PHYSICAL REVIEW B VOLUME 28, NUMBER 2 15 JULY 1983

Frequency shift and attenuation length of a Rayleigh wave due to surface roughness

A. G. Eguiluz and A. A. Maradudin
Department of Physics, University of California—Irvine, Irvine, California 92717
(Received 24 March 1982; revised manuscript received 7 February 1983)

On the basis of Rayleigh's method we study the effect of surface roughness on the dispersion relation of a Rayleigh wave on an isotropic medium. The stress-free boundary conditions (applied at the rough interface) lead us to an integral eigenvalue equation for the elastic displacement field. That equation reduces to a simpler algebraic equation upon averaging the displacement field over an ensemble of rough surfaces. We obtain explicit expressions for the roughness-induced perturbation of the flat-surface dispersion relation $\omega_0(q_{\|})=c_R q_{\|}$. (Here c_R is the speed of a Rayleigh wave and $q_{\|}$ is a two-dimensional wave vector.) The real part of this perturbation measures the shift in the frequency away from $\omega_0(q_{\|})$ while its imaginary part measures the lifetime (or the inverse attenuation length) of the Rayleigh wave. We present detailed numerical results for both the shift and the lifetime. We show that the decay of a Rayleigh wave into bulk elastic waves is a much more efficient mechanism than its decay into other Rayleigh surface waves. An explanation for this disagreement with the earlier results of Maradudin and Mills is presented.

I. INTRODUCTION

A Rayleigh wave is a solution of the equations of motion of a semi-infinite elastic medium, bounded by a planar, stress-free surface. It propagates in a wavelike fashion in directions parallel to the surface, but its amplitude decays exponentially with increasing distance into the medium from the surface. It is, therefore, a wave localized at the surface of the elastic medium—a surface wave.

Rayleigh waves are of current theoretical and experimental interest both because of their utility in device applications, and because through studies of their propagation one can examine properties of solids in the near vicinity of their surface.

As Rayleigh waves propagate along solid surfaces they are attenuated. An important mechanism for this attenuation is surface roughness, which is found in varying degrees on all solid surfaces, even carefully prepared ones. The first study of this attenuation mechanism was carried out by Urazakov and Fal'kovskii,[1] on the basis of Rayleigh's method.[2] In this method the surface-localized elastic displacement field obtained from the equations of motion in the region beyond the maximum amplitude of the surface-roughness profile is continued in to the surface itself, and the stress-free boundary conditions are applied to this field along the surface. In a subsequent work Maradudin and Mills[3] used a Green's-function method to solve the same problem. Both sets of authors find that the attenuation rate is proportional to the fifth power of the frequency of the Rayleigh wave, in the limit that the wavelength of the Rayleigh wave is longer than the transverse correlation length of the surface roughness. The latter is a measure of the mean distance between consecutive peaks and valleys on the surface. The frequency variation of the attenuation rate then becomes much slower than the ω_R^5 law as the wavelength of the Rayleigh wave becomes comparable to, or shorter than, the transverse correlation length.

In this paper we examine the propagation of Rayleigh waves over a rough surface by Rayleigh's method. There are two principal reasons for doing so. The first is that the two preceding studies of this problem, by Urazakov and Fal'kovskii[1] and by Maradudin and Mills,[3] were carried out independently by different methods. It is of interest to compare the two approaches to this problem, and the results to which they give rise, but such a comparison requires more explicit results from the Rayleigh approach than are provided in the paper by Urazakov and Fal'kovskii. The second reason for the present calculation is the fact that as a Rayleigh wave propagates along a rough surface not only is it attenuated but its frequency changes as well. The roughness-induced shift in the frequency of the Rayleigh wave can be obtained by Rayleigh's method, along with the attenuation. This question was not studied by Urazakov and Fal'kovskii, however. The Green's-function method of Maradudin and Mills yields the attenuation but not the frequency shift. Knowledge of the frequency dependence of the roughness-induced shift in the Rayleigh-wave frequency provides an additional means for the study of properties of a solid in the vicinity of its surface or of the surface itself. Thus in the present work we will obtain both the roughness-induced frequency shift and attenuation of a Rayleigh wave.

In order to obtain results that are as explicit as possible, the work in this paper is based on an isotropic elastic medium, for which most of the calculations can be carried out analytically.

The outline of this paper is as follows. In Sec. II we formulate the problem of the propagation of a Rayleigh wave at the surface of an isotropic elastic medium in the case that the surface is rough. The stress-free boundary conditions, applied at each point of the rough surface, lead us to an eigenvalue problem for the coefficients of the solution of the equation of motion for the elastic displacement field. That eigenvalue problem consists of a homogeneous integral equation for the aforementioned coefficients. The surface-roughness profile—a random function—enters the kernel of this integral equation [which is valid to $O(\delta^2)$, where δ is the root-mean-square departure of the surface from flatness]. Averaging over an ensemble of rough surfaces we reduce the eigenvalue problem to a much simpler algebraic (matrix) equation. From the solubility condition for such a matrix equation we obtain, in Sec. III, the dispersion relation for a Rayleigh wave in the presence of roughness. In particular, we obtain the roughness-induced perturbation $\Delta\omega(q_{\|})$ of the dispersion relation $\omega_0(q_{\|})$ that applies when the surface is flat. In Sec. IV we obtain explicit expressions for the real $[\nu_1(q_{\|})]$ and imaginary $[\nu_2(q_{\|})]$ parts of $\Delta\omega(q_{\|})$. The former measures the shift in the frequency of the Rayleigh wave away from $\omega_0(q_{\|})$. The latter measures the lifetime of the wave or, equivalently, its inverse attenuation length. We present detailed numerical results for both $\nu_1(q_{\|})$ and $\nu_2(q_{\|})$. We find that, contrary to the conclusions of Maradudin and Mills,[3] the bulk-wave channels provide a much more efficient mechanism for the decay of a Rayleigh wave than the surface-wave channel. We address this discrepancy in detail in Sec. IV and Appendix B, and show how the method of Ref. 3, if correctly implemented, yields results that agree with ours.

II. THE PROPAGATION OF A RAYLEIGH WAVE ALONG A ROUGH SURFACE

A. Statement of the problem

The normal modes of vibration of an elastic medium are solutions to the following homogeneous differential equation:

$$\sum_{\mu} L_{\alpha\mu}(\vec{x}\,|\,\omega)u_{\mu}(\vec{x}\,|\,\omega)=0\ , \tag{2.1a}$$

where $u_{\mu}(\vec{x}\,|\,\omega)$ is the frequency Fourier transform of the elastic displacement field, and the operator $L_{\alpha\beta}(\vec{x}\,|\,\omega)$ is given by

$$L_{\alpha\mu}(\vec{x}\,|\,\omega)=\omega^2\delta_{\alpha\mu}+\frac{1}{\rho}\sum_{\beta,\nu} c_{\alpha\beta\mu\nu}\frac{\partial^2}{\partial x_\beta \partial x_\nu}\ . \tag{2.1b}$$

In Eq. (2.1b) we have denoted by ρ the mass density of the elastic medium and by $c_{\alpha\beta\mu\nu}$ the fourth-rank elastic modulus tensor. Unless otherwise specified, in this paper greek indices denote Cartesian coordinates x_1,x_2,x_3 (or 1,2,3).

As indicated in the Introduction, in this paper we confine our attention to the case of an isotropic medium, for which the elastic modulus tensor is given by the equation

$$c_{\alpha\beta\mu\nu}=\rho[(c_l^2-2c_t^2)\delta_{\alpha\beta}\delta_{\mu\nu}+c_t^2(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu})]\ . \tag{2.2a}$$

Thus in this case we have that

$$L_{\alpha\mu}(\vec{x}\,|\,\omega)=\delta_{\alpha\mu}(\omega^2+c_t^2\nabla^2)+(c_l^2-c_t^2)\frac{\partial^2}{\partial x_\alpha \partial x_\mu}\ , \tag{2.2b}$$

where c_l and c_t are, respectively, the speeds of the longitudinal and transverse bulk sound waves.

In the case of an infinite, homogeneous medium, Eqs. (2.1) (together with the requirement that the elastic displacement field be well behaved at infinity) fully define the eigenvalue problem for the normal modes. Its solutions are the longitudinal and transverse bulk sound waves. In the case of a semi-infinite medium bounded by a single surface, the normal modes are those solutions to Eq. (2.1) that are well behaved at infinity and satisfy the stress-free boundary conditions at the surface.

In this paper we are concerned with *surface* modes, that is, the solutions to Eq. (2.1) brought about by the presence of a surface and described by a displacement field that is localized to the surface region. In the case of a planar, flat surface, the surface-wave normal mode is the Rayleigh wave. This wave propagates parallel to the surface with a (two-dimensional) wave vector $\vec{q}_{\|}$, its amplitude decays exponentially into the medium, and its dispersion relation is given by

$$\omega_0(q_{\|})=c_R q_{\|}\ , \tag{2.3}$$

where c_R, the speed of the Rayleigh wave, satisfies the equation[3]

$$4\left[1-\frac{c_R^2}{c_l^2}\right]^{1/2}\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}=\left[2-\frac{c_R^2}{c_t^2}\right]^2 . \quad (2.4)$$

We note that $c_R < c_t$ and, in general, $c_t < c_l$.

In this paper we consider the case that the surface is rough. The stress-free boundary conditions must then be satisfied at every point on an irregular surface. Now, in practice, the actual rough-surface profile is not known in detail. Thus the theoretical description of the problem assumes that the surface profile is a random function. One must then solve a random eigenvalue problem. The simplifying feature is that one is interested only in average quantities, such as the average displacement field $\langle u_\mu(\vec{x}\,|\,\omega)\rangle$, the average being carried out with respect to the ensemble of realizations of the surface-profile function. This averaging procedure restores both isotropy and infinitesimal translational invariance in the plane of the flat, nominal surface. Thus the surface-mode solution to Eq. (2.1) (which holds for both the actual displacement field and its average) can still be characterized by a wave vector $\vec{q}_{||}$ in the plane of the nominal surface, and its frequency $\omega_R(q_{||})$ depends only on the magnitude $q_{||}$ of the wave vector $\vec{q}_{||}$.

The present work deals with the case that the degree of roughness is small enough that a perturbation-theory approach is adequate.[4] We can then visualize the physical problem as one in which the rough profile is a source of scattering for a Rayleigh wave. That is, the roughness opens up decay channels for a Rayleigh wave, namely those provided by the bulk elastic modes and by other Rayleigh waves. Thus the Rayleigh wave is attenuated as it propagates along the surface and its frequency $\omega_R(q_{||})$ acquires an imaginary part. In addition, the real part of the Rayleigh wave frequency is shifted by the roughness relative to the frequency $\omega_0(q_{||})$ given by Eq. (2.3). Our objective in this paper is to calculate the real and imaginary parts of $\omega_R(q_{||})$.

B. The boundary-value problem

We assume that the isotropic elastic medium under consideration occupies the region $x_3 > \zeta(\vec{x}_{||})$, where $\zeta(\vec{x}_{||})$ is the surface-roughness profile function. Here $\vec{x}_{||}=x_1\hat{x}_1+x_2\hat{x}_2$ is the two-dimensional position vector in the plane of the nominal, flat surface (the plane $x_3=0$).

It is convenient to introduce the two-dimensional Fourier transform of the displacement field $u_\mu(\vec{q}_{||}\omega\,|\,x_3)$, such that

$$u_\mu(\vec{x}\,|\,\omega)=\int\frac{d^2q_{||}}{(2\pi)^2}e^{i\vec{q}_{||}\cdot\vec{x}_{||}}u_\mu(\vec{q}_{||}\omega\,|\,x_3) . \quad (2.5)$$

The equation of motion for $u_\mu(\vec{q}_{||}\omega\,|\,x_3)$ is readily obtained from Eqs. (2.1), (2.2), and (2.5); it can be cast in the following form:

$$\sum_\mu L_{\alpha\mu}(\vec{q}_{||}\omega\,|\,x_3)u_\mu(\vec{q}_{||}\omega\,|\,x_3)=0 , \quad (2.6)$$

where the elements of the tensor $L_{\alpha\mu}(\vec{q}_{||}\omega\,|\,x_3)$ are given by

$$L_{\alpha\mu}(\vec{q}_{||}\omega\,|\,x_3)=\delta_{\alpha\mu}\left[\omega^2+c_t^2\left[-q_{||}^2+\frac{d^2}{dx_3^2}\right]\right]+(c_l^2-c_t^2)\left[(1-\delta_{\alpha3})iq_\alpha+\delta_{\alpha3}\frac{d}{dx_3}\right]\left[(1-\delta_{\mu3})iq_\mu+\delta_{\mu3}\frac{d}{dx_3}\right] . \quad (2.7)$$

The solution to Eqs. (2.6) and (2.7) is

$$u_1(\vec{q}_{||}\omega\,|\,x_3)=\frac{q_1}{q_{||}}(e^{-\alpha_l x_3}A_1+e^{-\alpha_t x_3}A_2)-\frac{q_2}{q_{||}}e^{-\alpha_t x_3}A_3 , \quad (2.8)$$

$$u_2(\vec{q}_{||}\omega\,|\,x_3)=\frac{q_2}{q_{||}}(e^{-\alpha_l x_3}A_1+e^{-\alpha_t x_3}A_2)+\frac{q_1}{q_{||}}e^{-\alpha_t x_3}A_3 , \quad (2.9)$$

$$u_3(\vec{q}_{||}\omega\,|\,x_3)=i\frac{\alpha_l}{q_{||}}e^{-\alpha_l x_3}A_1+i\frac{q_{||}}{\alpha_t}e^{-\alpha_t x_3}A_2 , \quad (2.10)$$

where we have defined the decay constants $\alpha_l(q_{||}\,|\,\omega)$ and $\alpha_t(q_{||}\,|\,\omega)$ according to

$$\alpha_{l,t}(q_{||}\,|\,\omega)=\left[q_{||}^2-\frac{(\omega+i\eta)^2}{c_{l,t}^2}\right]^{1/2} \quad (2.11)$$

with $\eta\to 0+$ and the provision that $\mathrm{Re}\,\alpha_{l,t}>0$.

We emphasize that the coefficients A_1, A_2, and A_3 are functions of $\vec{q}_{||}$ and ω. They must be determined by applying the stress-free boundary conditions at the (rough) surface $x_3=\zeta(\vec{x}_{||})$, namely:

$$\sum_\beta T_{\alpha\beta}(\vec{x}\,|\,\omega)\hat{n}_\beta(\vec{x}_{||})\,\big|_{x_3=\zeta(\vec{x}_{||})}=0 , \quad (2.12)$$

where $\hat{n}(\vec{x}_{||})$, the unit vector normal to the surface at each point, is given by

$$\hat{n}(\vec{x}_{||})=\left[1+\left[\frac{\partial\zeta}{\partial x_1}\right]^2+\left[\frac{\partial\zeta}{\partial x_2}\right]^2\right]^{-1}\times\left[-\frac{\partial\zeta(\vec{x}_{||})}{\partial x_1},-\frac{\partial\zeta(\vec{x}_{||})}{\partial x_2},1\right], \quad (2.13)$$

and the stress tensor $T_{\alpha\beta}(\vec{x}\,|\,\omega)$ is, in the general case, defined by the equation

$$T_{\alpha\beta}(\vec{x}\,|\,\omega)=\sum_{\mu,\nu}c_{\alpha\beta\mu\nu}\frac{\partial}{\partial x_\nu}u_\mu(\vec{x}\,|\,\omega)\ . \quad (2.14)$$

For an isotropic medium, making use of Eq. (2.2a) in Eq. (2.14), we readily obtain the result that

$$T_{\alpha\beta}(\vec{x}\,|\,\omega)=\rho(c_l^2-2c_t^2)\delta_{\alpha\beta}\vec{\nabla}\cdot\vec{u}(\vec{x}\,|\,\omega)+\rho c_t^2\left[\frac{\partial}{\partial x_\beta}u_\alpha(\vec{x}\,|\,\omega)+\frac{\partial}{\partial x_\alpha}u_\beta(\vec{x}\,|\,\omega)\right]. \quad (2.15)$$

We now proceed to apply the stress-free boundary conditions (2.12). For brevity here we shall only outline the required steps. We substitute Eq. (2.15) in Eq. (2.12) and make the assumption that the roughness is small. Here this assumption means that we can write down expansions such as

$$u_\mu(\vec{q}_{||}\omega\,|\,x_3)\,|_{x_3=\zeta(\vec{x}_{||})}=u_\mu(\vec{q}_{||}\omega\,|\,0)+\left[\frac{d}{dx_3}u_\mu(\vec{q}_{||}\omega\,|\,x_3)\right]_{x_3=0}\zeta(\vec{x}_{||})+\frac{1}{2}\left[\frac{d^2}{dx_3^2}u_\mu(\vec{q}_{||}\omega\,|\,x_3)\right]_{x_3=0}\zeta^2(\vec{x}_{||})+\cdots\ , \quad (2.16)$$

and obtain meaningful results by keeping terms of up to $O(\zeta^2)$. We introduce the Fourier coefficients of the rough-surface profile, $\zeta(\vec{Q}_{||})$ [defined according to Eq. (2.5)], we take the Fourier transform of Eq. (2.12), and we make use of the result for $u_\mu(\vec{q}_{||}\omega\,|\,x_3)$ given by Eqs. (2.8)—(2.10). After considerable algebra, we find that Eq. (2.12) can be written, to $O(\zeta^2)$, as

$$\int\frac{d^2q_{||}}{(2\pi)^2}e^{i\vec{q}_{||}\cdot\vec{x}_{||}}\sum_\beta\left[M_{\alpha\beta}^{(0)}(\vec{q}_{||}\,|\,\omega)+\int\frac{d^2Q_{||}}{(2\pi)^2}e^{i\vec{Q}_{||}\cdot\vec{x}_{||}}\zeta(\vec{Q}_{||})X_{\alpha\beta}(\vec{Q}_{||};\vec{q}_{||}\,|\,\omega)+\int\frac{d^2Q_{||}}{(2\pi)^2}\int\frac{d^2P_{||}}{(2\pi)^2}e^{i(\vec{Q}_{||}+\vec{P}_{||})\cdot\vec{x}_{||}}\zeta(\vec{Q}_{||})\zeta(\vec{P}_{||})Z_{\alpha\beta}(\vec{Q}_{||};\vec{q}_{||}\,|\,\omega)\right]A_\beta(\vec{q}_{||}\,|\,\omega)=0\ . \quad (2.17)$$

In Eqs. (2.17) we have introduced the 3×3 matrices $\overleftrightarrow{M}^{(0)}(\vec{q}_{||}\,|\,\omega)$, $\overleftrightarrow{X}(\vec{Q}_{||};\vec{q}_{||}\,|\,\omega)$, and $\overleftrightarrow{Z}(\vec{Q}_{||};\vec{q}_{||}\,|\,\omega)$, whose elements we list below:

$$M_{11}^{(0)}(\vec{q}_{||}\,|\,\omega)=-2c_t^2\frac{q_1}{q_{||}}\alpha_l(q_{||})\ , \quad (2.18a)$$

$$M_{12}^{(0)}(\vec{q}_{||}\,|\,\omega)=-c_t^2\frac{q_1}{q_{||}\alpha_t(q_{||})}\left[2q_{||}^2-\frac{\omega^2}{c_t^2}\right], \quad (2.18b)$$

$$M_{13}^{(0)}(\vec{q}_{||}\,|\,\omega)=c_t^2\frac{q_2}{q_{||}}\alpha_t(q_{||})\ , \quad (2.18c)$$

$$M_{21}^{(0)}(\vec{q}_{||}\,|\,\omega)=-2c_t^2\frac{q_2}{q_{||}}\alpha_l(q_{||})\ , \quad (2.18d)$$

$$M_{22}^{(0)}(\vec{q}_{||}\,|\,\omega)=-c_t^2\frac{q_2}{q_{||}\alpha_t(q_{||})}\left[2q_{||}^2-\frac{\omega^2}{c_t^2}\right], \quad (2.18e)$$

$$M_{23}^{(0)}(\vec{q}_{||}\,|\,\omega)=-c_t^2\frac{q_1}{q_{||}}\alpha_t(q_{||})\ , \quad (2.18f)$$

$$M_{31}^{(0)}(\vec{q}_{||}\,|\,\omega)=-c_t^2\frac{i}{q_{||}}\left[2q_{||}^2-\frac{\omega^2}{c_t^2}\right], \quad (2.18g)$$

$$M_{32}^{(0)}(\vec{q}_{\|}\,|\,\omega)=-2ic_t^2 q_{\|}\ , \tag{2.18h}$$

$$M_{33}^{(0)}(\vec{q}_{\|}\,|\,\omega)=0\ , \tag{2.18i}$$

$$X_{11}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=\frac{c_t^2}{q_{\|}}\left[2q_1(\vec{Q}_{\|}\cdot\vec{q}_{\|})+2q_1\alpha_l^2(q_{\|})+Q_1\frac{\omega^2}{c_t^2}(1-2\lambda^2)\right]\ , \tag{2.19a}$$

$$X_{12}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=\frac{c_t^2}{q_{\|}}\left[2q_1(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_1\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right]\right]\ , \tag{2.19b}$$

$$X_{13}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-\frac{c_t^2}{q_{\|}}[q_2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_1(\vec{Q}_{\|}\times\vec{q}_{\|})_3+q_2\alpha_t^2(q_{\|})]\ , \tag{2.19c}$$

$$X_{21}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=\frac{c_t^2}{q_{\|}}\left[2q_2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+2q_2\alpha_l^2(q_{\|})+Q_2\frac{\omega^2}{c_t^2}(1-2\lambda^2)\right]\ , \tag{2.19d}$$

$$X_{22}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=\frac{c_t^2}{q_{\|}}\left[2q_2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_2\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right]\right]\ , \tag{2.19e}$$

$$X_{23}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=\frac{c_t^2}{q_{\|}}[q_1(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_1\alpha_t^2(q_{\|})-q_2(\vec{Q}_{\|}\times\vec{q}_{\|})_3]\ , \tag{2.19f}$$

$$X_{31}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=ic_t^2\frac{\alpha_l(q_{\|})}{q_{\|}}\left[2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+2q_{\|}^2-\frac{\omega^2}{c_t^2}\right]\ , \tag{2.19g}$$

$$X_{32}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=i\frac{c_t^2}{q_{\|}\alpha_t(q_{\|})}\left[\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right](\vec{Q}_{\|}\cdot\vec{q}_{\|})+2q_{\|}^2\alpha_t^2(q_{\|})\right]\ , \tag{2.19h}$$

$$X_{33}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-ic_t^2\frac{\alpha_t(q_{\|})}{q_{\|}}(\vec{Q}_{\|}\times\vec{q}_{\|})_3\ , \tag{2.19i}$$

$$Z_{11}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-c_t^2\frac{\alpha_l(q_{\|})}{q_{\|}}\left[2q_1(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_1\alpha_l^2(q_{\|})+Q_1\frac{\omega^2}{c_t^2}(1-2\lambda^2)\right]\ , \tag{2.20a}$$

$$Z_{12}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-c_t^2\frac{\alpha_t(q_{\|})}{q_{\|}}\left[2q_1(\vec{Q}_{\|}\cdot\vec{q}_{\|})+\frac{q_1}{2}\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right]\right]\ , \tag{2.20b}$$

$$Z_{13}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=c_t^2\frac{\alpha_t(q_{\|})}{q_{\|}}\left[q_2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_1(\vec{Q}_{\|}\times\vec{q}_{\|})_3+\frac{q_2}{2}\alpha_t^2(q_{\|})\right]\ , \tag{2.20c}$$

$$Z_{21}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-c_t^2\frac{\alpha_l(q_{\|})}{q_{\|}}\left[2q_2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_2\alpha_l^2(q_{\|})+Q_2\frac{\omega^2}{c_t^2}(1-\lambda^2)\right]\ , \tag{2.20d}$$

$$Z_{22}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-c_t^2\frac{\alpha_t(q_{\|})}{q_{\|}}\left[2q_2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+\frac{q_2}{2}\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right]\right]\ , \tag{2.20e}$$

$$Z_{23}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=c_t^2\frac{\alpha_t(q_{\|})}{q_{\|}}\left[q_2(\vec{Q}_{\|}\times\vec{q}_{\|})_3-q_1(\vec{Q}_{\|}\cdot\vec{q}_{\|})-\frac{q_1}{2}\alpha_t^2(q_{\|})\right]\ , \tag{2.20f}$$

$$Z_{31}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-ic_t^2\frac{\alpha_l^2(q_{\|})}{q_{\|}}\left[2(\vec{Q}_{\|}\cdot\vec{q}_{\|})+\left[q_{\|}^2-\frac{\omega^2}{2c_t^2}\right]\right]\ , \tag{2.20g}$$

$$Z_{32}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=-i\frac{c_t^2}{q_{\|}}\left[\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right](\vec{Q}_{\|}\cdot\vec{q}_{\|})+q_{\|}^2\alpha_t^2(q_{\|})\right]\ , \tag{2.20h}$$

$$Z_{33}(\vec{Q}_{\|};\vec{q}_{\|}\,|\,\omega)=ic_t^2\frac{\alpha_t^2(q_{\|})}{q_{\|}}(\vec{Q}_{\|}\times\vec{q}_{\|})_3\ . \tag{2.20i}$$

We note that in Eqs. (2.18)—(2.20) we have simplified the notation by setting $\alpha_{l,t}(\vec{q}_{||}\,|\,\omega)\equiv\alpha_{l,t}(q_{||})$, and we have called $c_t/c_l\equiv\lambda$.

We now equate to zero the coefficients of $\exp(i\vec{q}_{||}\cdot\vec{x}_{||})$ in Eq. (2.17). We thus obtain the following homogeneous integral equations for the coefficients $A_\alpha(\vec{q}_{||}\,|\,\omega)$:

$$\sum_\beta M^{(0)}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega)A_\beta(\vec{q}_{||}\,|\,\omega)+\sum_\beta\int\frac{d^2k_{||}}{(2\pi)^2}\zeta(\vec{q}_{||}-\vec{k}_{||})Y_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega)A_\beta(\vec{k}_{||}\,|\,\omega)$$
$$+\sum_\beta\int\frac{d^2k_{||}}{(2\pi)^2}\int\frac{d^2Q_{||}}{(2\pi)^2}\zeta(\vec{Q}_{||})\zeta(\vec{q}_{||}-\vec{Q}_{||}-\vec{k}_{||})Z_{\alpha\beta}(\vec{Q}_{||};\vec{k}_{||}\,|\,\omega)A_\beta(\vec{k}_{||}\,|\,\omega)=0\ , \quad (2.21)$$

where we have made the definition

$$Y_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega)=X_{\alpha\beta}(\vec{q}_{||}-\vec{k}_{||};\vec{k}_{||}\,|\,\omega)\ . \quad (2.22)$$

In conclusion, the stress-free boundary conditions given by Eq. (2.12) have led us to the eigenvalue equations (2.21) ($\alpha=1,2,3$) for the three unknown functions $A_\alpha(\vec{q}_{||}\,|\,\omega)$ [see Eqs. (2.8)—(2.10)]. The complications posed by Eq. (2.21) are twofold. First, it is an *integral* (matrix) equation. Second, the Fourier coefficients $\zeta(\vec{Q}_{||})$ of the surface-roughness profile are random variables. However, as we shall see below, the problem simplifies considerably because (as mentioned in Sec. II A) we do not need the coefficients $A_\alpha(\vec{q}_{||}\,|\,\omega)$ themselves but rather their average $\langle A_\alpha(\vec{q}_{||}\,|\,\omega)\rangle$ over the ensemble of realizations of the surface profile.

C. The eigenvalue problem for the average coefficients $\langle A_\alpha(\vec{q}_{||}\omega\,|\,x_3)\rangle$

In this section we transform the integral equation (2.21) into a simpler *algebraic* equation for the average coefficients $\langle A_\alpha(\vec{q}_{||}\omega\,|\,x_3)\rangle$. As we shall see below explicitly, this simplification is possible because of the fact that averaging over an ensemble of rough surfaces restores the translational invariance in the plane of the nominal surface.

This averaging is formally carried out by the operator P defined such that

$$PA_\alpha(\vec{q}_{||}\omega\,|\,x_3)=\langle A_\alpha(\vec{q}_{||}\omega\,|\,x_3)\rangle\ . \quad (2.23)$$

It is also convenient to introduce an operator Q according to

$$P+Q=I\ , \quad (2.24)$$

where I is the identity operator.

The demonstration that follows is simplified by writing down Eq. (2.21) symbolically as

$$(M^{(0)}+M)A=0\ . \quad (2.25)$$

Note that M in Eq. (2.21) is an integral, random operator.

We apply the operator P to Eq. (2.25) from the left. Noting the identity $A=(P+Q)A$, we have the result that

$$(M^{(0)}+\langle M\rangle)\langle A\rangle+PMQA=0\ . \quad (2.26)$$

We obtain an expression for QA (the fluctuating component of the variable A) by acting on Eq. (2.25) from the left with the operator Q. We thus have that

$$QA=-(M^{(0)}+QM)^{-1}QM\langle A\rangle\ . \quad (2.27)$$

Substituting Eq. (2.27) in the last term of Eq. (2.26) we obtain the result that

$$(M^{(0)}+\mathscr{P})\langle A\rangle=0\ , \quad (2.28)$$

where we have made the following definition:

$$\mathscr{P}\equiv\langle M\rangle-PM(M^{(0)}+QM)^{-1}QM\ . \quad (2.29)$$

We next expand the inverse matrix in Eq. (2.29) and keep only terms up to those quadratic in M. We thus have that

$$\mathscr{P}\cong\langle M\rangle-\langle M(M^{(0)})^{-1}M\rangle+\langle M\rangle(M^{(0)})^{-1}\langle M\rangle\ . \quad (2.30)$$

We now make use of the results given by Eqs. (2.28) and (2.30) in Eq. (2.21). This leads us to the result that

$$\sum_\beta\int d^2k_{||}[\delta(\vec{q}_{||}-\vec{k}_{||})M^{(0)}_{\alpha\beta}(\vec{k}_{||}\,|\,\omega)+\mathscr{P}_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega)]\langle A_\beta(\vec{k}_{||}\,|\,\omega)\rangle=0\ , \quad (2.31)$$

where we have called

$$\mathscr{P}_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega)=\int\frac{d^2Q_{||}}{(2\pi)^4}\langle\zeta(\vec{Q}_{||})\zeta(\vec{q}_{||}-\vec{Q}_{||}-\vec{k}_{||})\rangle Z_{\alpha\beta}(\vec{Q}_{||};\vec{k}_{||}\,|\,\omega)$$
$$-\sum_{\mu,\nu}\int\frac{d^2k'_{||}}{(2\pi)^4}\langle\zeta(\vec{q}_{||}-\vec{k}'_{||})\zeta(\vec{k}'_{||}-\vec{k}_{||})\rangle Y_{\alpha\mu}(\vec{q}_{||};\vec{k}'_{||}\,|\,\omega)\{[\vec{M}^{(0)}(\vec{k}'_{||}\,|\,\omega)]^{-1}\}_{\mu\nu}Y_{\nu\beta}(\vec{k}'_{||};\vec{k}_{||}\,|\,\omega)\ . \quad (2.32)$$

In Eq. (2.32) we have kept terms up to $O(\zeta^2)$ and have made use of the fact that, since we measure the coordinate x_3 from the plane of the nominal flat surface, we have that

$$\langle \zeta(\vec{x}_{||}) \rangle = 0 \ . \tag{2.33}$$

At this stage we must make an assumption about the statistical properties of the rough surface. We follow usual practice and assume that the surface-roughness profile is a stationary stochastic process, i.e., we assume that

$$\langle \zeta(\vec{x}_{||})\zeta(\vec{x}'_{||}) \rangle = \delta^2 W(|\vec{x}_{||} - \vec{x}'_{||}|) \ . \tag{2.34}$$

In Eq. (2.34) we have denoted by δ the root-mean-square departure of the surface from flatness. Note that $W(0)=1$.

In terms of the Fourier coefficients $\zeta(\vec{Q}_{||})$, Eq. (2.34) becomes

$$\langle \zeta(\vec{k}_{||})\zeta(\vec{k}'_{||}) \rangle = (2\pi)^2\delta^2 g(k_{||})\delta(\vec{k}_{||}+\vec{k}'_{||}) \ , \tag{2.35}$$

where $g(k_{||})$ is the two-dimensional Fourier transform of the correlation function $W(|\vec{x}_{||}|)$. With the assumption given by Eq. (2.35) it follows that

$$\mathscr{P}_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega) = \delta(\vec{q}_{||}-\vec{k}_{||})\mathscr{P}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) \ , \tag{2.36}$$

where, with the definitions

$$M^{(a)}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) = \int \frac{d^2Q_{||}}{(2\pi)^2} g(Q_{||}) Z_{\alpha\beta}(\vec{Q}_{||};\vec{q}_{||}\,|\,\omega) \tag{2.37}$$

and

$$M^{(b)}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) = -\int \frac{d^2k_{||}}{(2\pi)^2} g(|\vec{q}_{||}-\vec{k}_{||}|) \times R_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega) \ , \tag{2.38}$$

with

$$R_{\alpha\beta}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega) = \sum_{\mu,\nu} Y_{\alpha\mu}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega) \times \{[\overleftrightarrow{M}^{(0)}(\vec{k}_{||}\,|\,\omega)]^{-1}\}_{\mu\nu} \times Y_{\nu\beta}(\vec{k}_{||};\vec{q}_{||}\,|\,\omega) \ , \tag{2.39}$$

we have that

$$\mathscr{P}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) \equiv \delta^2 M^{(a)}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) + \delta^2 M^{(b)}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) \ . \tag{2.40}$$

We stress that the result given by Eq. (2.36) constitutes the statement that translational invariance has been restored by the averaging. Substituting Eq. (2.36) into Eq. (2.31) we obtain the following *algebraic* equation for the average coefficients $\langle A_\alpha(\vec{q}_{||}\,|\,\omega) \rangle$:

$$\sum_\beta [M^{(0)}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega) + \mathscr{P}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega)]\langle A_\beta(\vec{q}_{||}\,|\,\omega) \rangle = 0 \ . \tag{2.41}$$

For convenience in the presentation, the explicit expressions for the elements of $\mathscr{P}_{\alpha\beta}(\vec{q}_{||}\,|\,\omega)$ are given in Appendix A. The eigenvalue problem given by Eq. (2.41) is solved in the next section.

III. THE DISPERSION RELATION OF A RAYLEIGH WAVE PROPAGATING ON A ROUGH SURFACE

The dispersion relation of a Rayleigh wave propagating on a flat surface is given by Eq. (2.3). When the surface is randomly rough, Eq. (2.41) constitutes the eigenvalue problem for the coefficients $\langle A_\alpha(\vec{q}_{||}\,|\,\omega) \rangle$ of the displacement field given by Eqs. (2.8)–(2.10), averaged over the ensemble of realizations of the surface-profile function. The condition for nontrivial solutions ($\langle A_\alpha \rangle \neq 0$) to Eq. (2.41) is given by (in matrix notation):

$$\det[\overleftrightarrow{M}^{(0)}(\vec{q}_{||}\,|\,\omega) + \overleftrightarrow{\mathscr{P}}(\vec{q}_{||}\,|\,\omega)] = 0 \ . \tag{3.1}$$

For a given value of the wave vector $\vec{q}_{||}$, the solution to Eq. (3.1) defines the frequency of a Rayleigh wave, $\omega_R(q_{||})$. As indicated in Sec. II, the simple determinantal equation given by Eq. (3.1) was obtained because of the fact that our averaging procedure restores infinitesimal translational invariance in the plane of the nominal surface. Moreover, the solution $\omega_R(q_{||})$ must depend only on the *magnitude* $q_{||}$ of the wave vector $\vec{q}_{||}$ (our averaging also restores isotropy in the plane of the surface).

Since $\overleftrightarrow{\mathscr{P}}(\vec{q}_{||}\,|\,\omega)$ itself has been determined only to $O(\delta^2)$, with the aid of Eqs. (2.37)–(2.40) we rewrite Eq. (3.1) to $O(\delta^2)$ in the form

$$\det\overleftrightarrow{M}^{(0)}(q_{||}\,|\,\omega) + \delta^2\,\mathrm{Tr}\overleftrightarrow{\Sigma}(q_{||}\,|\,\omega) = 0 \ , \tag{3.2}$$

where

$$\overleftrightarrow{\Sigma}(q_{||}\,|\,\omega) = \overleftrightarrow{\Sigma}^{(a)}(q_{||}\,|\,\omega) + \overleftrightarrow{\Sigma}^{(b)}(q_{||}\,|\,\omega) \ , \tag{3.3a}$$

with

$$\overleftrightarrow{\Sigma}^{(a,b)}(q_{||}\,|\,\omega) = [\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{||}\,|\,\omega)]^T \cdot \overleftrightarrow{M}^{(a,b)}(\vec{q}_{||}\,|\,\omega) \ . \tag{3.3b}$$

In Eq. (3.3b) $\mathrm{cof}\overleftrightarrow{M}^{(0)}$ is the matrix of the cofactors of $\overleftrightarrow{M}^{(0)}$, and $(\mathrm{cof}\overleftrightarrow{M}^{(0)})^T$ is its transpose.

We remark that our notation emphasizes the fact that, although the matrices on the right-hand side of Eq. (3.3b) are functions of the full wave vector $\vec{q}_{||}$, the "self-energy" $\overleftrightarrow{\Sigma}(q_{||}\,|\,\omega)$ depends only on the magnitude $q_{||}$ of $\vec{q}_{||}$. (This is shown explicitly below.)

We can obtain an explicit solution of Eq. (3.2) for $\omega_R(q_{\|})$ by introducing the roughness-induced perturbation $\Delta\omega(q_{\|})$ of the flat-surface dispersion relation $\omega_0(q_{\|})=c_R q_{\|}$, according to the equation

$$\omega_R(q_{\|})=\omega_0(q_{\|})+\Delta\omega(q_{\|}) . \tag{3.4}$$

Since Eq. (3.2) is correct only to $O(\delta^2)$, we need a result for $\Delta\omega(q_{\|})$ only to the same order of accuracy.

Now, from the definitions given by Eqs. (2.18) and (A3) it follows that

$$\det\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)=iq_{\|}^2\Delta(q_{\|}\,|\,\omega) . \tag{3.5}$$

Thus from Eqs. (2.3), (2.4), and (A3) we have that

$$\det\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega_0(q_{\|}))=0 . \tag{3.6}$$

Expanding Eq. (3.2) in powers of $\Delta\omega(q_{\|})$ and keeping only the leading term, we are led to the following concise result for $\Delta\omega(q_{\|})$:

$$\Delta\omega(q_{\|})=-\delta^2\frac{\mathrm{Tr}\overleftrightarrow{\Sigma}(q_{\|}\,|\,\omega_0(q_{\|}))}{\left[\dfrac{d}{d\omega}\det\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)\right]_{\omega=\omega_0(q_{\|})}} . \tag{3.7}$$

From the real part of $\Delta\omega(q_{\|})$ we obtain the shift in the frequency of the Rayleigh wave and from its imaginary part we obtain its attenuation length.

We note that one convenient feature of the result given by Eq. (3.7) is that we only need to evaluate the *diagonal* elements of the matrix $\overleftrightarrow{\Sigma}$. [In fact, as we now show, only the 11 and 22 elements of $\overleftrightarrow{\Sigma}^{(b)}$ contribute to our final result, to $O(\delta^2)$.] From Eqs. (3.3b) and (A2) we can easily show the result that (recall that $\overleftrightarrow{B}^{-1}=(\mathrm{cof}\overleftrightarrow{B})^T/\det\overleftrightarrow{B}$):

$$\overleftrightarrow{\Sigma}^{(a)}(q_{\|}\,|\,\omega)=\tfrac{1}{2}\det\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)\times\begin{bmatrix}\alpha_l^2(q_{\|}) & 0 & 0\\ 0 & \alpha_t^2(q_{\|}) & 0\\ 0 & 0 & \alpha_t^2(q_{\|})\end{bmatrix} , \tag{3.8}$$

which, in view of Eq. (3.6), implies that $\overleftrightarrow{\Sigma}^{(a)}(q_{\|}\,|\,\omega)$ does not contribute to the result for $\Delta\omega(q_{\|})$ given by Eq. (3.7).

We now consider the evaluation of the trace of $\overleftrightarrow{\Sigma}^{(b)}(q_{\|}\,|\,\omega)$. Substituting in Eq. (3.3b) the definition of the matrix $\overleftrightarrow{M}^{(b)}(q_{\|}\,|\,\omega)$ [Eq. (2.38)] we have that

$$\overleftrightarrow{\Sigma}^{(b)}(q_{\|}\,|\,\omega)=-\int\frac{d^2k_{\|}}{(2\pi)^2}g(\,|\,\vec{q}_{\|}-\vec{k}_{\|}\,|\,)\times[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T\times\overleftrightarrow{R}(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega) . \tag{3.9}$$

We recall that the elements of the matrix $\overleftrightarrow{R}(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega)$ are given in Appendix A. The elements of the matrix $[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T$ are obtained from the definition of $\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)$, given by Eq. (2.18). We have that

$$[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T_{1;1,2}=-\frac{2i}{c_t^2}q_{\|}^3\alpha_t(q_{\|})q_{1,2} , \tag{3.10a}$$

$$[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T_{13}=\frac{1}{c_t^2}q_{\|}^3\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right] , \tag{3.10b}$$

$$[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T_{2;1,2}=\frac{i}{c_t^2}q_{\|}\alpha_t(q_{\|})\times\left[2q_{\|}^2-\frac{\omega^2}{c_t^2}\right]q_{1,2} , \tag{3.10c}$$

$$[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T_{23}=-\frac{2}{c_t^2}q_{\|}^3\alpha_l(q_{\|})\alpha_t(q_{\|}) , \tag{3.10d}$$

$$[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T_{3;1,2}=\frac{(-1)^{\nu+1}}{c_t^2}\frac{q_{2,1}}{q_{\|}\alpha_t(q_{\|})}\times\det\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)\quad(\nu=1,2) , \tag{3.10e}$$

$$[\mathrm{cof}\overleftrightarrow{M}^{(0)}(\vec{q}_{\|}\,|\,\omega)]^T_{33}=0 . \tag{3.10f}$$

From Eqs. (3.9), (3.10e), and (3.10f) we readily conclude that

$$\Sigma_{33}^{(b)}(q_{\|}\,|\,\omega)\propto\det\overleftrightarrow{M}(\vec{q}_{\|}\,|\,\omega) . \tag{3.11}$$

Thus $\Sigma_{33}^{(b)}(q_{\|}\,|\,\omega)$ does not contribute to the result for $\Delta\omega(q_{\|})$ given by Eq. (3.7).

The explicit evaluation of the two remaining elements on the diagonal of $\overleftrightarrow{\Sigma}^{(b)}(q_{\|}\,|\,\omega)$ is rather lengthy. For brevity we give here only the result we obtained for their sum upon setting $\omega=\omega_0(q_{\|})$. [We note that in the derivation of Eq. (3.12) some algebraic manipulations were necessary in order to identify some terms which are proportional to $\Delta(q_{\|}\,|\,\omega)$. Those terms do not contribute to Eq. (3.12).] We have that

$$\mathrm{Tr}\overleftrightarrow{\Sigma}^{(b)}(q_{\|}\,|\,\omega_0(q_{\|}))=-i\frac{c_R^2}{c_t^2}q_{\|}^3\int\frac{d^2k_{\|}}{(2\pi)^2}g(\,|\,\vec{q}_{\|}-\vec{k}_{\|}\,|\,)F_1(\vec{q}_{\|};\vec{k}_{\|})-i\frac{c_R^4}{c_t^4}q_{\|}^5\int\frac{d^2k_{\|}}{(2\pi)^2}g(\,|\,\vec{q}_{\|}-\vec{k}_{\|}\,|\,)\frac{F_2(\vec{q}_{\|};\vec{k}_{\|})}{\Delta(k_{\|}\,|\,\omega_0(q_{\|}))} , \tag{3.12}$$

where we have defined the functions $F_1(\vec{q}_{\|};\vec{k}_{\|})$ and $F_2(\vec{q}_{\|};\vec{k}_{\|})$ by the following equations [note that $\epsilon(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega)$ is given by Eq. (A5)]:

$$F_1(\vec{q}_{\|};\vec{k}_{\|})=\left[2-\frac{c_R^2}{c_t^2}\right]q_{\|}^3\left[\frac{c_R^2}{c_t^2}q_{\|}^2(1-\lambda^2)+(\vec{q}_{\|}\cdot\vec{k}_{\|})\left[\frac{c_R^2}{c_t^2}\frac{\epsilon(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega_0)}{2k_{\|}^2}-2(1-\lambda^2)\right]\right]$$
$$+\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\epsilon^2(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega_0)\frac{(\vec{q}_{\|}\times\vec{k}_{\|})^2}{k_{\|}^2\alpha_t(k_{\|}\,|\,\omega_0)} \tag{3.13}$$

and

$$F_2(\vec{q}_{\|};\vec{k}_{\|})=\frac{c_R^4}{c_t^4}\left[1-\frac{c_R^2}{c_l^2}\right]^{1/2}q_{\|}^6\alpha_l(k_{\|}\,|\,\omega_0)$$
$$+\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\frac{\alpha_t(k_{\|}\,|\,\omega_0)}{k_{\|}^2}[(\vec{q}_{\|}\cdot\vec{k}_{\|})\epsilon(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega_0)+2q_{\|}^2k_{\|}^2(1-2\lambda^2)]^2$$
$$-\frac{1}{2}\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}\right]\frac{q_{\|}^3}{k_{\|}^2}\left[2k_{\|}^2-\frac{c_R^2}{c_t^2}q_{\|}^2\right][(\vec{q}_{\|}\cdot\vec{k}_{\|})\epsilon(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega_0)+2q_{\|}^2k_{\|}^2(1-2\lambda^2)]\ . \tag{3.14}$$

In order to obtain explicit results for $\Delta\omega(q_{\|})$ we need an explicit, analytic expression for the function $g(\,|\vec{q}_{\|}-\vec{k}_{\|}|\,)$. Here we follow standard practice[3] and assume a Gaussian form for $W(\,|\vec{x}_{\|}-\vec{x}_{\|}'|\,)$, which gives us the following form for its Fourier transform:

$$g(\,|\vec{q}_{\|}-\vec{k}_{\|}|\,)=\pi a^2\exp\left[-\frac{a^2}{4}|\vec{q}_{\|}-\vec{k}_{\|}|^2\right]. \tag{3.15}$$

The parameter a introduced in Eq. (3.15), "the transverse correlation length," is a measure of the average distance between successive peaks and valleys in the rough-surface profile.

With the assumption given by Eq. (3.15), we can carry out the integrals over the angular variable in Eq. (3.12) in closed form. Substitution of Eqs. (3.13)–(3.15) in Eq. (3.12) yields the result that

$$\mathrm{Tr}\overleftrightarrow{\Sigma}^{(b)}(q_{\|}\,|\,\omega_0(q_{\|}))=-i\frac{a^2}{2}\frac{c_R^2}{c_t^2}q_{\|}^7\exp\left[-\frac{a^2q_{\|}^2}{4}\right]$$
$$\times\left[\int_0^\infty dk_{\|}k_{\|}\exp\left[-\frac{a^2k_{\|}^2}{4}\right]J^{(1)}(q_{\|}\,|\,k_{\|})\right.$$
$$\left.+\frac{c_R^2}{c_t^2}q_{\|}^2\int_0^\infty dk_{\|}k_{\|}\exp\left[-\frac{a^2k_{\|}^2}{4}\right]\frac{J^{(2)}(q_{\|}\,|\,k_{\|})}{\Delta(k_{\|}\,|\,\omega_0(q_{\|}))}\right], \tag{3.16}$$

where

$$J^{(1)}(q_{\|}\,|\,k_{\|})=a_0(q_{\|})I_0(z)+a_1(q_{\|}\,|\,k_{\|})\frac{I_1(z)}{z}+a_2(q_{\|}\,|\,k_{\|})\frac{I_2(z)}{z^2}\ , \tag{3.17}$$

$$J^{(2)}(q_{\|}\,|\,k_{\|})=b_0(q_{\|}\,|\,k_{\|})I_0(z)+b_1(q_{\|}\,|\,k_{\|})\frac{I_1(z)}{z}+b_2(q_{\|}\,|\,k_{\|})\frac{I_2(z)}{z^2}\ , \tag{3.18}$$

$I_0(z)$, $I_1(z)$, and $I_2(z)$ are modified Bessel functions, and we have called $z\equiv\frac{1}{2}a^2q_{\|}k_{\|}$. The coefficients $a_0(q_{\|}),\ldots,b_2(q_{\|}\,|\,k_{\|})$, introduced in Eqs. (3.17) and (3.18) are defined by the following equations:

$$a_0(q_{\|})=\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}\right](2-\lambda^2)q_{\|}\ , \tag{3.19}$$

$$a_1(q_{\|}\,|\,k_{\|})=\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\left[4k_{\|}^2+\frac{c_R^4}{c_t^4}q_{\|}^2\right]\frac{1}{a_t(k_{\|}\,|\,\omega_0)}$$
$$-\left[2-\frac{c_R^2}{c_t^2}\right]\left[\frac{c_R^2}{c_t^2}q_{\|}+\frac{z}{k_{\|}}\left[2(1-\lambda^2)k_{\|}^2+\frac{1}{2}\frac{c_R^4}{c_t^4}q_{\|}^2\right]\right], \tag{3.20}$$

$$a_2(q_{\parallel}\,|\,k_{\parallel})=-4\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\left[3k_{\parallel}+\frac{c_R^2}{c_t^2}q_{\parallel}z\right]\frac{k_{\parallel}}{\alpha_t(k_{\parallel}\,|\,\omega_0)}\,, \tag{3.21}$$

$$b_0(q_{\parallel}\,|\,k_{\parallel})=\frac{c_R^4}{c_t^4}\left[1-\frac{c_R^2}{c_l^2}\right]^{1/2}q_{\parallel}^2\alpha_l(k_{\parallel}\,|\,\omega_0)-2\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}\right](1-\lambda^2)q_{\parallel}\left[2k_{\parallel}^2-\frac{c_R^2}{c_t^2}q_{\parallel}^2\right]$$
$$+\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\alpha_t(k_{\parallel}\,|\,\omega_0)\left[16(1-\lambda^2)^2k_{\parallel}^2+\frac{c_R^4}{c_t^4}q_{\parallel}^2\right], \tag{3.22}$$

$$b_1(q_{\parallel}\,|\,k_{\parallel})=\frac{1}{2}\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}\right]\left[2k_{\parallel}^2-\frac{c_R^2}{c_t^2}q_{\parallel}^2\right]\left[2k_{\parallel}+\frac{c_R^2}{c_t^2}q_{\parallel}z\right]\frac{q_{\parallel}}{k_{\parallel}}$$
$$-\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\alpha_t(k_{\parallel}\,|\,\omega_0)\left[16(1-\lambda^2)k_{\parallel}^2+\frac{c_R^4}{c_t^4}q_{\parallel}^2+8\frac{c_R^2}{c_t^2}(1-\lambda^2)q_{\parallel}k_{\parallel}z\right], \tag{3.23}$$

$$b_2(q_{\parallel}\,|\,k_{\parallel})=4\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\alpha_t(k_{\parallel}\,|\,\omega_0)k_{\parallel}\left[3k_{\parallel}+\frac{c_R^2}{c_t^2}q_{\parallel}z\right]. \tag{3.24}$$

Finally, we substitute Eq. (3.16) in Eq. (3.7). [Recall Eq. (3.3a) and the comment that follows Eq. (3.8).] We obtain the result that

$$\Delta\omega(q_{\parallel})=\frac{a^2\delta^2}{8B_1}c_Rq_{\parallel}^2\exp\left[-\frac{a^2q_{\parallel}^2}{4}\right]\left[\int_0^{\infty}dk_{\parallel}k_{\parallel}\exp\left[-\frac{a^2k_{\parallel}^2}{4}\right]J^{(1)}(q_{\parallel}\,|\,k_{\parallel})\right.$$
$$\left.+\frac{c_R^2}{c_t^2}q_{\parallel}^2\int_0^{\infty}dk_{\parallel}k_{\parallel}\exp\left[-\frac{a^2k_{\parallel}^2}{4}\right]\frac{J^{(2)}(q_{\parallel}\,|\,k_{\parallel})}{\Delta(k_{\parallel}\,|\,\omega_0(q_{\parallel}))}\right], \tag{3.25}$$

where we have made use of Eqs. (3.5) and (A3) and have defined the coefficient B_1 according to

$$B_1=\frac{1}{(2-c_R^2/c_t^2)^2}\left\{\left[2-\frac{c_R^2}{c_t^2}\right]^3-4\left[1+\lambda^2\left[1-2\frac{c_r^2}{c_t^2}\right]\right]\right\}. \tag{3.26}$$

In the next section we present a detailed analysis of Eq. (3.25) which, as indicated before, gives the effects of surface roughness on the dispersion relation of a Rayleigh wave.

IV. NUMERICAL RESULTS

As a prelude to our numerical analysis of Eq. (3.25), it is convenient to extract explicitly the real $[\nu_1(q_{\parallel})]$ and imaginary $[\nu_2(q_{\parallel})]$ parts of $\Delta\omega(q_{\parallel})$, defined such that

$$\Delta\omega(q_{\parallel})=\nu_1(q_{\parallel})-i\nu_2(q_{\parallel})\,. \tag{4.1}$$

Since the energy carried by a given elastic mode is a quadratic function of its displacement field,[3] the lifetime $\tau(q_{\parallel})$ of a Rayleigh wave of wave vector $\vec{q}_{\parallel}$ equals $[2\nu_2(q_{\parallel})]^{-1}$. The attenuation length $l(q_{\parallel})$, defined as the distance over which the energy of the Rayleigh wave decays to $1/e$ of its initial value, is given by

$$l(q_{\parallel})=c_R\tau(q_{\parallel})=\frac{c_R}{2\nu_2(q_{\parallel})}\,, \tag{4.2}$$

where c_R, the speed of the Rayleigh wave (equal to its group velocity to zeroth order in the roughness) is defined by Eq. (2.4).

It is possible to obtain explicit expressions for the separate contributions to $\nu_2(q_{\parallel})$ from the individual decay channels opened up by the surface roughness. (These channels are bulk elastic waves of both polarizations and other surface Rayleigh waves.) The key point to note in this connection is that for $k_{\parallel}=q_{\parallel}$ the denominator of the second integral in Eq. (3.25) vanishes, i.e. [see Eqs. (3.5) and (3.6)],

$$\Delta(k_{\parallel}=q_{\parallel}\,|\,\omega_0(q_{\parallel}))=0\,. \tag{4.3}$$

From a physical viewpoint, Eq. (4.3) is a reflection of a process in which a Rayleigh wave of wave vector $\vec{q}_{\parallel}$ and frequency $\omega_R(q_{\parallel})$ is elastically scattered by the roughness into another Rayleigh wave with the same frequency and wave vector $\vec{q}\,'_{||}$ such that $|\vec{q}\,'_{||}|=q_{||}$. (Note that the wave-vector difference $\vec{q}\,'_{||}-\vec{q}_{||}$ is supplied by the surface roughness.)

Now, according to Eqs. (2.11) the frequency ω is to be interpreted as having a small positive imaginary part, i.e., $\omega=\omega+i\eta$, with $\eta\rightarrow 0+$. We then have that

$$\frac{1}{\Delta(k_{||}\,|\,\omega_0(q_{||})+i\eta)}=\left[\frac{1}{\Delta(k_{||}\,|\,\omega_0(q_{||}))}\right]_P-i\pi\,\mathrm{sgn}B_1\frac{1}{\left|\frac{d}{dk_{||}}\Delta(k_{||}\,|\,\omega_0(q_{||}))\right|_{k_{||}=q_{||}}}\delta(k_{||}-q_{||})\;, \tag{4.4}$$

where B_1 is defined by Eq. (3.26) and $\mathrm{sgn}x=\pm 1$ for $x\gtrless 0$. Note that, when substituted in Eq. (3.25), the second term in Eq. (4.4) immediately gives one contribution to $\nu_2(q_{||})$, namely that due to the surface channel (see below).

To complete the determination of $\nu_1(q_{||})$ and $\nu_2(q_{||})$ from Eq. (3.25) we need to obtain the real and imaginary parts of the functions $J^{(1)}(q_{||}\,|\,k_{||})$ and $J^{(2)}(q_{||}\,|\,k_{||})$, defined by Eqs. (3.17) and (3.18), respectively. From these definitions and the expressions for the coefficients appearing in them, Eqs. (3.19)–(3.24), we see that the real and imaginary parts of $J^{(1)}(q_{||}\,|\,k_{||})$ and $J^{(2)}(q_{||}\,|\,k_{||})$ are directly determined by the functions $\alpha_{l,t}(k_{||}\,|\,\omega_0(q_{||}))$, defined by Eq. (2.11). For our present purposes it is convenient to write down the equation

$$\alpha_{l,t}(k_{||}\,|\,\omega_0(q_{||}))=\Theta\left[k_{||}-\frac{c_R}{c_{l,t}}q_{||}\right]\beta_{l,t}(k_{||}\,|\,\omega_0(q_{||}))-i\Theta\left[\frac{c_R}{c_{l,t}}q_{||}-k_{||}\right]\gamma_{l,t}(k_{||}\,|\,\omega_0(q_{||}))\;, \tag{4.5}$$

where we have introduced the functions $\beta_{l,t}(k_{||}\,|\,\omega_0(q_{||}))$ and $\gamma_{l,t}(k_{||}\,|\,\omega_0(q_{||}))$, given by

$$\beta_{l,t}(k_{||}\,|\,\omega_0(q_{||}))=\left[k_{||}^2-\frac{c_R^2}{c_{l,t}^2}q_{||}^2\right]^{1/2},\quad \gamma_{l,t}(k_{||}\,|\,\omega_0(q_{||}))=\left[\frac{c_R^2}{c_{l,t}^2}q_{||}^2-k_{||}^2\right]^{1/2}. \tag{4.6}$$

In Eq. (4.5) $\Theta(x)$ denotes the unit step function. Note that in the $k_{||}$ intervals in which the step functions in Eq. (4.5) are nonzero, the functions $\beta_{l,t}$ and $\gamma_{l,t}$ are real.

Next we note that $c_R<c_t<c_l$. The function $\Delta(k_{||}\,|\,\omega_0(q_{||}))$ can therefore be separated into its real and imaginary parts according to

$$\begin{aligned}\Delta(k_{||}\,|\,\omega_0(q_{||}))&=-\left[4k_{||}^2\gamma_l(k_{||}\,|\,\omega_0)\gamma_t(k_{||}\,|\,\omega_0)+\left[2k_{||}^2-\frac{c_R^2}{c_t^2}q_{||}^2\right]^2\right]\\&\equiv-\widetilde{\Delta}(k_{||}\,|\,\omega_0(q_{||}))\;,\quad k_{||}<(c_R/c_l)q_{||}\end{aligned} \tag{4.7a}$$

$$\begin{aligned}\Delta(k_{||}\,|\,\omega_0(q_{||}))&=-\left[\left[2k_{||}^2-\frac{c_R^2}{c_t^2}q_{||}^2\right]^2+i\,4k_{||}^2\beta_l(k_{||}\,|\,\omega_0)\gamma_t(k_{||}\,|\,\omega_0)\right]\\&\equiv-[\Delta^{(1)}(k_{||}\,|\,\omega_0(q_{||}))+i\Delta^{(2)}(k_{||}\,|\,\omega_0(q_{||}))]\;,\quad (c_R/c_l)q_{||}<k_{||}<(c_R/c_t)q_{||}\end{aligned} \tag{4.7b}$$

$$\Delta(k_{||}\,|\,\omega_0(q_{||}))=4k_{||}^2\beta_l(k_{||}\,|\,\omega_0)\beta_t(k_{||}\,|\,\omega_0)-\left[2k_{||}^2-\frac{c_R^2}{c_t^2}q_{||}^2\right]^2,\quad k_{||}>(c_R/c_t)q_{||}\;. \tag{4.7c}$$

Note that the last interval is the one to which Eq. (4.4) applies.

The above results and definitions provide all the necessary ingredients to obtain $\nu_1(q_{||})$ and $\nu_2(q_{||})$. These functions can be written as follows:

$$\nu_1(q_{\|})=a^2\delta^2\frac{c_R}{8B_1}q_{\|}^2\exp\left[-\frac{a^2q_{\|}^2}{4}\right]\left\{\int_0^\infty dk_{\|}k_{\|}\exp\left[-\frac{a^2k_{\|}^2}{4}\right]\left[f_1(q_{\|}\,|\,k_{\|})+\Theta\left[k_{\|}-\frac{c_R}{c_t}q_{\|}\right]f_2(q_{\|}\,|\,k_{\|})\right.\right.$$

$$+\Theta\left[\frac{c_R}{c_t}q_{\|}-k_{\|}\right]\Theta\left[k_{\|}-\frac{c_R}{c_l}q_{\|}\right]f_3(q_{\|}\,|\,k_{\|})$$

$$\left.+\Theta\left[\frac{c_R}{c_l}q_{\|}-k_{\|}\right]f_4(q_{\|}\,|\,k_{\|})\right]$$

$$\left.+\int_{(c_R/c_t)q_{\|}}^\infty dk_{\|}k_{\|}\exp\left[-\frac{a^2k_{\|}^2}{4}\right]f_0(q_{\|}\,|\,k_{\|})\left[\frac{1}{\Delta(k_{\|}\,|\,\omega_0(q_{\|}))}\right]_P\right\}$$

(4.8)

and

$$\nu_2(q_{\|})=a^2\delta^2\frac{c_R}{8B_1}q_{\|}^2\exp\left[-\frac{a^2q_{\|}^2}{4}\right]\left\{\int_0^{(c_R/c_t)q_{\|}} dk_{\|}k_{\|}\exp\left[-\frac{a^2k_{\|}^2}{4}\right]\left[g_1(q_{\|}\,|\,k_{\|})+\Theta\left[k_{\|}-\frac{c_R}{c_l}q_{\|}\right]g_2(q_{\|}\,|\,k_{\|})\right.\right.$$

$$\left.+\Theta\left[\frac{c_R}{c_l}q_{\|}-k_{\|}\right]g_3(q_{\|}\,|\,k_{\|})\right]$$

$$\left.+\frac{\pi}{4B_2}\frac{c_R^2}{c_t^2}\operatorname{sgn}B_1\exp\left[-\frac{a^2q_{\|}^2}{4}\right]J^{(2)}(q_{\|}\,|\,q_{\|})\right\}. \quad (4.9)$$

In Eqs. (4.8) and (4.9) we have made the following definitions (for brevity we do not show the arguments of most functions):

$$f_0(q_{\|}\,|\,k_{\|})=\frac{c_R^2}{c_t^2}q_{\|}^2\left[(b_0^{(a)}+\beta_t b_0^{(b)}+\beta_l b_0^{(c)})I_0(z)+(b_1^{(a)}+\beta_t b_1^{(b)})\frac{I_1(z)}{z}+\beta_t b_2^{(b)}\frac{I_2(z)}{z^2}\right], \quad (4.10a)$$

$$f_1(q_{\|}\,|\,k_{\|})=a_0I_0(z)+a_1^{(a)}\frac{I_1(z)}{z}, \quad (4.10b)$$

$$f_2(q_{\|}\,|\,k_{\|})=\frac{1}{\beta_t}\left[a_1^{(b)}\frac{I_1(z)}{z}+a_2^{(b)}\frac{I_2(z)}{z^2}\right], \quad (4.10c)$$

$$f_3(q_{\|}\,|\,k_{\|})=-\frac{c_R^2}{c_t^2}q_{\|}^2\frac{1}{(\Delta^{(1)})^2+(\Delta^{(2)})^2}\left[\Delta^{(1)}\left[(b_0^{(a)}+\beta_l b_0^{(c)})I_0(z)+b_1^{(a)}\frac{I_1(z)}{z}\right]\right.$$

$$\left.-\Delta^{(2)}\gamma_t\left[b_0^{(b)}I_0(z)+b_1^{(b)}\frac{I_1(z)}{z}+b_2^{(b)}\frac{I_2(z)}{z^2}\right]\right], \quad (4.10d)$$

$$f_4(q_{\|}\,|\,k_{\|})=-\frac{c_R^2}{c_t^2}q_{\|}^2\frac{1}{\tilde{\Delta}(k_{\|}\,|\,\omega_0)}\left[b_0^{(a)}I_0(z)+b_1^{(a)}\frac{I_1(z)}{z}\right], \quad (4.10e)$$

$$g_1(q_{||} \mid k_{||}) = -\frac{1}{\gamma_t}\left[a_1^{(b)}\frac{I_1(z)}{z} + a_2^{(b)}\frac{I_2(z)}{z^2}\right], \tag{4.11a}$$

$$g_2(q_{||} \mid k_{||}) = -\frac{c_R^2}{c_t^2}q_{||}^2\frac{1}{(\Delta^{(1)})^2+(\Delta^{(2)})^2}\left[\Delta^{(1)}\gamma_t\left[b_0^{(b)}I_0(z)+b_1^{(b)}\frac{I_1(z)}{z}+b_2^{(b)}\frac{I_2(z)}{z^2}\right] + \Delta^{(2)}\left[(b_0^{(a)}+\beta_l b_0^{(c)})I_0(z)+b_1^{(a)}\frac{I_1(z)}{z}\right]\right], \tag{4.11b}$$

$$g_3(q_{||} \mid k_{||}) = -\frac{c_R^2}{c_t^2}q_{||}^2\frac{1}{\widetilde{\Delta}(k_{||} \mid \omega_0)}\left[(\gamma_t b_0^{(b)}+\gamma_l b_0^{(c)})I_0(z)+\gamma_t b_1^{(b)}\frac{I_1(z)}{z}+\gamma_t b_2^{(b)}\frac{I_2(z)}{z^2}\right], \tag{4.11c}$$

where

$$a_1^{(a)}(q_{||} \mid k_{||}) = -\left[2-\frac{c_R^2}{c_t^2}\right]\left[\frac{c_R^2}{c_t^2}q_{||}+\frac{z}{k_{||}}\left[2(1-\lambda^2)k_{||}^2+\frac{1}{2}\frac{c_R^4}{c_t^4}q_{||}^2\right]\right], \tag{4.12a}$$

$$a_1^{(b)}(q_{||} \mid k_{||}) = \left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\left[4k_{||}^2+\frac{c_R^4}{c_t^4}q_{||}^2\right], \tag{4.12b}$$

$$a_2^{(b)}(q_{||} \mid k_{||}) = -4\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}k_{||}\left[3k_{||}+\frac{c_R^2}{c_t^2}q_{||}z\right], \tag{4.13}$$

$$b_0^{(a)}(q_{||} \mid k_{||}) = -2\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}\right](1-\lambda^2)q_{||}\left[2k_{||}^2-\frac{c_R^2}{c_t^2}q_{||}^2\right], \tag{4.14a}$$

$$b_0^{(b)}(q_{||} \mid k_{||}) = \left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}\left[16(1-\lambda^2)^2k_{||}^2+\frac{c_R^4}{c_t^4}q_{||}^2\right], \tag{4.14b}$$

$$b_0^{(c)}(q_{||} \mid k_{||}) = \frac{c_R^4}{c_t^4}\left[1-\frac{c_R^2}{c_l^2}\right]^{1/2}q_{||}^2, \tag{4.14c}$$

$$b_1^{(a)}(q_{||} \mid k_{||}) = \frac{1}{2}\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}\right]\left[2k_{||}^2-\frac{c_R^2}{c_t^2}q_{||}^2\right]\left[2k_{||}+\frac{c_R^2}{c_t^2}q_{||}z\right]\frac{q_{||}}{k_{||}}, \tag{4.15a}$$

$$b_1^{(b)}(q_{||} \mid k_{||}) = -\left[1-\frac{c_R^2}{c_R^2}\right]^{1/2}\left[16(1-\lambda^2)k_{||}^2+\frac{c_R^4}{c_t^4}q_{||}^2+8\frac{c_R^2}{c_t^2}(1-\lambda^2)q_{||}k_{||}z\right], \tag{4.15b}$$

$$b_2^{(b)}(q_{||} \mid k_{||}) = 4\left[1-\frac{c_R^2}{c_t^2}\right]^{1/2}k_{||}\left[3k_{||}+\frac{c_R^2}{c_t^2}q_{||}z\right]. \tag{4.16}$$

The parameter B_2 introduced in Eq. (4.24) is given by

$$B_2 = \frac{4}{(2-c_R^2/c_t^2)^2}\left|\left[2-\frac{c_R^2}{c_t^2}(1+\lambda^2)\right]-\frac{1}{8}\left[2+\frac{c_R^2}{c_t^2}\right]\left[2-\frac{c_R^2}{c_t^2}\right]^3\right|. \tag{4.17}$$

The principal-part integral that enters Eq. (4.8) is defined according to

$$P\int_{(c_R/c_t)q_{||}}^{\infty} dk_{||}\cdots = \lim_{\epsilon\to 0}\left[\int_{(c_R/c_t)q_{||}}^{q_{||}-\epsilon} dk_{||}\cdots + \int_{q_{||}+\epsilon}^{\infty} dk_{||}\cdots\right]. \tag{4.18}$$

Let us first consider the shift $\nu_1(q_{||})$ given by Eq. (4.8). It can be written conveniently in the form

$$\frac{\nu_1(q_{||})}{\omega_0(q_{||})} = \left[\frac{\delta^2}{a^2}\right]\omega_1(aq_{||}). \tag{4.19}$$

Thus the ratio of the shift $\nu_1(q_{||})$ and the frequency $\omega_0(q_{||}) = c_R q_{||}$ of a Rayleigh wave propagating on a flat surface is given by the product of the "universal" function $\omega_1(aq_{\parallel})$ and the ratio $(\delta/a)^2$ (which characterizes the rough-surface profile).

In Fig. 1 we give the result for $-\omega_1(aq_{\parallel})$ that is obtained carrying out the integrals required in Eq. (4.8) numerically.[5] The frequency shift turns out to be negative for all values of $aq_{\parallel}$. As shown in Fig. 1, it has an extremum for $aq_{\parallel} \gtrsim 1$, a result that was to be expected on physical grounds. The sharp increase of $-\omega_1$ for larger values of $aq_{\parallel}$ has the following origin. It is easy to show that, for example, the last integral in Eq. (4.8) is of the form

$$P\int_{c_R/c_t}^{\infty} dx\, x \exp\left[-z_0\left[\frac{x^2}{2}-x\right]\right]\frac{1}{f(x)}\ , \quad (4.20)$$

where $f(x)$ is a bounded function of $x = k_{\parallel}/q_{\parallel}$ with a simple zero at $x=1$. Here we have called

$$z_0 = \tfrac{1}{2}(aq_{\parallel})^2\ . \quad (4.21)$$

We can rewrite the integral (4.20) according to

$$P\int_{c_R/c_t}^{\infty} dx \cdots = P\int_{c_R/c_t}^{2} dx \cdots + \int_{2}^{\infty} dx \cdots\ . \quad (4.22)$$

(We recall that the ratio c_R/c_t is always smaller than unity.) In the first (second) integral on the right-hand side of Eq. (4.22) the argument of the exponential that enters Eq. (4.20) is positive (negative). Thus, while the second integral is vanishingly small for $z_0 \gg 1$, the first one grows exponentially in that limit. This exponential blow-up is compensated for by the overall factor of $\exp(-z_0/2)$ that enters Eq. (4.8). However, these are also factors of z_0^2 in the definition of $\omega_1(aq_{\parallel})$. They account for the rapid increase of the magnitude of $-\omega_1(aq_{\parallel})$ for $aq_{\parallel} > 1$, as shown in Fig. 1.

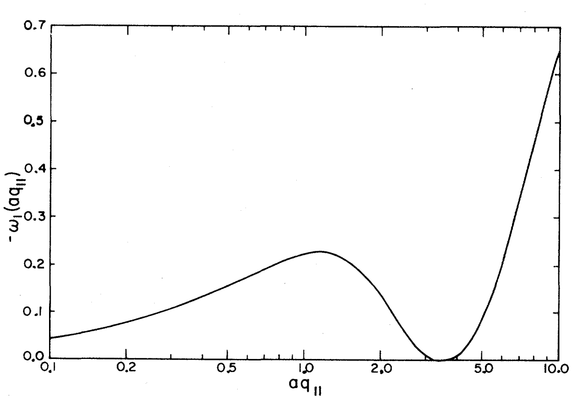

FIG. 1. Shift in the frequency of a Rayleigh wave due to surface roughness as a function of the product $aq_{\parallel}$. Note that the acutal shift (in units of $c_R q_{\parallel}$) is given by $\nu_1 = (\delta^2/a^2)\omega_1$ [see Eq. (4.19)].

We now turn to the analysis of $\nu_2(q_{\parallel})$, given by Eq. (4.9). As was done above in the case of $\nu_1(q_{\parallel})$, it is useful to rewrite $\nu_2(q_{\parallel})$ as

$$\frac{\nu_2(q_{\parallel})}{\omega_0(q_{\parallel})} = \left[\frac{\delta^2}{a^2}\right]\omega_2(aq_{\parallel})\ . \quad (4.23)$$

Note that, in terms of $\omega_2(aq_{\parallel})$, the inverse attenuation length [Eq. (4.2)] is given by

$$l^{-1}(q_{\parallel}) = 2\left[\frac{\delta^2}{a^2}\right] q_{\parallel}\omega_2(aq_{\parallel})\ . \quad (4.24)$$

From the discussion following Eq. (4.2) we readily conclude that the terms involving integrals in Eq. (4.9) give the contribution to $\omega_2(aq_{\parallel})$ due to the bulk channels $[\omega_2^{(B)}(aq_{\parallel})]$, while the last term in Eq. (4.9) gives the corresponding contribution from the surface channel $[\omega_2^{(S)}(aq_{\parallel})]$. Now, in order to establish as detailed a comparison as possible with the work of Maradudin and Mills[3] (see below), it is useful to display fully the latter contribution to $\omega_2(aq_{\parallel})$. After a little algebra, we obtain the result that

$$\omega_2^{(S)}(aq_{\parallel}) = C z_0^2 e^{-z_0} G(z_0)\ , \quad (4.25)$$

where we have called

$$C \equiv \frac{\pi}{8\,|\,B_1\,|\,B_2}\frac{c_R^2}{c_t^2}\ , \quad (4.26)$$

with B_1 and B_2 defined by Eqs. (3.26) and (4.17), respectively, and z_0 by Eq. (4.21). The function $G(z_0)$ is defined by

$$G(z_0) = G_1(z_0) + G_2(z_0)\ , \quad (4.27)$$

where

$$G_1(z_0) \equiv 4\left[1-\frac{c_R^2}{c_t^2}\right]\left[4(1-\lambda^2)^2 I_0(z_0) - 4(1-\lambda^2)\frac{I_1(z_0)}{z_0} + 3\frac{I_2(z_0)}{z_0^2}\right]\ , \quad (4.28a)$$

$$G_2(z_0) \equiv C_0 I_0(z_0) + C_1\frac{I_1(z_0)}{z_0} + C_2\frac{I_2(z_0)}{z_0^2}\ . \quad (4.28b)$$

The coefficients introduced in Eq. (4.28b) are defined by the equations

$$C_0=\frac{c_R^2}{c_t^2}\left[\frac{c_R^2}{c_t^2}\left[2-\frac{c_R^2}{c_t^2}(1+\lambda^2)\right]\right.$$
$$\left.-2(1-\lambda^2)\left[2-\frac{c_R^2}{c_t^2}\right]^2\right], \quad (4.29a)$$

$$C_1=\frac{c_R^2}{c_t^2}\left[\frac{1}{2}\left[2-\frac{c_R^2}{c_t^2}\right]^2\left[2+\frac{c_R^2}{c_t^2}z_0\right]\right.$$
$$\left.-\left[1-\frac{c_R^2}{c_t^2}\right]\left[\frac{c_R^2}{c_t^2}+8(1-\lambda^2)z_0\right]\right], \quad (4.29b)$$

$$C_2=4\frac{c_R^2}{c_t^2}\left[1-\frac{c_R^2}{c_t^2}\right]z_0 . \quad (4.29c)$$

In Figs. 2 and 3 we show the results we obtained from $\omega_2^{(B)}(aq_{\parallel})$, $\omega_2^{(S)}(aq_{\parallel})$, and their sum, $\omega_2(aq_{\parallel})$. In agreement with earlier results,[3] it is straightforward to show that $\omega_2(aq_{\parallel})\sim(aq_{\parallel})^4$ for $aq_{\parallel}\to 0$. The main new qualitative feature of our results is that $\omega_2^{(B)}>>\omega_2^{(S)}$. Thus the effect of surface roughness on the lifetime of a Rayleigh wave (or on its inverse attenuation length) is due almost entirely to the decay of the Rayleigh wave into bulk sound waves. This result is in direct contradiction with the earlier results of Maradudin and Mills.[3] The result that is obtained from Ref. 3 for $\tilde{\omega}^{(S)}(aq_{\parallel})$ is

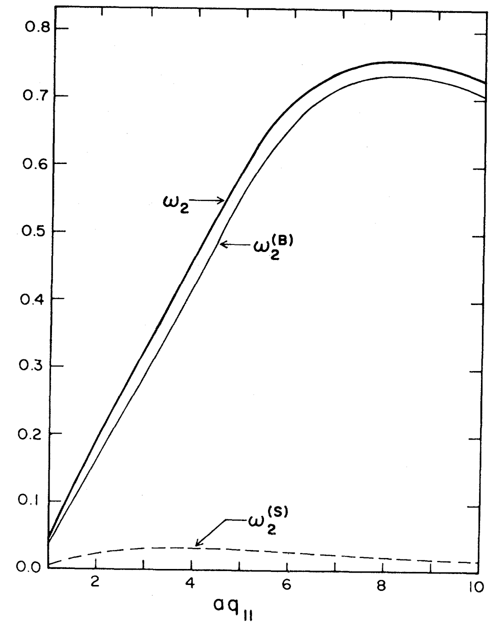

FIG. 3. Same as Fig. 2 but for larger values of the product $aq_{\parallel}$.

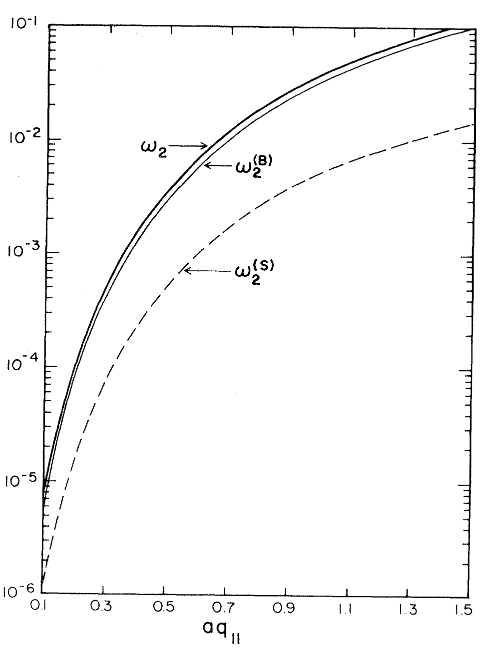

FIG. 2. Imaginary part of the perturbation due to surface roughness of the Rayleigh-wave dispersion relation [cf. Eq. (4.23)] for small and intermediate values of the product $aq_{\parallel}$. The figure shows that the contribution to $\omega_2(aq_{\parallel})$ from the bulk channels ($\omega_2^{(B)}$) dominates that due to the surface channel ($\omega_2^{(S)}$).

$$\tilde{\omega}_2^{(S)}(aq_{\parallel})=Cz_0^2e^{-z_0}G_1(z_0) . \quad (4.30)$$

Comparing Eqs. (4.25) and (4.30) we see that they differ by the presence of the function $G_2(z_0)$ in the definition of $G(z_0)$ [see Eq. (4.27)]. Now, it turns out that the functions $G_1(z_0)$ and $G_2(z_0)$ are of the same order of magnitude and of opposite sign. Thus the contribution from the surface channel to ω_2 or l^{-1} is drastically reduced, compared to the results of Ref. 3, and the decay into bulk modes is the more efficient mechanism provided by surface roughness for the attenuation of a Rayleigh wave.

The origin of this discrepancy is explained in detail in Appendix B, where we also indicate how the method of Ref. 3, if correctly implemented, does give rise to results for the attenuation length which agree with those obtained in the present work.

We finish this paper by giving a few representative numbers for the shift $\nu_1(q_{\parallel})$ and the mean free path $l(q_{\parallel})$. Assuming a ratio $\delta/a=0.3$, we have that the relative downward shift of the frequency of a Rayleigh wave is 0.4% for $aq_{\parallel}=0.1$, 2% for $aq_{\parallel}=1.0$, and 5.8% for $aq_{\parallel}=10.0$. As for the mean free path $l(q_{\parallel})$, assuming the rather typical values $\omega=10^8$ sec^{-1} and $c_R=3\times10^5$ cm sec^{-1}, for which we have $q_{\parallel}=333.33$ cm^{-1} (the corresponding wavelength is $\lambda = 2\pi/q_{\parallel} \cong 0.02$ cm), we obtain the results that $l = 2.6\times 10^3$ cm for $aq_{\parallel} = 0.1$, $l = 0.42$ cm for $aq_{\parallel} = 1$, and $l = 0.02$ cm for $aq_{\parallel} = 10.0$. (Note that while in the first case $l \gg \lambda$, in the last one $l \cong \lambda$.) Of course, the mean free path is reduced if we increase the ratio δ^2/a^2. However, in the context of our perturbation theory, values of this ratio much greater than the one used in this example are not expected to be meaningful.

ACKNOWLEDGMENT

This work was supported in part by U. S. Army Research Office Grant No. DAAG 29-78-G0108.

APPENDIX A

The matrix $\mathscr{P}(\vec{q}_{\parallel}\,|\,\omega)$ is defined by Eq. (2.40). The matrices $\overleftrightarrow{M}^{(a)}(\vec{q}_{\parallel}\,|\,\omega)$ and $\overleftrightarrow{M}^{(b)}(\vec{q}_{\parallel}\,|\,\omega)$ entering that equation are defined by Eqs. (2.37) and (2.38), respectively.

We consider first the matrix $\overleftrightarrow{M}^{(a)}(\vec{q}_{\parallel}\,|\,\omega)$. We note that because $g(Q_{\parallel})$ depends only on the magnitude of the wave vector $\vec{Q}_{\parallel}$, all the terms linear in $\vec{Q}_{\parallel}$ in the definition of $Z_{\alpha\beta}(\vec{Q}_{\parallel};\vec{q}_{\parallel}\,|\,\omega)$ [see Eqs. (2.20)] give a nonvanishing contribution to the integral in Eq. (2.37). Moreover, the remaining terms in Eq. (2.20) are independent of $Q_{\parallel}$. Thus noting that the condition $W(0)=1$ implies that

$$\int \frac{d^2Q_{\parallel}}{(2\pi)^2} g(Q_{\parallel}) = 1\ , \qquad \text{(A1)}$$

we have the simple results that

$$M_{\alpha 1}^{(a)}(\vec{q}_{\parallel}\,|\,\omega) = \tfrac{1}{2}\alpha_l^2(q_{\parallel}) M_{\alpha 1}^{(0)}(\vec{q}_{\parallel}\,|\,\omega)\ , \qquad \text{(A2a)}$$

$$M_{\alpha 2}^{(a)}(\vec{q}_{\parallel}\,|\,\omega) = \tfrac{1}{2}\alpha_t^2(q_{\parallel}) M_{\alpha 2}^{(0)}(\vec{q}_{\parallel}\,|\,\omega)\ , \qquad \text{(A2b)}$$

$$M_{\alpha 3}^{(a)}(\vec{q}_{\parallel}\,|\,\omega) = \tfrac{1}{2}\alpha_t^2(q_{\parallel}) M_{\alpha 3}^{(0)}(\vec{q}_{\parallel}\,|\,\omega)\ , \qquad \text{(A2c)}$$

where $\alpha = 1,2,3$ and the elements of the matrix $\overleftrightarrow{M}^{(0)}(\vec{q}_{\parallel}\,|\,\omega)$ are given by Eqs. (2.18).

The algebra involved in carrying out the matrix products required in the definition given by Eq. (2.39) is extremely lengthy. Here we simply give the final result. It is convenient to make the following definitions:

$$\Delta(k_{\parallel}\,|\,\omega) = 4k_{\parallel}^2\alpha_l(k_{\parallel})\alpha_t(k_{\parallel}) - \left[2k_{\parallel}^2 - \frac{\omega^2}{c_t^2}\right]^2\ , \qquad \text{(A3)}$$

$$\beta(k_{\parallel}\,|\,\omega) = 2k_{\parallel}^2 - \frac{\omega^2}{c_t^2} - 2\alpha_l(k_{\parallel})\alpha_t(k_{\parallel})\ , \qquad \text{(A4)}$$

$$\epsilon(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega) = 2(\vec{q}_{\parallel}\cdot\vec{k}_{\parallel}) - \frac{\omega^2}{c_t^2}\ , \qquad \text{(A5)}$$

$$\Theta_1(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega) = k_2(\vec{q}_{\parallel}\cdot\vec{k}_{\parallel}) + k_1(\vec{q}_{\parallel}\times\vec{k}_{\parallel})_3 - k_2\frac{\omega^2}{c_t^2}\ , \qquad \text{(A6)}$$

$$\Theta_2(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega) = k_1(\vec{q}_{\parallel}\cdot\vec{k}_{\parallel}) - k_2(\vec{q}_{\parallel}\times\vec{k}_{\parallel})_3 - k_1\frac{\omega^2}{c_t^2}\ , \qquad \text{(A7)}$$

$$\Lambda_1(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega) = 2k_1(\vec{q}_{\parallel}\cdot\vec{k}_{\parallel}) + 2q_1k_{\parallel}^2(1-2\lambda^2) - k_1\frac{\omega^2}{c_t^2}\ , \qquad \text{(A8)}$$

$$\Lambda_2(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega) = 2k_2(\vec{q}_{\parallel}\cdot\vec{k}_{\parallel}) + 2q_2k_{\parallel}^2(1-2\lambda^2) - k_2\frac{\omega^2}{c_t^2}\ . \qquad \text{(A9)}$$

With these definitions, we can express the elements of the matrix $\overleftrightarrow{R}(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega)$ as follows (for brevity we suppress the explicit reference to the frequency ω from most functions):

$$R_{11}(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega) = \frac{c_t^2\epsilon(\vec{q}_{\parallel};\vec{k}_{\parallel})}{q_{\parallel}}\left[q_1(1-2\lambda^2)\alpha_l(q_{\parallel}) - \frac{\Theta_1(\vec{q}_{\parallel};\vec{k}_{\parallel})}{k_{\parallel}^2\alpha_t(k_{\parallel})}(\vec{q}_{\parallel}\times\vec{k}_{\parallel})_3\right] + \frac{c_t^2}{q_{\parallel}}\frac{\Lambda_1(\vec{q}_{\parallel};\vec{k}_{\parallel})}{\Delta(k_{\parallel}\,|\,\omega)}\left[\alpha_l(q_{\parallel})\beta(k_{\parallel})\epsilon(\vec{q}_{\parallel};\vec{k}_{\parallel}) - \frac{\omega^2}{c_t^2}\frac{\alpha_t(k_{\parallel})}{k_{\parallel}^2}\left[(\vec{q}_{\parallel}\cdot\vec{k}_{\parallel})\epsilon(\vec{q}_{\parallel};\vec{k}_{\parallel}) + \frac{\omega^2}{c_t^2}k_{\parallel}^2(1-2\lambda^2)\right]\right]\ , \qquad \text{(A10)}$$

$$R_{12}(\vec{q}_{||};\vec{k}_{||}\omega)=\frac{c_t^2}{q_{||}}\left[\frac{-\epsilon(\vec{q}_{||};\vec{k}_{||})\Theta_1(\vec{q}_{||};\vec{k}_{||})}{k_{||}^2\alpha_t(k_{||})}(\vec{q}_{||}\times\vec{k}_{||})_3+\frac{q_1(1-2\lambda^2)}{\alpha_t(q_{||})}\left[q_{||}^2\epsilon(\vec{q}_{||};\vec{k}_{||})-\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]\right]$$
$$+\frac{c_t^2}{q_{||}}\frac{\Lambda_1(\vec{q}_{||};\vec{k}_{||})}{\Delta(k_{||}\mid\omega)}\left[\frac{\beta(k_{||})}{\alpha_t(q_{||})}\left[q_{||}^2\epsilon(\vec{q}_{||};\vec{k}_{||})-\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]-\frac{\omega^2}{c_t^2}\frac{\epsilon(\vec{q}_{||};\vec{k}_{||})\alpha_t(k_{||})}{k_{||}^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right], \tag{A11}$$

$$R_{13}(\vec{q}_{||};\vec{k}_{||}\mid\omega)=\frac{c_t^2}{q_{||}}\left[q_1(1-2\lambda^2)\alpha_t(q_{||})(\vec{q}_{||}\times\vec{k}_{||})_3-\frac{\Theta_1(\vec{q}_{||};\vec{k}_{||})}{k_{||}^2\alpha_t(k_{||})}\left[(\vec{q}_{||}\times\vec{k}_{||})_3^2-(\vec{q}_{||}\cdot\vec{k}_{||})^2+\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]\right]$$
$$+\frac{c_t^2}{q_{||}}\frac{\Lambda_1(\vec{q}_{||};\vec{k}_{||})}{\Delta(k_{||}\mid\omega)}(\vec{q}_{||}\times\vec{k}_{||})_3\left[\alpha_t(q_{||})\beta(k_{||})-\frac{\omega^2}{c_t^2}\frac{\epsilon(\vec{q}_{||};\vec{k}_{||})\alpha_t(k_{||})}{k_{||}^2}\right], \tag{A12}$$

$$R_{21}(\vec{q}_{||};\vec{k}_{||}\mid\omega)=\frac{c_t^2\epsilon(\vec{q}_{||};\vec{k}_{||})}{q_{||}}\left[q_2(1-2\lambda^2)\alpha_l(q_{||})+\frac{\Theta_2(\vec{q}_{||};\vec{k}_{||})}{k_{||}^2\alpha_t(k_{||})}(\vec{q}_{||}\times\vec{k}_{||})_3\right]$$
$$+\frac{c_t^2}{q_{||}}\frac{\Lambda_2(\vec{q}_{||};\vec{k}_{||})}{\Delta(k_{||}\mid\omega)}\left[\alpha_l(q_{||})\beta(k_{||})\epsilon(\vec{q}_{||};\vec{k}_{||})\right.$$
$$\left.-\frac{\omega^2}{c_t^2}\frac{\alpha_t(k_{||})}{k_{||}^2}\left[(\vec{q}_{||}\cdot\vec{k}_{||})\epsilon(\vec{q}_{||};\vec{k}_{||})+\frac{\omega^2}{c_t^2}k_{||}^2(1-2\lambda^2)\right]\right], \tag{A13}$$

$$R_{22}(\vec{q}_{||};\vec{k}_{||}\mid\omega)=\frac{c_t^2}{q_{||}}\left[\frac{\epsilon(\vec{q}_{||};\vec{k}_{||})\Theta_2(\vec{q}_{||};\vec{k}_{||})}{k_{||}^2\alpha_t(k_{||})}(\vec{q}_{||}\times\vec{k}_{||})_3+\frac{q_2(1-2\lambda^2)}{\alpha_t(q_{||})}\left[q_{||}^2\epsilon(\vec{q}_{||};\vec{k}_{||})-\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]\right]$$
$$+\frac{c_t^2}{q_{||}}\frac{\Lambda_2(\vec{q}_{||};\vec{k}_{||})}{\Delta(k_{||}\mid\omega)}\left[\frac{\beta(k_{||})}{\alpha_t(q_{||})}\left[q_{||}^2\epsilon(\vec{q}_{||};\vec{k}_{||})-\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]-\frac{\omega^2}{c_t^2}\frac{\epsilon(\vec{q}_{||};\vec{k}_{||})\alpha_t(k_{||})}{k_{||}^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right], \tag{A14}$$

$$R_{23}(\vec{q}_{||};\vec{k}_{||}\mid\omega)=\frac{c_t^2}{q_{||}}\left[q_2(1-2\lambda^2)\alpha_t(q_{||})(\vec{q}_{||}\times\vec{k}_{||})_3+\frac{\Theta_2(\vec{q}_{||};\vec{k}_{||})}{k_{||}^2\alpha_t(k_{||})}\left[(\vec{q}_{||}\times\vec{k}_{||})_3^2-(\vec{q}_{||}\cdot\vec{k}_{||})^2+\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]\right]$$
$$+\frac{c_t^2}{q_{||}}\frac{\Lambda_2(\vec{q}_{||};\vec{k}_{||})}{\Delta(k_{||}\mid\omega)}(\vec{q}_{||}\times\vec{k}_{||})_3\left[\alpha_t(q_{||})\beta(k_{||})-\frac{\omega^2}{c_t^2}\frac{\epsilon(\vec{q}_{||};\vec{k}_{||})\alpha_t(k_{||})}{k_{||}^2}\right], \tag{A15}$$

$$R_{31}(\vec{q}_{||};\vec{k}_{||}\mid\omega)=-i\frac{c_t^2}{q_{||}}\left[q_{||}^2\epsilon(\vec{q}_{||};\vec{k}_{||})+\frac{\omega^2}{c_t^2}(1-2\lambda^2)(\vec{q}_{||}\cdot\vec{k}_{||})\right]$$
$$+i\frac{\omega^2}{q_{||}}\frac{1}{\Delta(k_{||}\mid\omega)}\left[\frac{\omega^2}{c_t^2}\epsilon(\vec{q}_{||};\vec{k}_{||})\alpha_l(q_{||})\alpha_l(k_{||})\right.$$
$$\left.-\beta(k_{||})\left[\frac{\omega^2}{c_t^2}k_{||}^2(1-2\lambda^2)+(\vec{q}_{||}\cdot\vec{k}_{||})\epsilon(\vec{q}_{||};\vec{k}_{||})\right]\right], \tag{A16}$$

$$R_{32}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega)=-ic_t^2 q_{||}\epsilon(\vec{q}_{||};\vec{k}_{||})+i\frac{\omega^2}{q_{||}}\frac{1}{\Delta(k_{||}\,|\,\omega)}\left[-\beta(k_{||})(\vec{q}_{||}\cdot\vec{k}_{||})\epsilon(\vec{q}_{||};\vec{k}_{||}) + \frac{\omega^2}{c_t^2}\frac{\alpha_l(k_{||})}{\alpha_t(q_{||})}\left[q_{||}^2\epsilon(\vec{q}_{||};\vec{k}_{||})-\frac{\omega^2}{c_t^2}(\vec{q}_{||}\cdot\vec{k}_{||})\right]\right], \tag{A17}$$

$$R_{33}(\vec{q}_{||};\vec{k}_{||}\,|\,\omega)=-ic_t^2 q_{||}(\vec{q}_{||}\times\vec{k}_{||})_3+i\frac{\omega^2}{q_{||}}\frac{(\vec{q}_{||}\times\vec{k}_{||})_3}{\Delta(k_{||}\,|\,\omega)}\left[-\beta(k_{||})\epsilon(\vec{q}_{||};\vec{k}_{||})+\frac{\omega^2}{c_t^2}\alpha_l(k_{||})\alpha_t(q_{||})\right]. \tag{A18}$$

We note that the results given by Eqs. (A10)—(A18) consist of two different types of terms, namely those that have the function $\Delta(k_{||}\,|\,\omega)$ in the denominator and those that do not. This point is of relevance in Sec. IV, since for a given value of the wave vector $\vec{k}_{||}$, the root of the equation $\Delta(k_{||}\,|\,\omega)=0$ occurs at the frequency $\omega_0(k_{||})$ of a Rayleigh wave propagating on a flat surface. [This statement can be easily verified using Eqs. (2.3), (2.4), and (A3).]

APPENDIX B

In this appendix we explain the discrepancy between the result obtained in this paper for the inverse attenuation length of Rayleigh waves propagating on a rough surface and the corresponding result obtained by Maradudin and Mills.[3]

The boundary conditions (2.12)—(2.14) can be expanded in powers of $\zeta(\vec{x}_{||})$ to yield a set of effective boundary conditions on the surface $x_3=0$ instead of on the true surface $x_3=\zeta(\vec{x}_{||})$. To first order in $\zeta(\vec{x}_{||})$ these effective boundary conditions are $(\alpha=1,2,3)$

$$\sum_{\mu,\nu}c_{\alpha3\mu\nu}\frac{\partial u_\mu(\vec{x}\,|\,\omega)}{\partial x_\nu}\bigg|_{x_3=0}=\sum_{\delta=1}^{2}\sum_{\mu,\nu}c_{\alpha\delta\mu\nu}\frac{\partial\zeta(\vec{x}_{||})}{\partial x_\delta}\frac{\partial u_\mu(\vec{x}\,|\,\omega)}{\partial x_\nu}\bigg|_{x_3=0}-\sum_{\mu,\nu}c_{\alpha3\mu\nu}\zeta(\vec{x}_{||})\frac{\partial^2 u_\mu(\vec{x}\,|\,\omega)}{\partial x_3\partial x_\nu}\bigg|_{x_3=0}. \tag{B1}$$

In what follows the summation index δ will be understood to assume only the values of 1 and 2. Since the vanishing of the left-hand side of this equation expresses the stress-free boundary conditions on the planar surface of a semi-infinite elastic medium, we see from Eq. (B1) that the effects of surface roughness can be expressed as the presence of nonzero stresses, given by the right-hand side of Eq. (B1), acting on the planar surface $x_3=0$.

We are thus led to seek a solution of the equations of motion (2.1) in the elastic medium that now satisfies the boundary conditions (B1). For this purpose we introduce a Green's tensor[3] $D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)$ as the solution of the inhomogeneous equation

$$\sum_{\mu}L_{\alpha\mu}(\vec{x}\,|\,\omega)D_{\mu\gamma}(\vec{x};\vec{y}\,|\,\omega)=\delta_{\alpha\gamma}\delta(\vec{x}-\vec{y})\ ,\quad x_3>0\ ,\quad y_3>0\ ,\quad \alpha,\gamma=1,2,3 \tag{B2}$$

that satisfies the boundary conditions

$$\sum_{\mu,\nu}c_{\alpha3\mu\nu}\frac{\partial}{\partial x_\nu}D_{\mu\gamma}(\vec{x};\vec{y}\,|\,\omega)\bigg|_{x_3=0}=0\ ,\quad \alpha=1,2,3 \tag{B3}$$

on the surface $x_3=0$, together with exponentially decaying or outgoing wave boundary conditions as $x_3\to\infty$.

The following generalization of Green's theorem is valid for any two vector functions $\vec{U}(\vec{x}\,|\,\omega)$ and $\vec{V}(\vec{x}\,|\,\omega)$ defined within and on the boundary of a volume V whose surface is denoted by Σ (Ref. 6):

$$\sum_{\alpha,\mu}\int_V d^3x[U_\alpha(\vec{x}\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)V_\mu(\vec{x}\,|\,\omega)-V_\alpha(\vec{x}\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)U_\mu(\vec{x}\,|\,\omega)]$$
$$=\frac{1}{\rho}\sum_{\alpha,\beta,\mu,\nu}\int_\Sigma dS\,\hat{n}_\beta c_{\alpha\beta\mu\nu}\left[U_\alpha(\vec{x}\,|\,\omega)\frac{\partial}{\partial x_\nu}V_\mu(\vec{x}\,|\,\omega)-V_\alpha(\vec{x}\,|\,\omega)\frac{\partial}{\partial x_\nu}U_\mu(\vec{x}\,|\,\omega)\right], \tag{B4}$$

where $\hat{n}$ is the unit vector normal to the surface Σ, directed from inside the volume V to outside it.

If we now let the volume V be the half-space $x_3>0$, the surface Σ consists of the plane $x_3=0$ and a hemisphere in the upper half-space whose radius is allowed to become infinite. Then, with $U_\alpha(\vec{x}\,|\,\omega)\equiv u_\alpha(\vec{x}\,|\,\omega)$ and $V_\alpha(\vec{x}\,|\,\omega)\equiv D_{\alpha\gamma}(\vec{x};\vec{y}\,|\,\omega)$ for fixed $\vec{y}$ and γ, we obtain from Eq. (B4)

$$u_\gamma(\vec{y}\,|\,\omega)\Theta_v(\vec{y}) = -\frac{1}{\rho}\sum_\alpha \int d^2x_{||} \left[u_\alpha(\vec{x}\,|\,\omega)\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\frac{\partial}{\partial x_\nu} D_{\mu\nu}(\vec{x};\vec{y}\,|\,\omega) - D_{\alpha\gamma}(\vec{x};\vec{y}\,|\,\omega)\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\frac{\partial}{\partial x_\nu} u_\mu(\vec{x}\,|\,\omega)\right]_{x_3=0}, \tag{B5}$$

where $\Theta_v(\vec{y})$ equals unity if $\vec{y}$ is in the volume V ($y_3>0$) and vanishes otherwise. Since we are seeking a displacement field $\vec{u}(\vec{x}\,|\,\omega)$ that is exponentially localized to the surface $x_3=0$ (a Rayleigh wave), the integral over the hemispherical surface in the space $x_3>0$ gives no contribution to the right-hand side of Eq. (B4) in the limit as its radius becomes infinite.

Equation (B5) can be simplified by noting that the first term on the right-hand side vanishes (for $y_3>0$), in view of the boundary condition (B3) satisfied by the Green's function $D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)$. We next use the fact that the Green's tensor is symmetric,[6]

$$D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega) = D_{\beta\alpha}(\vec{y};\vec{x}\,|\,\omega), \tag{B6}$$

and the result given by Eq. (B1) to rewrite Eq. (B5) for $y_3>0$ as

$$u_\gamma(\vec{y}\,|\,\omega) = \frac{1}{\rho}\sum_\alpha \int d^2x_{||} D_{\gamma\alpha}(\vec{y};\vec{x}\,|\,\omega)\,|_{x_3=0} \left[\sum_\delta \sum_{\mu,\nu} c_{\alpha\delta\mu\nu}\frac{\partial\zeta(\vec{x}_{||})}{\partial x_\delta}\frac{\partial u_\mu(\vec{x}\,|\,\omega)}{\partial x_\nu} - \sum_{\mu,\nu} c_{\alpha 3\mu\nu}\zeta(\vec{x}_{||})\frac{\partial^2 u_\mu(\vec{x}\,|\,\omega)}{\partial x_3 \partial x_\nu}\right]_{x_3=0}. \tag{B7}$$

The result given by Eq. (B7) is a particular solution of the equations of motion of an elastic medium that satisfies the effective boundary conditons (B1). To it we can add any other solution of the equations of motion. If we wish to study how a Rayleigh wave on a flat surface is scattered by surface roughness, we choose for this additional solution $u_\gamma^{(0)}(\vec{y}\,|\,\omega)$, which satisfies stress-free boundary condtions on the flat surface $x_3=0$

$$\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\frac{\partial u_\mu^{(0)}(\vec{y}\,|\,\omega)}{\partial y_\nu}\Bigg|_{y_3=0} = 0, \quad \alpha=1,2,3 \tag{B8}$$

and represents the incident Rayleigh wave. With some changes of variables we thus have as the integral equation describing the scattering of a Rayleigh wave by surface roughness

$$u_\alpha(\vec{x}\,|\,\omega) = u_\alpha^{(0)}(\vec{x}\,|\,\omega) + \frac{1}{\rho}\sum_\beta \int d^2y_{||} D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)\,|_{y_3=0} \left[\sum_\delta \sum_{\gamma,\nu} c_{\beta\delta\gamma\nu}\frac{\partial\zeta(\vec{y}_{||})}{\partial y_\delta}\frac{\partial u_\gamma(\vec{y}\,|\,\omega)}{\partial y_\nu} - \sum_{\gamma,\nu} c_{\beta 3\gamma\nu}\zeta(\vec{y}_{||})\frac{\partial^2 u_\gamma(\vec{y}\,|\,\omega)}{\partial y_3 \partial y_\nu}\right]_{y_3=0}. \tag{B9}$$

We can now compare this equation with the one that provided the starting point for the analysis by Maradudin and Mills.[3] Equation (2.7) of their paper, rewritten in time-independent form, is

$$u_\alpha(\vec{x}\,|\,\omega) = u_\alpha^{(0)}(\vec{x}\,|\,\omega) - \sum_{\beta\gamma}\int d^3y\, D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega) L_{\beta\gamma}^{(1)}(\vec{y}) u_\gamma(\vec{y}\,|\,\omega), \tag{B10}$$

where the operator $L_{\beta\gamma}^{(1)}(\vec{y})$ is given by

$$L_{\beta\gamma}^{(1)}(\vec{y}) = -\frac{1}{\rho}\delta(y_3)\sum_{\delta\nu} c_{\beta\delta\gamma\nu}\frac{\partial\zeta(\vec{y}_{||})}{\partial y_\delta}\frac{\partial}{\partial y_\nu} - \frac{1}{\rho}\delta'(y_3)\sum_\nu c_{\beta 3\gamma\nu}\zeta(\vec{y}_{||})\frac{\partial}{\partial y_\nu} - \frac{1}{\rho}\delta(y_3)\sum_{\mu\nu} c_{\beta\mu\gamma\nu}\zeta(\vec{y}_{||})\frac{\partial^2}{\partial y_\mu \partial y_\nu}. \tag{B11}$$

When Eq. (B11) is substituted into Eq. (B10) we obtain

$$\begin{aligned} u_\alpha(\vec{x}\,|\,\omega) = {} & u_\alpha^{(0)}(\vec{x}\,|\,\omega) + \frac{1}{\rho}\sum_\beta \int d^2y_{||} D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)\,|_{y_3=0} \sum_\delta \sum_{\gamma,\nu} c_{\beta\delta\gamma\nu}\frac{\partial\zeta(\vec{y}_{||})}{\partial y_\delta}\frac{\partial u_\gamma(\vec{y}\,|\,\omega)}{\partial y_\nu}\Bigg|_{y_3=0} \\ & + \frac{1}{\rho}\sum_\beta \int d^3y\, D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)\delta'(y_3)\sum_{\gamma,\nu} c_{\beta 3\gamma\nu}\frac{\partial u_\gamma(\vec{y}\,|\,\omega)}{\partial y_\nu} \\ & + \frac{1}{\rho}\sum_\beta \int d^2y_{||} D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)\,|_{y_3=0} \sum_{\gamma,\mu,\nu} c_{\beta\mu\gamma\nu}\zeta(\vec{y}_{||})\frac{\partial^2 u_\gamma(\vec{y}\,|\,\omega)}{\partial y_\mu \partial y_\nu}\Bigg|_{y_3=0}. \end{aligned} \tag{B12}$$

A comparison of Eq. (B12) with Eq. (B9) reveals the following. The first of the integral terms on the right-hand side of each of these equations is the same. The second integral term on the right-hand side of Eq. (B12) would agree exactly with the corresponding term on the right-hand side of Eq. (B9) if the projective property of the delta function,

$$\delta'(y_3)f(y_3) = -\delta(y_3)\left[\frac{df(y_3)}{dy_3}\right]_{y_3=0}, \quad \text{(B13)}$$

were used, and only the factor $\partial u_\gamma(\vec{y}\,|\,\omega)/\partial y_\nu$ played the role of the function $f(y_3)$ in Eq. (B12). Indeed, this is what should have been done in Ref. 3, because Eq. (B10) was obtained by applying a Green's-function approach to an inhomogeneous partial differential equation in which

$$-\sum_\gamma L^{(1)}_{\alpha\gamma}(\vec{x})u_\gamma(\vec{x}\,|\,\omega)$$

plays the role of the inhomogeneous, or source term. If this source term had been transformed first with the use of Eq. (B13), to represent a boundary condition on the plane $x_3=0$, and *then* the differential equation had been converted into an integral equation with the use of Green's function, the first two integral terms on the right-hand side of Eq. (B12) would have coincided with the corresponding terms on the right-hand side of Eq. (B9). As it was, the second integral term on the right-hand side of Eq. (B12) was transformed with the aid of Eq. (B13) in Ref. 3 with the product $D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)\partial u_\gamma(\vec{y}\,|\,\omega)/\partial y_\nu$ playing the role of the function $f(y_3)$. In this way terms containing

$$\left[\frac{\partial D_{\alpha\beta}(\vec{x};\vec{y}\,|\,\omega)}{\partial y_3}\frac{\partial u_\gamma(\vec{y}\,|\,\omega)}{\partial y_\nu}\right]_{y_3=0}$$

were introduced into the right-hand side of Eq. (B12) that have no counterpart in Eq. (B9). Finally, the third integral term on the right-hand side of Eq. (B12) has no counterpart on the right-hand side of Eq. (B9). Since this term represents a modification of the equations of motion in the region of space occupied by the elastic medium, which in fact require no modification, rather than an (effective) boundary condition, it should not have appeared on the right-hand side of Eq. (B12) in the work of Ref. 3.

It is thus the fact that additional terms [beyond those that appear on the right-hand side of Eq. (B9)] are incorrectly present on the right-hand side of Eq. (B12) (which was the starting point for the work in Ref. 3) that accounts for the differences between the results of that reference and those of the present paper for the attenuation of Rayleigh waves on a randomly rough surface. Indeed, if Eq. (B9) is solved for the scattered wave in first Born approximation, and the inverse attenuation length is obtained by a conservation of energy argument, just as this was done in Ref. 3, the result obtained coincides exactly with the result, Eq. (4.24), obtained here. The details of that calculation will be presented elsewhere.

[1]E. I. Urazakov and L. A. Fal'kovskii, Zh. Eksp. Teor. Fiz. 63, 2297 (1972) [Sov. Phys.—JETP 36, 1214 (1973)].

[2]Lord Rayleigh, Philos. Mag. 14, 70 (1907); *Theory of Sound,* 2nd ed. (Dover, New York, 1945), p. 89.

[3]A. A. Maradudin and D. L. Mills, Ann. Phys. (N.Y.) 100, 262 (1976).

[4]Progress has recently been made in the study of the effects of large-amplitude surface roughness in the case that the rough profile is a periodic function (grating). Within the present context of elastic waves, see N. E. Glass, R. Loudon, and A. A. Maradudin, Phys. Rev. B 24, 6843 (1981) and N. E. Glass and A. A. Maradudin, Electron. Lett. 77, 773 (1981).

[5]We note that the principal-part integral required in Eq. (4.8) was computed in two different ways, both methods yielding the same results. (a) We used the definition given by Eq. (4.18). The parameter ϵ was chosen sufficiently small so that the result of the computation was independent of ϵ. (b) Noting the result that

$$\int_a^b dx\frac{1}{x-c} = \ln\left|\frac{b-c}{a-c}\right|,$$

we set

$$P\int_a^b dx\frac{f(x)}{\Delta(x)} \cong \sum_{n=0}^{N-1}\frac{f(\tilde{x}_n)}{\left[\frac{d\Delta}{dx}\right]_{x=\tilde{x}_n}}\ln\left|\frac{x_{n+1}-c_n}{x_n-c_n}\right|.$$

Here $x_n=a+nh$, $h=(b-a)/N$, $\tilde{x}_n=x_n+\frac{1}{2}h$, and

$$c_n=\tilde{x}_n-\left[\frac{\Delta(x)}{d\Delta(x)/dx}\right]_{x=\tilde{x}_n}.$$

For sufficiently large values of N, the results so obtained are independent of N.

[6]A. G. Eguiluz and A. A. Maradudin, preceding paper, Phys. Rev. B 28, 711 (1983).