments,[4] we must include the effects of the continuous spectrum. From (9), one can readily obtain[4,7]

$$u^{(1)}(\chi,t)+-\frac{\epsilon}{2}R\int_{-\infty}^{\infty}dk\;G(k,\chi)[1-\cos(\omega_k t)] \quad , \tag{12}$$

where

$$G(k,\chi)=\frac{k\cos(k\chi)-\sin(k\chi)\tanh\chi}{(1+k^2)^2\sinh(\pi k/2)} \quad . \tag{13}$$

In Ref. 4, the integral in (12) is evaluated by contour integration and is reduced to an infinite series. However, that infinite series is only convergent if one is *outside* the light cone. Inside the light cone, one must use other techniques. For large times $u^{(1)}$ will conveniently separate into the two parts

$$u^{(1)}=-\frac{\epsilon}{2}R\int_{-\infty}^{\infty}dk\;G(k,\chi)+\frac{\epsilon}{2}R\int_{-\infty}^{\infty}dk\;G(k,\chi)\cos(\omega_k t) \quad . \tag{14}$$

The first part is time independent, and corresponds to a permanent change in the soliton's shape. The second term may be evaluated by stationary phase, and represents outward traveling radiation.

If instead we are interested in short-time scales, then we may expand (12) in a Taylor series, obtaining

$$u^{(1)}(\chi,t)=-\frac{\epsilon}{4}Rt^2\int_{-\infty}^{\infty}dk(1+k^2)G(k,\chi)+O(t^4) \quad , \tag{15}$$

which evaluates to

$$u^{(1)}(\chi,t)=\frac{\epsilon t^2}{4}R\left(2-\frac{\pi}{\cosh\chi}\right)+\cdots \quad . \tag{16}$$

Now from (2), (11), and (16), we have

$$u(x,t)=U_0\left(x\pm\frac{\pi}{8}\epsilon Rt^2\right)+\frac{\epsilon t^2}{4}R\left(2-\frac{\pi}{\cosh\chi}\right)+O(t^4) \quad . \tag{17}$$

Since we have evaluated (12) by a Taylor's series expansion in t, we may as well do the same for the soliton part, noting that $U_{0\chi}=\pm 2/\cosh\chi$. Whence

$$u(x,t)=U_0(x)+\tfrac{1}{2}\epsilon t^2 R+O(t^4) \quad . \tag{18}$$

Naturally, Eq. (18) is exactly the same result as that obtained in Ref. 4. However, I have obtained it via a different definition and interpretation. I interpret Eq. (17) as a Newtonian particle moving with a constant acceleration, and with radiation being created at a rate proportional to t^2 on short-time scales. What will be observed numerically is shown in Eq. (18). As shown by Eq. (18), the buildup of the above-created radiation will exactly cancel the soliton motion, causing the soliton to seem to hang motionless.

One can also explain this by considering the various time scale involved. In a Taylor series expansion as in Eq. (18), one is implicitly considering the response of the system on a very-short-time scale, at least faster than the time required for a signal to cross the width of the soliton. On such a short-time scale, for example, the left side of the soliton will not know what has happened on the right side of the soliton. Thus whatever happens on the right side cannot effect the left side. Thus each element of the sine-Gordon field will respond *independently* of all other elements. To make this clearer, consider now Eq. (1) on this short-time scale, and in the rest frame of the soliton. Since we start with an equilibrium state, we have $u_{xx}-\sin u=0$, at least on this time scale, which leaves only

$$u_{tt}=\epsilon R(x,t) \quad . \tag{19}$$

What (19) demonstrates is simply that the response of u at x is independent of what u is at another value of x. Each element of u is responding like a free particle, independent of all other elements, and its response is only determined by the value of the forcing term at the position of that element. In other words, the concepts of solitons and radiation are only of value when one is concerned with or interested in the intermediate or long-time behavior. On the short-time scales, the soliton concept is of less value than the field concept, as was demonstrated by Eq. (19).

I also suggest that a similar analysis of the $K\,dV$ equation may well explain Ko and Kuehl's[1] result, but that remains to be seen.

[1]K. Ko and H. H. Kuehl, Phys. Rev. Lett. 40, 233 (1978).
[2]D. J. Kaup and A. C. Newell, Proc. R. Soc. London, Ser. A 361, 413 (1978).
[3]H. H. Kuehl (private communication).
[4]G. Reinisch and J. C. Fernandez, Phys. Rev. B 24, 835 (1981).
[5]D. J. Kaup, SIAM J. Appl. Math. 31, 121 (1976).
[6]J. P. Keener and D. W. McLaughlin, Phys. Rev. A 16, 777 (1977); J. Math. Phys. 18, 2008 (1977).
[7]A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica D 1, 1 (1980).
[8]D. W. McLaughlin and A. C. Scott, Phys. Rev. A 18, 1652 (1978).
[9]*Solitons in Action,* edited by K. Lonngren and A. C. Scott (Academic, New York, 1978).
[10]*Solitons,* edited by R. K. Bullough and P. J. Caudrey (Springer, New York, 1980).
[11]M. B. Fogel, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. B 15, 1578 (1977).
[12]D. J. Kaup, Phys. Rev. 152, 1130 (1966), and references therein.