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Effective boundary conditions for a semi-infinite elastic medium bounded by a rough planar stress-free surface

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The problem of determining the static or dynamic elastic displacement field in a semi-infinite medium bounded by a rough, planar, stress-free surface is a random problem due to the stochastic nature of the surface-profile function. What is usually solved for in such problems is the displacement field in the medium averaged over the ensemble of realizations of the surface roughness. With the aid of Green's theorem we have replaced the true boundary conditions on the displacement field at the actual surface of the medium by effective boundary conditions satisfied by the average displacement field at the nominal, flat surface. The average field can then be obtained by solving an effective (nonstochastic) problem that is formally similar to the flat-surface problem. We apply this method to obtain the attenuation length and frequency shift of a Rayleigh wave due to surface roughness on an isotropic medium. The results are found to be in agreement with those of a calculation based on the boundary-matching method (Rayleigh's method).

I. INTRODUCTION

The study of dynamic and static properties of a semi-infinite elastic medium bounded by a rough, stress-free planar surface is made difficult by the necessity of satisfying the stress-free boundary conditions point by point along an irregular surface. It is attractive, therefore, to explore the possibility of replacing the original boundary conditions on the rough surface by modified, effective boundary conditions on the planar surface that is the nominal surface of the solid. In this paper we show how this can be done if it is not the actual displacement field in the solid that is being sought, but rather the displacement field averaged over the ensemble of realizations of the surface roughness. The limitation of the treatment to the mean displacement field in a solid is not overly restrictive. It suffices for the determination of the dispersion relation for Rayleigh waves propagating along a rough surface, and for obtaining the specular component of an elastic wave scattered from a rough surface, for example. There are other applications to which the results obtained in this paper can be put, e.g., the determination of the static and dynamic Green's tensors for an elastic half-space bounded by a rough surface.

Effective boundary conditions have been used in the study of other problems associated with rough surfaces. The derivation by Bass[1] of the boundary conditions for the averaged electromagnetic field at a statistically rough surface is an example of prior work of this type.

Although the derivation of the effective boundary conditions for the averaged displacement field will be presented here in complete generality, we will quickly specialize the results obtained to the case of isotropic elastic media, for which quite explicit results can be obtained.

The outline of this paper is as follows. In Sec. II we consider an arbitrary semi-infinite elastic medium bounded by a rough surface. The actual vector-displacement field satisfies the stress-free boundary conditions at every point on the rough surface. Because of the presence of roughness, the actual displacement field is a *random* field. We decompose it into two components: its average component mentioned above and its fluctuating component. With the assumption that the roughness is sufficiently small so that a perturbation theory is adequate, we are able to transform the original stress-free boundary conditions at the actual surface into new boundary conditions for the average component of the displacement field at the nominal flat surface. In physical terms, the new boundary conditions for the average field can be considered as defining an effective, fictitious stress field that, applied at the nominal flat surface, will give the same average results as the original boundary conditions.

In Sec. III we specialize the effective boundary conditions to the case of an isotropic elastic medium. The algebra involved in so doing is rather lengthy, but the final result can be given in a rather concise form.

We emphasize that, having obtained our effective boundary conditions, we have in effect taken into account the irregular nature of the surface (and the random nature of the problem) once and for all. For example, with our effective boundary conditions we can obtain the average Green's functions for a semi-infinite elastic medium bounded by a rough surface by a procedure that is formally identical to that developed by Maradudin and Mills[2] for the evaluation of the Green's functions appropriate to the case of a flat surface. That is, both sets of Green's functions obey the same differential equation and satisfy boundary conditions at the surface $x_3=0$. In the flat-surface case those boundary conditions are the stress-free boundary conditions. In the rough surface case they are the effective boundary conditions we obtain in this paper. We note that the aforementioned average Green's functions are a basic ingredient for the study of a number of problems of physical significance, such as the Brillouin scattering of light from a rough surface.

That application of our theory has been relegated to a future publication. In Sec. IV we outline a simpler illustration of our theory by obtaining the dispersion relation $\omega_R(q_{||})$ of a Rayleigh wave propagating along a rough surface. (Here $q_{||}$ is a wave vector parallel to the nominal surface.) We obtain an expression for the perturbation due to the roughness on the dispersion relation that applies when the surface is flat. That perturbation has both a real and an imaginary part. The real part measures the shift in the frequency away from the flat-surface result. The imaginary part measures the inverse attenuation length of the Rayleigh wave.

This paper is concluded with four Appendixes. In particular, in Appendix A we obtain one of the central results of our formalism, namely an expression for the fluctuating component of the elastic displacement at an arbitrary point inside the medium in terms of the values of the average displacement field at the surface $x_3=0$.

II. FORMULATION OF THE PROBLEM

We consider a general elastic medium that occupies the region $x_3>\zeta(\vec{x}_{||})$, where $\zeta(\vec{x}_{||})$ is the surface-roughness profile function (see Fig. 1). It is a random function of the two-dimensional position vector $\vec{x}_{||}=\hat{x}_1x_1+\hat{x}_2x_2$, where $\hat{x}_1$ and $\hat{x}_2$ are two mutually perpendicular unit vectors in the plane $x_3=0$.

The medium is characterized by the general, linear stress-strain relation

$$T_{\alpha\beta}(\vec{x}\mid\omega)=\sum_{\mu,\nu}c_{\alpha\beta\mu\nu}\frac{\partial}{\partial x_\nu}u_\mu(\vec{x}\mid\omega)\ . \tag{2.1}$$

Here $u_\mu(\vec{x}\mid\omega)$ is the frequency Fourier transform of the elastic displacement field, $T_{\alpha\beta}(\vec{x}\mid\omega)$ is the stress tensor of the elastic medium, and $c_{\alpha\beta\mu\nu}$ is the

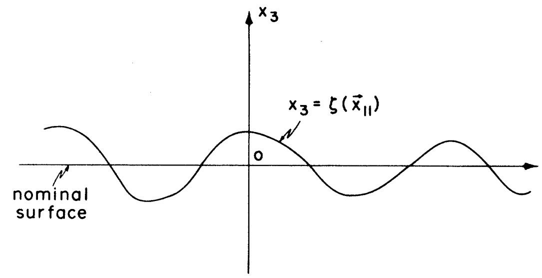

FIG. 1. A randomly rough surface profile.

fourth-rank elastic modulus tensor, which is assumed to be independent of position right up to the (rough) surface. Unless otherwise indicated, in this paper Greek indices denote Cartesian components x_1,x_2,x_3 (or 1,2,3).

The equation of motion for $u_\mu(\vec{x}\mid\omega)$ is given by

$$\sum_\mu L_{\alpha\mu}(\vec{x}\mid\omega)u_\mu(\vec{x}\mid\omega)=0\ , \tag{2.2}$$

where the differential operator $L_{\alpha\mu}(\vec{x}\mid\omega)$ is defined by

$$L_{\alpha\mu}(\vec{x}\mid\omega)=\omega^2\delta_{\alpha\mu}+\frac{1}{\rho}\sum_{\beta,\nu}c_{\alpha\beta\mu\nu}\frac{\partial^2}{\partial x_\beta\partial x_\nu}\ , \tag{2.3}$$

ρ being the mass density of the medium. Equation (2.2) applies at every point inside the medium. It is supplemented by the stress-free boundary conditions at the actual, rough, surface:

$$\sum_\beta T_{\alpha\beta}(\vec{x}\mid\omega)\hat{n}_\beta\Bigg|_{x_3=\zeta(\vec{x}_{||})}=0\ , \tag{2.4}$$

where $\hat{n}_\beta$, the unit vector normal to the surface at each point, has components proportional to the vector

$$\left[-\frac{\partial\zeta(\vec{x}_{||})}{\partial x_1},-\frac{\partial\zeta(\vec{x}_{||})}{\partial x_2},1\right]\ . \tag{2.5}$$

We now make the assumption that the roughness is "small," in the sense that it is possible to expand all the dynamical variables in a Taylor series as follows ($\vec{x}=\vec{x}_{||}+\hat{x}_3x_3$):

$$F(\vec{x})\,|_{x_3=\zeta(\vec{x}_{||})}=F(\vec{x}_{||};0)+\zeta(\vec{x}_{||})\left[\frac{\partial}{\partial x_3}F(\vec{x})\right]_{x_3=0}+\tfrac{1}{2}\zeta^2(\vec{x}_{||})\left[\frac{\partial^2}{\partial x_3^2}F(\vec{x})\right]_{x_3=0}+\cdots\ , \tag{2.6}$$

and obtain meaningful results by keeping only terms of up to $O(\zeta^2)$. Thus we substitute (2.5) into Eq. (2.4) and expand the stress tensor according to Eq. (2.6). In this way we obtain the result that

$$\sum_{\mu,\nu}\sum_{\delta} c_{\alpha\delta\mu\nu}\left[\frac{\partial}{\partial x_\delta}\zeta(\vec{x}_{||})\right]\left[\left[\frac{\partial}{\partial x_\nu}+\zeta(\vec{x}_{||})\frac{\partial^2}{\partial x_3\partial x_\nu}\right]u_\mu(\vec{x}\,|\,\omega)\right]_{x_3=0}$$
$$-\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\left[\left[\frac{\partial}{\partial x_\nu}+\zeta(\vec{x}_{||})\frac{\partial^2}{\partial x_3\partial x_\nu}+\tfrac{1}{2}\zeta^2(\vec{x}_{||})\frac{\partial^3}{\partial x_3^2\partial x_\nu}\right]u_\mu(\vec{x}\,|\,\omega)\right]_{x_3=0}=0\ . \qquad (2.7)$$

In Eq. (2.7) and in the rest of this paper, the Greek index δ denotes the coordinates x_1 and x_2 only. This convention exploits the fact that x_1 and x_2 enter all the equations in an equivalent way.

Now, as mentioned in the Introduction, because of the random nature of the surface-profile function, the quantity of interest is not the displacement $u_\mu(\vec{x}\,|\,\omega)$ itself but its average $\langle u_\mu(\vec{x}\,|\,\omega)\rangle$ over the ensemble of realizations of the surface roughness. It is then useful to introduce the operator P (in this context called the "smoothing operator") such that

$$Pu_\mu(\vec{x}\,|\,\omega)=\langle u_\mu(\vec{x}\,|\,\omega)\rangle\ , \qquad (2.8)$$

together with the operator Q such that

$$P+Q=1\ . \qquad (2.9)$$

With these definitions we obtain the useful identity

$$u_\mu \equiv (P+Q)u_\mu \qquad (2.10a)$$
$$=\langle u_\mu\rangle + Qu_\mu\ . \qquad (2.10b)$$

Thus Qu_μ describes the random fluctuation of u_μ about $\langle u_\mu\rangle$. We also note that an expansion of u_μ of the form $u_\mu=u_0+u_1\zeta+\cdots$ in Eq. (2.10b) indicates that the vector $Qu_\mu(\vec{x};\omega)$ is of $O(\zeta)$. This conclusion will be used below repeatedly. We next note the following results:

$$P\zeta(\vec{x}_{||})=0\ , \qquad (2.11)$$
$$P\zeta^2(\vec{x}_{||})=\delta^2\ , \qquad (2.12)$$
$$P\frac{\partial}{\partial x_\delta}\zeta(\vec{x}_{||})=0\ , \qquad (2.13a)$$
$$P\zeta(\vec{x}_{||})\frac{\partial}{\partial x_\delta}\zeta(\vec{x}_{||})=0\ . \qquad (2.13b)$$

Equation (2.11) states that it is possible to define a flat, nominal surface (in fact, we measure the coordinate x_3 from that surface). In Eq. (2.12) we have introduced the root-mean-square departure of the surface from flatness δ, which is one of the two parameters that characterizes the ensemble of rough profiles. (The other, the transverse correlation length a, is introduced explicitly in Ref. 3.) Finally, Eqs. (2.13) show that the limit involved in differentiating ζ can be interchanged with the averaging operator P, i.e., that averaging commutes with differentiation.

We now act on Eq. (2.7) from the left with the operator P. Utilizing the identity (2.10b) and Eqs. (2.11)–(2.13) leads us to the result that

$$\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\left[\frac{\partial}{\partial x_\nu}\langle u_\mu(\vec{x}\,|\,\omega)\rangle\right]_{x_3=0}=\sum_{\mu,\nu}\sum_{\delta} c_{\alpha\delta\mu\nu}P\left[\frac{\partial\zeta(\vec{x}_{||})}{\partial x_\delta}\frac{\partial}{\partial x_\nu}Qu_\mu(\vec{x}\,|\,\omega)\right]_{x_3=0}$$
$$-\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\left[P\left[\zeta(\vec{x}_{||})\frac{\partial^2}{\partial x_3\partial x_\nu}Qu_\mu(\vec{x}\,|\,\omega)\right]_{x_3=0}+\tfrac{1}{2}\delta^2\left[\frac{\partial^3}{\partial x_3^2\partial x_\nu}\langle u_\mu(\vec{x}\,|\,\omega)\rangle\right]_{x_3=0}\right]\ . \qquad (2.14)$$

In Eq. (2.14) we have kept all terms through $O(\delta^2)$.

At this point it is instructive to note that, in the case of a medium bounded by a perfectly flat surface, the boundary condition for $u_\mu(\vec{x}\,|\,\omega)$ is [see Eqs. (2.1), (2.4), and (2.5)] as follows:

$$\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\left[\frac{\partial}{\partial x_\nu}u_\mu(\vec{x}\,|\,\omega)\right]_{x_3=0}=0\ . \qquad (2.15)$$

Thus Eq. (2.14) implies that if we want to work with the *average* of the elastic displacement field *ab initio,* the stress-free boundary condition (2.15) that applies in the absence of roughness is effectively replaced by Eq. (2.14), whose right-hand side may be interpreted as defining an effective (fictitious) stress that is applied at the average, nominal surface.

Now, Eq. (2.14) is not the result we are seeking, since in the first two terms of the right-hand side there enters the fluctuating component of the displacement vector Qu_μ. We thus need to relate Qu_μ

and $\langle u_\mu \rangle$. Since Qu_μ enters Eq. (2.14) multiplied by $\zeta(\vec{x}_{||})$ (or its derivatives), we need to obtain such a relation to $O(\zeta)$ only (this simplifies the problem considerably).

We proceed to establish a boundary condition for Qu_μ by acting on Eq. (2.7) from the left with the operator Q. Recalling Eqs. (2.10)–(2.13) and the comment following Eq. (2.10), we are led to the result [valid to $O(\zeta)$] that

$$\sum_\mu c_{\alpha 3\mu\nu}\left[\frac{\partial}{\partial x_\nu}Qu_\mu(\vec{x}\mid\omega)\right]_{x_3=0}=\sum_{\mu,\nu}\left[\beta_{\alpha\mu\nu}(\vec{x})\frac{\partial}{\partial x_\nu}\langle u_\mu(\vec{x}\mid\omega)\rangle\right]_{x_3=0}, \quad (2.16)$$

where

$$B_{\alpha\mu\nu}(\vec{x})=-c_{\alpha 3\mu\nu}\zeta(\vec{x}_{||})\frac{\partial}{\partial x_3}+\sum_\delta c_{\alpha\delta\mu\nu}\left[\frac{\partial}{\partial x_\delta}\zeta(\vec{x}_{||})\right]. \quad (2.17)$$

We next notice that the vector Qu_μ satisfies the same differential equation as u_μ itself, namely Eq. (2.2). A generalization of Green's theorem for the differential operator $L_{\alpha\mu}$, expressing Qu_μ in terms of its boundary value [that is, according to Eq. (2.16), in terms of $\langle u_\mu\rangle$] suggests itself. This question is addressed in Appendix A, where we obtain the result [see Eq. (A13)] that

$$Qu_\mu(\vec{x}\mid\omega)=\frac{1}{\rho}\sum_{\alpha,\beta,\gamma}\int d^2x'_{||}D_{\mu\alpha}(\vec{x};\vec{x}'_{||},0\mid\omega)\left[B_{\alpha\beta\gamma}(\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\mid\omega)\rangle\right]_{x'_3=0}, \quad (2.18)$$

where the integral runs over the plane $x'_3=0$. The "flat-surface" Green's functions $D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)$ (Ref. 2) are defined by Eqs. (A6) and (A7).

Substituting Eqs. (2.18) into Eq. (2.14) leads us to the desired effective boundary conditions for the average displacement field ($\lambda=1,2,3$):

$$\begin{aligned}\sum_{\mu,\nu}c_{\lambda 3\mu\nu}\left[\frac{\partial}{\partial x_\nu}\langle u_\mu(\vec{x}\mid\omega)\rangle\right]_{x_3=0}=&-\frac{\delta^2}{2}\sum_{\mu,\nu}c_{\lambda 3\mu\nu}\left[\frac{\partial^3}{\partial x_3^2\partial x_\nu}\langle u_\mu(\vec{x}\mid\omega)\rangle\right]_{x_3=0}\\&+\frac{\delta^2}{\rho}\sum_{\substack{\mu,\nu\\ \alpha,\beta,\gamma}}\sum_\delta c_{\lambda\delta\mu\nu}\int d^2x'_{||}\left[\frac{\partial}{\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}'_{||},0\mid\omega)\right]_{x_3=0}\\&\times\left[F^{(\delta)}_{\alpha\beta\gamma}(\vec{x}_{||};\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\mid\omega)\rangle\right]_{x'_3=0}\\&-\frac{\delta^2}{\rho}\sum_{\substack{\mu,\nu\\ \alpha,\beta,\gamma}}c_{\lambda 3\mu\nu}\int d^2x'_{||}\left[\frac{\partial^2}{\partial x_3\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}'_{||},0\mid\omega)\right]_{x_3=0}\\&\times\left[F^{(3)}_{\alpha\beta\gamma}(\vec{x}_{||};\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\mid\omega)\rangle\right]_{x'_3=0}, \end{aligned}\quad (2.19)$$

where

$$F^{(1)}_{\alpha\beta\gamma}(\vec{x}_{||};\vec{x}')=-c_{\alpha 3\beta\gamma}\frac{\partial}{\partial x_1}W(\mid\vec{x}_{||}-\vec{x}'_{||}\mid)\frac{\partial}{\partial x'_3}+\sum_\delta c_{\alpha\delta\beta\gamma}\frac{\partial^2}{\partial x_1\partial x'_\delta}W(\mid\vec{x}_{||}-\vec{x}'_{||}\mid), \quad (2.20)$$

$$F^{(2)}_{\alpha\beta\gamma}(\vec{x}_{||};\vec{x}')=-c_{\alpha 3\beta\gamma}\frac{\partial}{\partial x_2}W(\mid\vec{x}_{||}-\vec{x}'_{||}\mid)\frac{\partial}{\partial x'_3}+\sum_\delta c_{\alpha\delta\beta\gamma}\frac{\partial^2}{\partial x_2\partial x'_\delta}W(\mid\vec{x}_{||}-\vec{x}'_{||}\mid), \quad (2.21)$$

and

$$F^{(3)}_{\alpha\beta\gamma}(\vec{x}_{||};\vec{x}')=-c_{\alpha 3\beta\gamma}W(\mid\vec{x}_{||}-\vec{x}'_{||}\mid)\frac{\partial}{\partial x'_3}+\sum_\delta c_{\alpha\delta\beta\gamma}\frac{\partial}{\partial x'_\delta}W(\mid\vec{x}_{||}-\vec{x}'_{||}\mid). \quad (2.22)$$

Here we have introduced the correlation function

$$\langle \zeta(\vec{x}_{\parallel})\zeta(\vec{x}'_{\parallel})\rangle = \delta^2 W(|\vec{x}_{\parallel}-\vec{x}'_{\parallel}|) , \tag{2.23}$$

chosen such that $W(0)=1$. [This ensures that Eq. (2.23) is consistent with Eq. (2.12).]

We note that Eq. (2.23) embodies all the information we need about the statistical properties of the rough surface [in addition to the existence of a flat nominal surface; see Eq. (2.11)]. Equation (2.23) states that the surface-roughness profile is a stationary stochastic process. We remark that the stationarity assumption (the assumption that the correlation function W depends on the vector difference between the points $\vec{x}_{\parallel}$ and $\vec{x}'_{\parallel}$ and not on their separate locations on the surface) has the consequence that infinitesimal translational invariance is restored to the problem for quantities that are averages over an ensemble of rough surfaces. This fact will be exploited in Sec. III when we obtain (for the case of an isotropic medium) effective boundary conditions for the Fourier transform of the displacement field.[4]

As indicated in the comment following Eq. (2.15), the right-hand side of the effective boundary conditions given by Eq. (2.19) can be thought of as defining fictitious stresses that, applied at the flat nominal surface, will give the correct average results. We emphasize that although those effective stresses have rather complicated definitions, Eq. (2.19) offers the following advantageous approach to the solution of problems involving rough surfaces. This is that with the procedure leading to Eq. (2.19) we have effectively taken into account the irregular nature of the boundary surface once and for all. Having obtained Eq. (2.19), the calculation of average displacements, Green's functions, etc., can be formally carried out as if the surface were flat (and the problem nonrandom). We illustrate the use of our effective boundary conditions in Sec. IV.

III. ISOTROPIC MEDIUM

A. Effective boundary conditions in configuration space

The effective boundary conditions (2.19) apply for an arbitrary elastic medium. It is of interest to analyze them in detail for special cases. Thus in this section we consider the case of an isotropic medium. (The medium is isotropic from the point of view of its bulk elastic properties.) Such a medium is a particular case of a medium with cubic symmetry for which[5] $c_{12}=c_{11}-2c_{44}$, with $c_{11}=\rho c_l^2$ and $c_{44}=\rho c_t^2$, where c_l and c_t are, respectively, the speeds of longitudinal and transverse elastic waves in the bulk. Alternatively, for an isotropic medium we have that the elastic modulus tensor $c_{\alpha\beta\mu\nu}$ is given by the equation

$$c_{\alpha\beta\mu\nu}=\rho[(c_l^2-2c_t^2)\delta_{\alpha\beta}\delta_{\mu\nu} + c_t^2(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu})] . \tag{3.1}$$

We shall illustrate the algebra involved in going from Eq. (2.19) to Eq. (3.24) by considering one typical term of Eq. (2.19) in detail, namely,

$$I \equiv \frac{\delta^2}{\rho} \sum_{\substack{\mu,\nu,\\ \alpha,\beta,\gamma}} c_{11\mu\nu} \int d^2x'_{\parallel} \left[\frac{\partial}{\partial x_\nu} D_{\mu\alpha}(\vec{x};\vec{x}'_{\parallel},0\,|\,\omega) \right]_{x_3=0} \left[F^{(1)}_{\alpha\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\,|\,\omega)\rangle \right]_{x'_3=0} . \tag{3.2}$$

We note that in the case of an isotropic medium, the Green's functions $D_{\mu\nu}(\vec{x}\vec{x}'\,|\,\omega)$ that enter Eq. (3.2) have been calculated by Maradudin and Mills.[2] (These authors actually calculated the appropriate Fourier transforms of these Green's functions.)

We begin by carrying out the sums over β and γ in Eq. (3.2). Now

$$\begin{aligned}
\sum_{\beta,\gamma} F^{(1)}_{1\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\,|\,\omega)\rangle
&= \sum_{\beta,\gamma}\left[c_{11\beta\gamma}\frac{\partial^2 W}{\partial x_1 \partial x'_1} + c_{12\beta\gamma}\frac{\partial^2 W}{\partial x_1\partial x'_2} - c_{13\beta\gamma}\frac{\partial W}{\partial x_1}\frac{\partial}{\partial x'_3}\right]\frac{\partial\langle u_\beta\rangle}{\partial x'_\gamma} \\
&= \rho\left[c_l^2\frac{\partial\langle u_1\rangle}{\partial x'_1} + (c_l^2-2c_t^2)\frac{\partial\langle u_2\rangle}{\partial x'_2} + (c_l^2-2c_t^2)\frac{\partial\langle u_3\rangle}{\partial x'_3}\right]\frac{\partial^2}{\partial x_1\partial x'_1}W(|\vec{x}_{\parallel}-\vec{x}'_{\parallel}|) \\
&\quad + \rho\left[c_t^2\frac{\partial\langle u_1\rangle}{\partial x'_2} + c_t^2\frac{\partial\langle u_2\rangle}{\partial x'_1}\right]\frac{\partial^2}{\partial x_1\partial x'_2}W(|\vec{x}_{\parallel}-\vec{x}'_{\parallel}|) - \rho\left[c_t^2\frac{\partial^2\langle u_1\rangle}{\partial x'^2_3} + c_t^2\frac{\partial^2\langle u_3\rangle}{\partial x'_3\partial x'_1}\right]\frac{\partial}{\partial x_1}W(|\vec{x}_{\parallel}-\vec{x}'_{\parallel}|) .
\end{aligned} \tag{3.3}$$

We now make the following definitions:

$$\hat{a}_{11}(\vec{x}_{\parallel};\vec{x}')=c_l^2\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_1'}+c_t^2\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_2'}-c_t^2W(r_{\parallel})\frac{\partial^2}{\partial x_3'^2}\,, \tag{3.4}$$

$$\hat{a}_{12}(\vec{x}_{\parallel};\vec{x}')=(c_l^2-2c_t^2)\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_2'}+c_t^2\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_1'}\,, \tag{3.5}$$

$$\hat{a}_{13}(\vec{x}_{\parallel};\vec{x}')=(c_l^2-2c_t^2)\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_3'}-c_t^2W(r_{\parallel})\frac{\partial^2}{\partial x_3'\partial x_1'}\,, \tag{3.6}$$

where we have set $r_{\parallel}=|\vec{x}_{\parallel}-\vec{x}_{\parallel}'|$. Substituting these definitions into Eq. (3.3) we have the result that

$$\sum_{\beta,\gamma}\left[F_{\alpha\beta\gamma}^{(1)}(\vec{x}_{\parallel};\vec{x}')\frac{\partial}{\partial x_\gamma'}\langle u_\beta(\vec{x}'\,|\,\omega)\rangle\right]_{x_3'=0}=\rho\frac{\partial}{\partial x_1}\sum_{\gamma}[\hat{a}_{\alpha\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x_3'=0}\,, \tag{3.7}$$

where we have generalized the result by letting α take on the values 1, 2, and 3. The elements of the 3×3 differential operator $\hat{a}_{\alpha\gamma}(\vec{x}_{\parallel};\vec{x}')$ for $\alpha=2,3$ are defined by the equations

$$\hat{a}_{21}(\vec{x}_{\parallel};\vec{x}')=c_t^2\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_2'}+(c_l^2-2c_t^2)\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_1'}\,, \tag{3.8}$$

$$\hat{a}_{22}(\vec{x}_{\parallel};\vec{x}')=c_t^2\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_1'}+c_l^2\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_2'}-c_t^2W(r_{\parallel})\frac{\partial^2}{\partial x_3'^2}\,, \tag{3.9}$$

$$\hat{a}_{23}(\vec{x}_{\parallel};\vec{x}')=(c_l^2-2c_t^2)\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_3'}-c_t^2W(r_{\parallel})\frac{\partial^2}{\partial x_3'\partial x_2'}\,, \tag{3.10}$$

$$\hat{a}_{31}(\vec{x}_{\parallel};\vec{x}')=c_t^2\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_3'}-(c_l^2-2c_t^2)W(r_{\parallel})\frac{\partial^2}{\partial x_3'\partial x_1'}\,, \tag{3.11}$$

$$\hat{a}_{32}(\vec{x}_{\parallel};\vec{x}')=c_t^2\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_3'}-(c_l^2-2c_t^2)W(r_{\parallel})\frac{\partial^2}{\partial x_3'\partial x_2'}\,, \tag{3.12}$$

$$\hat{a}_{33}(\vec{x}_{\parallel};\vec{x}')=c_t^2\frac{\partial W(r_{\parallel})}{\partial x_1'}\frac{\partial}{\partial x_1'}+c_t^2\frac{\partial W(r_{\parallel})}{\partial x_2'}\frac{\partial}{\partial x_2'}-c_l^2W(r_{\parallel})\frac{\partial^2}{\partial x_3'^2}\,. \tag{3.13}$$

Making the additional definition that

$$b_{1\alpha}^{(1)}(\vec{x}_{\parallel};\vec{x}_{\parallel}'\,|\,\omega)=\frac{1}{\rho}\sum_{\mu,\nu}c_{11\mu\nu}\left[\frac{\partial}{\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}_{\parallel}',0\,|\,\omega)\right]_{x_3=0}\,, \tag{3.14}$$

and utilizing the result given by Eq. (3.7) in Eq. (3.2) we have

$$I=\delta^2\rho\sum_{\beta,\gamma}\int d^2x_{\parallel}'\,b_{1\beta}^{(1)}(\vec{x}_{\parallel};\vec{x}_{\parallel}'\,|\,\omega)\frac{\partial}{\partial x_1}[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x_3'=0}\,. \tag{3.15}$$

Note that the original expression (3.2) consists of a sum over five indices, whereas the result (3.15) consists of a sum over two indices only.

Proceeding in a similar fashion, we can generalize Eq. (3.15) as follows. For $\lambda=1,2,3$ and $\delta=1,2$ we have that

$$\frac{\delta^2}{\rho}\sum_{\substack{\mu,\nu\\ \alpha,\beta,\gamma}}c_{\lambda\delta\mu\nu}\int d^2x_{\parallel}'\left[\frac{\partial}{\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}_{\parallel}',0\,|\,\omega)\right]_{x_3=0}\left[F_{\alpha\beta\gamma}^{(\delta)}(\vec{x}_{\parallel};\vec{x}')\frac{\partial}{\partial x_\gamma'}\langle u_\beta(\vec{x}'\,|\,\omega)\rangle\right]_{x_3'=0}$$
$$=\delta^2\rho\int d^2x_{\parallel}'\sum_{\beta,\gamma}b_{\lambda\beta}^{(\delta)}(\vec{x}_{\parallel};\vec{x}_{\parallel}'\,|\,\omega)\frac{\partial}{\partial x_\delta}[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x_3'=0}\,, \tag{3.16}$$

and, for $\lambda=1,2,3$

$$\frac{\delta^2}{\rho}\sum_{\substack{\mu,\nu\\ \alpha,\beta,\gamma}} c_{\lambda 3\mu\nu}\int d^2x'_{\parallel}\left[\frac{\partial^2}{\partial x_3\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}'_{\parallel},0\,|\,\omega)\right]_{x_3=0}\left[F^{(3)}_{\alpha\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\,|\,\omega)\rangle\right]_{x'_3=0}$$
$$=\delta^2\rho\int d^2x'_{\parallel}\sum_{\beta,\gamma} b^{(3)}_{\lambda\beta}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x'_3=0}\,. \quad (3.17)$$

Here we have made the definitions [cf. Eq. (3.14)]:

$$b^{(\delta)}_{\lambda\alpha}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)=\frac{1}{\rho}\sum_{\mu,\nu}c_{\lambda\delta\mu\nu}\left[\frac{\partial}{\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}'_{\parallel},0\,|\,\omega)\right]_{x_3=0} \quad (3.18)$$

and

$$b^{(3)}_{\lambda\alpha}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)=\frac{1}{\rho}\sum_{\mu,\nu}c_{\lambda 3\mu\nu}\left[\frac{\partial^2}{\partial x_3\partial x_\nu}D_{\mu\alpha}(\vec{x};\vec{x}'_{\parallel},0\,|\,\omega)\right]_{x_3=0}\,. \quad (3.19)$$

Explicit expressions for the matrices $b^{(\beta)}_{\lambda\alpha}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)$, with $\beta=1,2,3$, are obtained by combining the definitions (3.18) and (3.19) with Eq. (3.1) and the results of Ref. 2 for the flat-surface Green's functions. For brevity those expressions (or their Fourier transforms to be defined below) are not given in this paper. They are, however, implicit in the results for a related matrix that actually enters our final result, given by Eq. (3.35) [the matrix $c_{\lambda\beta}$, defined by Eq. (3.36)].

Consider now the boundary condition that results from Eq. (2.19) for $\lambda=1$. With the results (3.16) and (3.17) at hand, we easily obtain the result that

$$\rho c_t^2\left[\frac{\partial}{\partial x_3}\langle u_1(\vec{x}\,|\,\omega)\rangle+\frac{\partial}{\partial x_1}\langle u_3(\vec{x}\,|\,\omega)\rangle\right]_{x_3=0}$$
$$=-\frac{\delta^2}{2}\rho c_t^2\left[\frac{\partial^3}{\partial x_3^3}\langle u_1(\vec{x}\,|\,\omega)\rangle+\frac{\partial^3}{\partial x_3^2\partial x_1}\langle u_3(\vec{x}\,|\,\omega)\rangle\right]_{x_3=0}$$
$$+\delta^2\rho\sum_{\beta\gamma\delta}\int d^2x'_{\parallel}b^{(\delta)}_{1\beta}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)\frac{\partial}{\partial x_\delta}[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x'_3=0}$$
$$-\delta^2\rho\sum_{\beta\gamma}\int d^2x'_{\parallel}b^{(3)}_{1\beta}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x'_3=0}\,, \quad (3.20)$$

where we recall that the index δ assumes only the values 1 and 2. The boundary conditions for $\lambda=2,3$ can be written down without difficulty. It is, however, useful to give the boundary condition more concisely, viz., in vector form, by introducing the 3×3 differential operator $\hat{f}_{\lambda\beta}(\vec{x})$ according to

$$\hat{f}_{11}(\vec{x})=c_t^2\frac{\partial}{\partial x_3},\quad \hat{f}_{12}(\vec{x})=0,\quad \hat{f}_{13}(\vec{x})=c_t^2\frac{\partial}{\partial x_1}\,, \quad (3.21)$$

$$\hat{f}_{21}(\vec{x})=0,\quad \hat{f}_{22}(\vec{x})=c_t^2\frac{\partial}{\partial x_3},\quad \hat{f}_{23}(\vec{x})=c_t^2\frac{\partial}{\partial x_2}\,, \quad (3.22)$$

$$\hat{f}_{31}(\vec{x})=(c_l^2-2c_t^2)\frac{\partial}{\partial x_1},\quad \hat{f}_{32}(\vec{x})=(c_l^2-2c_t^2)\frac{\partial}{\partial x_2},\quad \hat{f}_{33}(\vec{x})=c_l^2\frac{\partial}{\partial x_3}\,. \quad (3.23)$$

We are thus led to the following effective boundary conditions that the average displacement field must satisfy on the flat nominal surface bounding an isotropic elastic medium ($\lambda=1,2,3$):

$$\sum_{\beta}[\hat{f}_{\lambda\beta}(\vec{x})\langle u_\beta(\vec{x}\,|\,\omega)\rangle]_{x_3=0}=-\frac{\delta^2}{2}\sum_{\beta}\left[\hat{f}_{\lambda\beta}(\vec{x})\frac{d^2}{dx_3^2}\langle u_\beta(\vec{x}\,|\,\omega)\rangle\right]_{x_3=0}$$
$$+\delta^2\sum_{\beta,\gamma,\delta}\int d^2x'_{\parallel}b^{(\delta)}_{\lambda\beta}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)\frac{\partial}{\partial x_\delta}[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x'_3=0}$$
$$-\delta^2\sum_{\beta,\gamma}\int d^2x'_{\parallel}b^{(3)}_{\lambda\beta}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\,|\,\omega)[\hat{a}_{\beta\gamma}(\vec{x}_{\parallel};\vec{x}')\langle u_\gamma(\vec{x}'\,|\,\omega)\rangle]_{x'_3=0}\,. \quad (3.24)$$

The left-hand side of Eq. (3.24) equals $\rho^{-1}T_{\lambda 3}(\vec{x}\mid\omega)$ evaluated at $x_3=0$, which vanishes in the case of a truly flat surface by virtue of the stress-free boundary conditions. The effects of surface roughness on the boundary conditions for the average field are given by the right-hand side of Eq. (3.24). We note that whereas the first term on the right-hand side of Eq. (3.24) is independent of the correlation function $W(\mid\vec{x}_{\parallel}-\vec{x}'_{\parallel}\mid)$, this function enters the remaining two terms in Eq. (3.24) via the operators $\hat{a}_{\beta\gamma}(\vec{x}_{\parallel},\vec{x}')$, defined by Eqs. (3.4)—(3.6) and (3.8)—(3.13). In order to obtain more explicit expressions for the effective stresses given by the right-hand side of Eq. (3.24) we must make use of an important physical feature of the problem, namely that of translational invariance in the plane of the surface.

B. Effective boundary conditions for the Fourier transform $\langle u_\alpha(\vec{k}_{\parallel}\omega\mid x_3)\rangle$ of the average displacement field

The averaging procedure implied by the operator P ($Pu=\langle u\rangle$) restores translational invariance in the plane of the flat, nominal surface. This suggests the convenience of transforming Eqs. (3.24) into boundary conditions for the Fourier coefficients $\langle u_\alpha(\vec{q}_{\parallel}\omega\mid x_3)\rangle$, defined by the relation

$$\langle u_\alpha(\vec{x}\mid\omega)\rangle=\int\frac{d^2q_{\parallel}}{(2\pi)^2}e^{i\vec{q}_{\parallel}\cdot\vec{x}_{\parallel}}\langle u_\alpha(\vec{q}_{\parallel}\omega\mid x_3)\rangle\ , \tag{3.25}$$

where $\vec{q}_{\parallel}$ is a two-dimensional wave vector in the plane of the surface.

We note that the "flat-surface" Green's functions $D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)$ possess the property that

$$D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)=D_{\mu\nu}(\vec{x}_{\parallel}-\vec{x}'_{\parallel};x_3x'_3\mid\omega)\ . \tag{3.26a}$$

Thus they can be Fourier transformed as follows:

$$D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)=\int\frac{d^2k_{\parallel}}{(2\pi)^2}e^{i\vec{k}_{\parallel}\cdot(\vec{x}_{\parallel}-\vec{x}'_{\parallel})}D_{\mu\nu}(\vec{k}_{\parallel}\omega\mid x_3x'_3)\ . \tag{3.26b}$$

We now outline the steps needed to Fourier transform Eq. (3.24) by considering one typical term entering that equation, namely,

$$J(\vec{x}_{\parallel}\mid\omega)\equiv\delta^2\int d^2x'_{\parallel}b_{11}^{(1)}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\mid\omega)\frac{\partial}{\partial x_1}[\hat{a}_{11}(\vec{x}_{\parallel};\vec{x}')\langle u_1(\vec{x}'\mid\omega)\rangle]_{x'_3=0}\ . \tag{3.27}$$

According to the definition given by Eqs. (3.14) and (3.26b), we can define the Fourier transform $b_{11}^{(1)}(\vec{k}_{\parallel}\mid\omega)$ of $b_{11}^{(1)}(\vec{x}_{\parallel};\vec{x}'_{\parallel}\mid\omega)$ via a relation of the form of Eq. (3.26b). Thus, making use of Eqs. (3.4) and (3.25) and introducing the Fourier transform $g(Q_{\parallel})$ of the correlation function $W(\mid\vec{x}_{\parallel}-\vec{x}'_{\parallel}\mid)$, we are led to the result that

$$J(\vec{x}_{\parallel}\mid\omega)=\delta^2\int\frac{d^2q_{\parallel}}{(2\pi)^2}e^{i\vec{q}_{\parallel}\cdot\vec{x}_{\parallel}}J(\vec{q}_{\parallel}\mid\omega)\ , \tag{3.28a}$$

where we have called

$$J(\vec{q}_{\parallel}\mid\omega)=i\int\frac{d^2k_{\parallel}}{(2\pi)^2}g(\mid\vec{q}_{\parallel}-\vec{k}_{\parallel}\mid)b_{11}^{(1)}(\vec{k}_{\parallel}\mid\omega)(q_1-k_1)[\hat{a}_{11}(\vec{q}_{\parallel};\vec{k}_{\parallel}\mid x'_3)\langle u_1(\vec{q}_{\parallel}\omega\mid x'_3)\rangle]_{x'_3=0}\ , \tag{3.28b}$$

with

$$\hat{a}_{11}(\vec{q}_{\parallel};\vec{k}_{\parallel}\mid x'_3)=c_l^2(q_1-k_1)q_1+c_t^2(q_2-k_2)q_2-c_t^2\frac{d^2}{dx_3'^2}\ . \tag{3.29}$$

Note that the fact that we were able to cast our result for $J(\vec{x}_{\parallel}\mid\omega)$ in the form (3.28a) is, in effect, the statement that averaging over the roughness restores translational invariance in the plane of the nominal surface. Equation (3.28a) ensures that the final result can be given as a boundary condition for the Fourier coefficients $\langle u_\alpha(\vec{q}_{\parallel}\omega\mid x_3)\rangle$. [We stress that the same property would not hold for the Fourier coefficients of the displacement field $u_\alpha(\vec{q}_{\parallel}\omega\mid x_3)$ itself.]

The preceding argument can be easily generalized to obtain the results that

$$\delta^2 \sum_{\beta,\gamma,\delta} \int d^2x'_{||} b^{(\delta)}_{\lambda\beta}(\vec{x}_{||};\vec{x}'_{||} \mid \omega) \frac{\partial}{\partial x_\delta} [\hat{a}_{\beta\gamma}(\vec{x}_{||};\vec{x}') \langle u_\gamma(\vec{x}' \mid \omega) \rangle]_{x'_3=0}$$

$$= \delta^2 \int \frac{d^2q_{||}}{(2\pi)^2} e^{i\vec{q}_{||}\cdot\vec{x}_{||}} \sum_{\beta,\gamma,\delta} \int \frac{d^2k_{||}}{(2\pi)^2} g(\,|\vec{q}_{||}-\vec{k}_{||}|\,)$$

$$\times \{ i(q_\delta - k_\delta) b^{(\delta)}_{\lambda\beta}(\vec{k}_{||} \mid \omega) [\hat{a}_{\beta\gamma}(\vec{q}_{||};\vec{k}_{||} \mid \vec{x}'_3) \langle u_\gamma(\vec{q}_{||}\omega \mid x'_3) \rangle] \}_{x'_3=0} \tag{3.30a}$$

and

$$\delta^2 \sum_{\beta,\gamma} \int d^2x'_{||} b^{(3)}_{\lambda\beta}(\vec{x}_{||};\vec{x}'_{||} \mid \omega) [\hat{a}_{\beta\gamma}(\vec{x}_{||};\vec{x}') \langle u_\gamma(\vec{x}' \mid \omega) \rangle]_{x'_3=0}$$

$$= \delta^2 \int \frac{d^2q_{||}}{(2\pi)^2} e^{i\vec{q}_{||}\cdot\vec{x}_{||}} \sum_{\beta\gamma} \int \frac{d^2k_{||}}{(2\pi)^2} g(\,|\vec{q}_{||}-\vec{k}_{||}|\,) \{ b^{(3)}_{\lambda\beta}(\vec{k}_{||} \mid \omega) [\hat{a}_{\beta\gamma}(\vec{q}_{||};\vec{k}_{||} \mid \vec{x}'_3) \langle u_\gamma(\vec{q}_{||}\omega \mid x'_3) \rangle]_{x'_3=0} \} . \tag{3.30b}$$

For convenience in the presentation, the definitions of the remaining elements of the 3×3 differential operator $\hat{a}_{\mu\nu}(\vec{q}_{||};\vec{k}_{||} \mid x'_3)$ [whose 11 element was given in Eq. (3.29)] are given in Appendix C.

We now turn to the left-hand side of the boundary conditions (3.24). Recalling Eq. (3.25), it is easy to show that

$$\sum_\beta [\hat{f}_{\lambda\beta}(\vec{x}) \langle u_\beta(\vec{x} \mid \omega) \rangle]_{x_3=0} = \int \frac{d^2q_{||}}{(2\pi)^2} e^{i\vec{q}_{||}\cdot\vec{x}_{||}} \sum_\beta [\hat{f}_{\lambda\beta}(\vec{q}_{||};x_3) \langle u_\beta(\vec{q}_{||}\omega \mid x_3) \rangle]_{x_3=0} , \tag{3.31}$$

where

$$\hat{f}_{11}(\vec{q}_{||};x_3) = c_t^2 \frac{d}{dx_3}, \quad \hat{f}_{12}(\vec{q}_{||};x_3) = 0, \quad \hat{f}_{13}(\vec{q}_{||};x_3) = iq_1 c_t^2 , \tag{3.32}$$

$$\hat{f}_{21}(\vec{q}_{||};x_3) = 0, \quad \hat{f}_{22}(\vec{q}_{||};x_3) = c_t^2 \frac{d}{dx_3}, \quad \hat{f}_{23}(\vec{q}_{||};x_3) = iq_2 c_t^2 , \tag{3.33}$$

$$\hat{f}_{31}(\vec{q}_{||};x_3) = iq_1(c_l^2 - 2c_t^2), \quad \hat{f}_{32}(\vec{q}_{||};x_3) = iq_2(c_l^2 - 2c_t^2), \quad \hat{f}_{33}(\vec{q}_{||};x_3) = c_l^2 \frac{d}{dx_3} . \tag{3.34}$$

Finally, we note that the first term on the right-hand side of Eq. (3.24) is obtained immediately from Eq. (3.31).

Substituting Eqs. (3.30) and (3.31) in Eq. (3.24) we can equate the coefficients of $(2\pi)^{-2}\exp(i\vec{q}_{||}\cdot\vec{x}_{||})$ in the integrands on both sides of the equation [see comment below Eq. (3.29)]. We thus obtain the following boundary conditions for the Fourier coefficients $\langle u_\alpha(\vec{q}_{||}\omega \mid x_3) \rangle$:

$$\sum_\beta [\hat{f}_{\lambda\beta}(\vec{q}_{||};x_3) \langle u_\beta(\vec{q}_{||}\omega \mid x_3) \rangle]_{x_3=0}$$

$$= -\frac{\delta^2}{2} \sum_\beta \left[\hat{f}_{\lambda\beta}(\vec{q}_{||};x_3) \frac{d^2}{dx_3^2} \langle u_\beta(\vec{q}_{||}\omega \mid x_3) \rangle \right]_{x_3=0}$$

$$+ \delta^2 \sum_{\beta,\gamma} \int \frac{d^2k_{||}}{(2\pi)^2} g(\,|\vec{q}_{||}-\vec{k}_{||}|\,) c_{\lambda\beta}(\vec{q}_{||};\vec{k}_{||} \mid \omega) [\hat{a}_{\beta\gamma}(\vec{q}_{||};\vec{k}_{||} \mid x'_3) \langle u_\gamma(\vec{q}_{||}\omega \mid x'_3) \rangle]_{x'_3=0} , \tag{3.35}$$

where we have defined the tensor $c_{\lambda\beta}(\vec{q}_{||};\vec{k}_{||} \mid \omega)$ according to

$$c_{\lambda\beta}(\vec{q}_{||};\vec{k}_{||} \mid \omega) = -b^{(3)}_{\lambda\beta}(\vec{k}_{||} \mid \omega) + \sum_\delta i(q_\delta - k_\delta) b^{(\delta)}_{\lambda\beta}(\vec{k}_{||} \mid \omega) . \tag{3.36}$$

For convenience in the presentation we give the explicit expressions for the elements of $c_{\lambda\beta}(\vec{q}_{||};\vec{k}_{||} \mid \omega)$ in Appendix C. It is of interest to note that for a given wave vector $\vec{k}_{||}$ the elements of this tensor [and thus the contributions to the "effective stresses" given by the integral in Eq. (3.35)] have a pole at the frequency $\omega=c_R k_{||}$ of a Rayleigh wave propagating on a flat, stress-free surface.[3] (Here c_R is the speed of such a wave.)

In conclusion, Eq. (3.35) gives, for $\lambda=1,2,3$, the effective boundary conditions for the Fourier coefficients of the average displacement field $\langle u_\alpha(\vec{q}_{||}\omega\,|\,x_3)\rangle$ at the surface $x_3=0$ in the case of an isotropic medium. As indicated at the end of Sec. II, the convenience of these boundary conditions stems from the fact that they provide a procedure for calculating the quantities of physical interest (viz., averages over an ensemble of realizations of the surface-roughness profile) that is formally identical to what one does in the case of a flat surface. We outline a simple example of the use of the boundary conditions given by Eqs. (3.35) in the next section.

IV. EXAMPLE: RAYLEIGH SURFACE WAVES

Equation (3.35) gives, for $\lambda=1,2,3$, the boundary conditions that must be satisfied by the Fourier transform of the average displacement field, $\langle u_\alpha(\vec{q}_{||}\omega\,|\,x_3)\rangle$, present in an isotropic medium bounded by a planar, rough surface. In this section we apply Eq. (3.35) to the particular case of the free oscillations of such a medium. In particular, we outline the derivation of a result giving the frequency shift and the attenuation length of a Rayleigh surface wave brought about by the presence of surface roughness, a problem of both theoretical and experimental interest.[3]

In the present case of an isotropic medium, the differential operator $L_{\alpha\mu}(\vec{x}\,|\,\omega)$, given by Eq. (2.3), adopts the simpler form

$$L_{\alpha\mu}(\vec{x}\,|\,\omega)=(\omega^2+c_t^2\nabla^2)\delta_{\alpha\mu}+(c_l^2-c_t^2)\frac{\partial^2}{\partial x_\alpha \partial x_\beta} . \tag{4.1}$$

The equation of motion for the coefficients $\langle u_\alpha(\vec{q}_{||}\omega\,|\,x_3)\rangle$ can be obtained by Fourier transforming Eqs. (2.2) and (4.1), with the result that

$$\sum_\mu L_{\alpha\mu}(\vec{q}_{||}\omega\,|\,x_3)\langle u_\mu(\vec{q}_{||}\omega\,|\,x_3)\rangle=0 , \tag{4.2}$$

where the elements of the differential operator $L_{\alpha\mu}(\vec{q}_{||}\omega\,|\,x_3)$ are given by

$$L_{11}(\vec{q}_{||}\omega\,|\,x_3)=\omega^2-(c_l^2q_1^2+c_t^2q_2^2)+c_t^2\frac{d^2}{dx_3^2} , \tag{4.3}$$

$$L_{12}(\vec{q}_{||}\omega\,|\,x_3)=-(c_l^2-c_t^2)q_1q_2 =L_{21}(\vec{q}_{||}\omega\,|\,x_3) , \tag{4.4}$$

$$L_{13}(\vec{q}_{||}\omega\,|\,x_3)=i(c_l^2-c_t^2)q_1\frac{d}{dx_3} =L_{31}(\vec{q}_{||}\omega\,|\,x_3) , \tag{4.5}$$

$$L_{22}(\vec{q}_{||}\omega\,|\,x_3)=\omega^2-(c_t^2q_1^2+c_l^2q_2^2)+c_t^2\frac{d^2}{dx_3^2} , \tag{4.6}$$

$$L_{23}(\vec{q}_{||}\omega\,|\,x_3)=i(c_l^2-c_t^2)q_2\frac{d}{dx_3} =L_{32}(\vec{q}_{||}\omega\,|\,x_3) , \tag{4.7}$$

$$L_{33}(\vec{q}_{||}\omega\,|\,x_3)=\omega^2-c_t^2q_{||}^2+c_l^2\frac{d^2}{dx_3^2} . \tag{4.8}$$

Equation (4.2) is solved without difficulty. The solution that vanishes as $x_3\to\infty$ is given by[3]

$$\langle u_1(\vec{q}_{||}\omega\,|\,x_3)\rangle=\frac{q_1}{q_{||}}(e^{-\alpha_l x_3}\langle A_1\rangle+e^{-\alpha_t x_3}\langle A_2\rangle) -\frac{q_2}{q_{||}}e^{-\alpha_t x_3}\langle A_3\rangle , \tag{4.9}$$

$$\langle u_2(\vec{q}_{||}\omega\,|\,x_3)\rangle=\frac{q_2}{q_{||}}(e^{-\alpha_l x_3}\langle A_1\rangle+e^{-\alpha_t x_3}\langle A_2\rangle) +\frac{q_1}{q_{||}}e^{-\alpha_t x_3}\langle A_3\rangle , \tag{4.10}$$

$$\langle u_3(\vec{q}_{||}\omega\,|\,x_3)\rangle=i\frac{\alpha_l}{q_{||}}e^{-\alpha_l x_3}\langle A_1\rangle +i\frac{q_{||}}{\alpha_t}e^{-\alpha_t x_3}\langle A_2\rangle , \tag{4.11}$$

where $\alpha_l(q_{||}\,|\,\omega)$ and $\alpha_t(q_{||}\,|\,\omega)$ are defined by Eqs. (C25) and (C26). The coefficients $A_\alpha(\vec{q}_{||}\,|\,\omega)$, or rather their averages $\langle A_\alpha(\vec{q}_{||}\,|\,\omega)\rangle$, are to be determined by applying the effective boundary conditions given by Eq. (3.35).

Our program is then to substitute Eqs. (4.9)–(4.11) in Eq. (3.35) and, making use of the results given in Appendix C, obtain the eigenvalue equation that the coefficients $\langle A_\alpha(q_{||}\omega)\rangle$ must satisfy [Eq. (4.12)]. The amount of algebra needed to implement this program is extremely lengthy (although straightforward) and here we will bypass it entirely. The final result agrees with Eq. (2.50) of Ref. 3. It has the following form:

$$\sum_\beta [M_{\alpha\beta}^{(0)}(\vec{q}_{||}\omega)+\mathscr{P}_{\alpha\beta}(\vec{q}_{||}\omega)]\langle A_\beta(\vec{q}_{||}\omega)\rangle=0 , \tag{4.12}$$

where the matrices $M_{\alpha\beta}^{(0)}(\vec{q}_{||}\omega)$ and $\mathscr{P}_{\alpha\beta}(\vec{q}_{||}\omega)$ are defined by Eqs. (2.18) and (2.40) of Ref. 3. It seems worth pointing out that we consider this agreement with the results of Ref. 3 as significant, in view of the algebraic intricacy of either method, and in view of the fact (explained in Appendix B of Ref. 3) that an earlier method[2] gave different (erroneous) results.

Let us comment briefly on Eq. (4.12). The matrix $\mathscr{P}_{\alpha\beta}(\vec{q}_{||}\omega)$ is of $O(\delta^2)$. Thus in the case of a flat surface the solubility condition for Eq. (4.12) is simply $\det|\overleftrightarrow{M}^{(0)}(\vec{q}_{||}\omega)|=0$, whose solution gives the dispersion relation $\omega_0(q_{||})=c_R q_{||}$ of a Rayleigh wave propagating on a flat surface. [The speed c_R of the Rayleigh wave is defined by Eq. (2.4) of Ref. 3.] In the case of a randomly rough surface the solution to the equation $\det[\overleftrightarrow{M}^{(0)}(q_{||}\omega)+\overleftrightarrow{\mathscr{P}}(\vec{q}_{||}\omega)]=0$ can be written in the form

$$\omega_R(q_{||})=\omega_0(q_{||})+\Delta\omega(q_{||}) , \tag{4.13}$$

which defines the perturbation $\Delta\omega(q_{||})$ of the Rayleigh wave dispersion relation due to the roughness.

In Ref. 3 we present a detailed analysis of $\Delta\omega(q_{||})$ in the case that the correlation function $W(|\vec{x}_{||}-\vec{x}\,'_{||}|)$ is assumed to be a Gaussian. From the real part of $\Delta\omega(q_{||})$ we obtain the shift in the frequency of a Rayleigh wave due to the presence of roughness and from its imaginary part we obtain the lifetime of such a wave (or its inverse attenuation length). In this regard we note that the surface roughness opens up three channels for the decay of a given Rayleigh wave, namely the channels provided by the transverse and longitudinal bulk modes and by other Rayleigh waves. In Ref. 3 we study the contributions from all three channels to both the real and imaginary parts of $\Delta\omega(q_{||})$.

In conclusion, we have given a simple illustration of the method developed in Secs. II and III for the study of dynamical properties of a semi-infinite elastic medium bounded by a rough, planar surface. In the present example we have applied the boundary conditions given by Eq. (3.35) to the solution of the equation of motion for the average displacement field given by Eqs. (4.9)—(4.11). We stress that what we have done in this section is completely analogous to what one does to obtain the dispersion relation $\omega_0(q_{||})=c_R q_{||}$ in the case of a flat surface. The conceptual simplicity afforded by our method is apparent.

As mentioned in the Introduction, our method can be applied to more general problems dealing with the interaction of a rough surface with various external probes. The key role in such problems is played by the Green's functions for the semi-infinite medium bounded by a rough surface. The average of these Green's functions over an ensemble of rough surfaces can be calculated rather directly by using our effective boundary conditions. Finally we note that, as indicated above, the results obtained in this section exactly agree with those obtained[3] using Rayleigh's method.[6]

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APPENDIX A: GREEN'S THEOREM FOR THE OPERATOR $L_{\alpha\beta}$

In this appendix we obtain a formal result for the vector $Qu_\mu(\vec{x}\,|\,\omega)$, the fluctuating component of the total displacement field $u_\mu(\vec{x}\,|\,\omega)$. Consider two fields $U_\alpha(\vec{x}\,|\,\omega)$ and $V_\alpha(\vec{x}\,|\,\omega)$. We proceed to evaluate the integral

$$I\equiv\sum_{\alpha,\mu}\int_V d^3x[U_\alpha(\vec{x}\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)V_\mu(\vec{x}\,|\,\omega)-V_\alpha(\vec{x}\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)U_\mu(\vec{x}\,|\,\omega)] , \tag{A1}$$

where

$$L_{\alpha\mu}(\vec{x}\,|\,\omega)=\omega^2\delta_{\alpha\mu}+\frac{1}{\rho}\sum_{\beta,\nu}c_{\alpha\beta\mu\nu}\frac{\partial^2}{\partial x_\beta\partial x_\nu} \tag{A2}$$

is the differential operator that governs the dynamical behavior of an elastic medium [see Eq. (2.2)]. The volume of integration, V, will be specified below. Substituting Eq. (A2) into Eq. (A1) and using the divergence (Gauss) theorem leads us to the result that

$$I=\frac{1}{\rho}\sum_{\alpha,\beta,\mu,\nu}\oiint_S dS\,\hat{n}_\beta c_{\alpha\beta\mu\nu}\left[U_\alpha\frac{\partial}{\partial x_\nu}V_\mu-V_\alpha\frac{\partial}{\partial x_\nu}U_\mu\right]$$

$$-\frac{1}{\rho}\sum_{\alpha,\beta,\mu,\nu}\int_V d^3x\,c_{\alpha\beta\mu\nu}\left[\frac{\partial}{\partial x_\beta}U_\alpha\frac{\partial}{\partial x_\nu}V_\mu-\frac{\partial}{\partial x_\beta}V_\alpha\frac{\partial}{\partial x_\nu}U_\mu\right] , \tag{A3}$$

where S is the surface that encloses the volume V, and $\hat{n}$ is its normal unit vector (directed outwards). Making use of the symmetry relation

$$c_{\alpha\beta\mu\nu}=c_{\mu\nu\alpha\beta} , \tag{A4}$$

it is easy to show that the second line in Eq. (A3) vanishes. Thus we obtain the result that

$$I=\frac{1}{\rho}\sum_{\alpha,\beta,\mu,\nu}\oint\!\!\!\oint_S dS\,\hat{n}_\beta c_{\alpha\beta\mu\nu}\left[U_\alpha(\vec{x}\mid\omega)\frac{\partial}{\partial x_\nu}V_\mu(\vec{x}\mid\omega)-V_\alpha(\vec{x}\mid\omega)\frac{\partial}{\partial x_\nu}U_\mu(\vec{x}\mid\omega)\right] . \tag{A5}$$

We define a Green's-function tensor $D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)$ such that

$$\sum_\mu L_{\alpha\mu}(\vec{x}\mid\omega)D_{\mu\gamma}(\vec{x};\vec{x}'\mid\omega)=\delta_{\alpha\gamma}\delta(\vec{x}-\vec{x}') \tag{A6}$$

(where we assume $x_3,x_3'>0$) together with the boundary conditions

$$\sum_{\mu,\nu} c_{\alpha 3\mu\nu}\frac{\partial}{\partial x_\nu}D_{\mu\gamma}(\vec{x};\vec{x}'\mid\omega)\Bigg|_{x_3=0}=0 . \tag{A7}$$

Equations (A6) and (A7) define the "flat-surface" Green's functions calculated by Maradudin and Mills.[2] We next define the quantity I_γ by

$$I_\gamma=\sum_{\alpha,\mu}\int_V d^3x[Qu_\alpha(\vec{x}\mid\omega)L_{\alpha\mu}(\vec{x}\mid\omega)D_{\mu\gamma}(\vec{x};\vec{x}'\mid\omega)-D_{\alpha\gamma}(\vec{x};\vec{x}'\mid\omega)L_{\alpha\mu}(\vec{x}\mid\omega)Qu_\mu(\vec{x}\mid\omega)] , \tag{A8}$$

where the volume V is the whole upper half-space $x_3\geq 0$. Now, the vector $QU_\alpha(\vec{x}\mid\omega)$ satisfies the same (homogeneous) equation as $u_\alpha(\vec{x}\mid\omega)$, namely Eq. (2.2). Thus from Eqs. (A6) and (A8) we have that

$$I_\gamma=Qu_\gamma(\vec{x}'\mid\omega) . \tag{A9}$$

We now make use of the result (A5), with the replacements $U\to Qu$, $V\to D$. We further assume that $u_\mu(\vec{x}\mid\omega)\to 0$ as $x_3\to\infty$. Noting that the unit vector normal to the surface $x_3=0$ is $\hat{n}=(0,0,-1)$, we obtain the result that

$$\begin{aligned}Qu_\gamma(\vec{x}'\mid\omega)=&-\frac{1}{\rho}\sum_{\alpha,\mu,\nu}c_{\alpha3\mu\nu}\int d^2x_{||}Qu_\alpha(\vec{x}\mid\omega)\frac{\partial}{\partial x_\nu}D_{\mu\gamma}(\vec{x};\vec{x}'\mid\omega)\Bigg|_{x_3=0}\\&+\frac{1}{\rho}\sum_{\alpha,\mu,\nu}c_{\alpha3\mu\nu}\int d^2x_{||}D_{\alpha\gamma}(\vec{x};\vec{x}'\mid\omega)\frac{\partial}{\partial x_\nu}Qu_\mu(\vec{x}\mid\omega)\Bigg|_{x_3=0} .\end{aligned} \tag{A10}$$

Furthermore, because of the stress-free boundary condition (A7) we imposed on the Green's function $D_{\mu\nu}$, the first term in Eq. (A10) vanishes. This is very convenient, since utilizing the boundary condition (2.16) in the second term gives then a result for the vector Qu_μ in terms of (derivatives of) the *average* field $\langle u_\mu(\vec{x};\omega)\rangle$ alone, namely,

$$Qu_\gamma(\vec{x}'\mid\omega)=\frac{1}{\rho}\sum_{\alpha,\mu,\nu}\int d^2x_{||}\left[D_{\alpha\gamma}(\vec{x};\vec{x}'\mid\omega)B_{\alpha\mu\nu}(\vec{x})\frac{\partial}{\partial x_\nu}\langle u_\mu(\vec{x}\mid\omega)\rangle\right]_{x_3=0} . \tag{A11}$$

We note that this result shows explicitly [see Eq. (2.17)] that the vector Qu_μ is of $O(\zeta)$ [as suggested below Eq. (2.10)]. We emphasize that Eq. (A10) holds for any Green's-function tensor satisfying Eq. (A6), e.g., the much simpler *infinite* medium Green's functions, for which, obviously, there is no analog for the boundary conditions (A7). The reason we chose the *semi*-infinite medium Green's functions, with the stress-free boundary conditions (A7), is that our choice provides us with an expression for Qu solely in terms of $\langle u\rangle$ [Eq. (A11)].

Finally, we make use of the reciprocity relation

$$D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)=D_{\nu\mu}(\vec{x}';\vec{x}\mid\omega) \tag{A12}$$

(see Appendix B), and rewrite Eq. (A11) in a form more convenient for Sec. II, namely,

$$Qu_\mu(\vec{x}\mid\omega)=\frac{1}{\rho}\sum_{\alpha,\beta,\gamma}\int d^2\vec{x}'_{||}\left[D_{\mu\alpha}(\vec{x};\vec{x}'\mid\omega)B_{\alpha\beta\gamma}(\vec{x}')\frac{\partial}{\partial x'_\gamma}\langle u_\beta(\vec{x}'\mid\omega)\rangle\right]_{x_3'=0} . \tag{A13}$$

APPENDIX B: PROOF OF EQ. (A12)

In this appendix we prove the reciprocity relation

$$D_{\mu\nu}(\vec{x};\vec{x}'\,|\,\omega)=D_{\nu\mu}(\vec{x}';\vec{x}\,|\,\omega) \tag{B1}$$

for the "flat-surface" Green's functions $D_{\mu\nu}(\vec{x};\vec{x}'\,|\,\omega)$. These Green's functions satisfy the differential equation given by Eqs. (A6) and (A2) with the stress-free boundary conditions given by Eq. (A7).

We introduce the functions $J_{\gamma\nu}(\vec{x}';\vec{x}''\,|\,\omega)$ by the equation

$$J_{\gamma\nu}=\sum_{\alpha,\mu}\int_V d^3x[D_{\alpha\nu}(\vec{x};\vec{x}''\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)D_{\mu\gamma}(\vec{x};\vec{x}'\,|\,\omega)-D_{\alpha\gamma}(\vec{x};\vec{x}'\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)D_{\mu\nu}(\vec{x};\vec{x}''\,|\,\omega)]\ , \tag{B2}$$

where the operator $L_{\alpha\mu}(\vec{x}\,|\,\omega)$ is given by Eq. (A2). Substituting Eq. (A2) in Eq. (B2) we have that

$$J_{\gamma\nu}=\frac{1}{\rho}\sum_{\alpha,\beta,\mu,\delta}c_{\alpha\beta\mu\delta}\int_V d^3x\left[D_{\alpha\nu}(\vec{x};\vec{x}''\,|\,\omega)\frac{\partial^2}{\partial x_\beta\partial x_\delta}D_{\mu\gamma}(\vec{x};\vec{x}'\,|\,\omega)-D_{\alpha\gamma}(\vec{x};\vec{x}'\,|\,\omega)\frac{\partial^2}{\partial x_\beta\partial x_\delta}D_{\mu\nu}(\vec{x};\vec{x}''\,|\,\omega)\right]. \tag{B3}$$

The procedure that leads us from Eq. (B3) to Eq. (B4) below is as follows. We integrate Eq. (B3) by parts once. We use the symmetry relation (A4) for the elastic moduli tensor $c_{\alpha\beta\mu\nu}$ plus appropriate changes of dummy variables and obtain the result that the "volume" terms we have after the integration by parts cancel each other. Finally, using the divergence (Gauss) theorem in the "surface" terms we obtain the result that

$$J_{\gamma\nu}=\frac{1}{\rho}\sum_{\alpha,\beta,\mu,\delta}c_{\alpha\beta\mu\delta}\oiint_S dS\,\hat{n}_\beta\left[D_{\alpha\nu}(\vec{x};\vec{x}''\,|\,\omega)\frac{\partial}{\partial x_\delta}D_{\mu\gamma}(\vec{x};\vec{x}'\,|\,\omega)-D_{\alpha\gamma}(\vec{x};\vec{x}'\,|\,\omega)\frac{\partial}{\partial x_\delta}D_{\mu\nu}(\vec{x};\vec{x}''\,|\,\omega)\right]. \tag{B4}$$

We can obtain an alternative result for $J_{\gamma\nu}(\vec{x}';\vec{x}''\,|\,\omega)$ in the following way. Multiplying Eq. (A6) from the left by $D_{\alpha\nu}(\vec{x};\vec{x}''\,|\,\omega)$ and summing over α we have that

$$\sum_{\alpha,\mu}D_{\alpha\nu}(\vec{x};\vec{x}''\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)D_{\mu\gamma}(\vec{x};\vec{x}'\,|\,\omega)=D_{\gamma\nu}(\vec{x}';\vec{x}''\,|\,\omega)\delta(\vec{x}-\vec{x}')\ . \tag{B5}$$

In a similar way we can obtain the result that

$$\sum_{\alpha,\mu}D_{\alpha\gamma}(\vec{x};\vec{x}'\,|\,\omega)L_{\alpha\mu}(\vec{x}\,|\,\omega)D_{\mu\nu}(\vec{x};\vec{x}''\,|\,\omega)=D_{\nu\gamma}(\vec{x}'';\vec{x}'\,|\,\omega)\delta(\vec{x}-\vec{x}'')\ . \tag{B6}$$

We next subtract Eq. (B6) from Eq. (B5) and integrate over $\vec{x}$ ($\vec{x}\in V$). With the requirement that both $\vec{x}'$ and $\vec{x}''$ lie within V, we have that

$$J_{\gamma\nu}=D_{\gamma\nu}(\vec{x}';\vec{x}''\,|\,\omega)-D_{\nu\gamma}(\vec{x}'';\vec{x}'\,|\,\omega)\ . \tag{B7}$$

We consider now the case that the volume V of integration is the upper half-space $x_3>0$. In this case the unit vector $\hat{n}$ is given by $\hat{n}=(0,0,-1)$. From Eqs. (B4) and (B7) it then follows that

$$\begin{aligned}D_{\gamma\nu}(\vec{x}';\vec{x}''\,|\,\omega)-D_{\nu\gamma}(\vec{x}'';\vec{x}'\,|\,\omega)=&-\frac{1}{\rho}\sum_\alpha\int d^2x_{\|}\left[D_{\alpha\nu}(\vec{x};\vec{x}''\,|\,\omega)\sum_{\mu,\delta}c_{\alpha3\mu\delta}\frac{\partial}{\partial x_\delta}D_{\mu\gamma}(\vec{x};\vec{x}'\,|\,\omega)\right]_{x_3=0}\\&+\frac{1}{\rho}\sum_\alpha\int d^2x_{\|}\left[D_{\alpha\gamma}(\vec{x};\vec{x}'\,|\,\omega)\sum_{\mu,\delta}c_{\alpha3\mu\delta}\frac{\partial}{\partial x_\delta}D_{\mu\nu}(\vec{x};\vec{x}''\,|\,\omega)\right]_{x_3=0}.\end{aligned} \tag{B8}$$

At this point we make use of the assumption that the Green's functions $D_{\mu\nu}(\vec{x};\vec{x}'\,|\,\omega)$ satisfy the stress-free boundary conditions given by Eq. (A7). Thus both lines on the right-hand side of Eq. (B8) vanish, which proves Eq. (B1).

APPENDIX C: THE MATRICES $\vec{a}(\vec{q}_{\|};\vec{k}_{\|}\,|\,x_3')$ and $\vec{c}(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega)$

In this appendix we give the explicit expressions for the elements of the matrices $\hat{a}_{\alpha\beta}(\vec{q}_{\|};\vec{k}_{\|}\,|\,x_3')$ and $c_{\alpha\beta}(\vec{q}_{\|};\vec{k}_{\|}\,|\,\omega)$. The former is defined implicitly by Eqs. (3.30) [its 11 element was given in Eq. (3.29)]. The latter is defined by Eq. (3.36). We have

$$\hat{a}_{11}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=c_l^2(q_1-k_1)q_1+c_t^2(q_2-k_2)q_2-c_t^2\frac{d^2}{dx_3'^2} , \tag{C1}$$

$$\hat{a}_{12}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=(c_l^2-2c_t^2)(q_1-k_1)q_2+c_t^2(q_2-k_2)q_1 , \tag{C2}$$

$$\hat{a}_{13}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=(-i)[(c_l^2-2c_t^2)(q_1-k_1)+c_t^2q_1]\frac{d}{dx_3'} , \tag{C3}$$

$$\hat{a}_{21}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=c_t^2(q_1-k_1)q_2+(c_l^2-2c_t^2)(q_2-k_2)q_1 , \tag{C4}$$

$$\hat{a}_{22}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=c_t^2(q_1-k_1)q_1+c_l^2(q_2-k_2)q_2-c_t^2\frac{d^2}{dx_3'^2} , \tag{C5}$$

$$\hat{a}_{23}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=(-i)[(c_l^2-2c_t^2)(q_2-k_2)+c_t^2q_2]\frac{d}{dx_3'} , \tag{C6}$$

$$\hat{a}_{31}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=(-i)[c_t^2(q_1-k_1)+(c_l^2-2c_t^2)q_1]\frac{d}{dx_3'} , \tag{C7}$$

$$\hat{a}_{32}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=(-i)[c_t^2(q_2-k_2)+(c_l^2-2c_t^2)q_2]\frac{d}{dx_3'} , \tag{C8}$$

$$\hat{a}_{33}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')=c_t^2(q_1-k_1)q_1+c_t^2(q_2-k_2)q_2-c_l^2\frac{d^2}{dx_3'^3} . \tag{C9}$$

Note that the elements of $\hat{a}_{\alpha\beta}(\vec{q}_{\|};\vec{k}_{\|} \mid x_3')$ are differential operators in x_3'.

The algebra involved in obtaining the elements of $c_{\alpha\beta}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)$ is very lengthy. [We must first obtain the elements of the matrices $b_{\alpha\beta}^{(\lambda)}$ for $\lambda=1,2,3$, and that requires a subtle discussion of the boundary conditions satisfied by the Green's functions $D_{\mu\nu}(\vec{x}\vec{x}' \mid \omega)$ when the arguments x_3',x_3 are set equal to zero *in that order;* see Appendix D.] We have

$$c_{11}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=\frac{1}{k_{\|}^2\alpha_t(k_{\|})}\left[k_2\Theta_1+\frac{\omega^2}{c_t^2}\frac{k_1\alpha_t^2(k_{\|})}{\Delta(k_{\|} \mid \omega)}\Lambda_1\right] , \tag{C10}$$

$$c_{12}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=\frac{1}{k_{\|}^2\alpha_t(k_{\|})}\left[-k_1\Theta_1+\frac{\omega^2}{c_t^2}\frac{k_2\alpha_t^2(k_{\|})}{\Delta(k_{\|} \mid \omega)}\Lambda_1\right] , \tag{C11}$$

$$c_{13}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=iq_1\left[1-2\frac{c_t^2}{c_l^2}\right]+i\frac{\beta(k_{\|})\Lambda_1}{\Delta(k_{\|} \mid \omega)} , \tag{C12}$$

$$c_{21}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=\frac{1}{k_{\|}^2\alpha_t(k_{\|})}\left[-k_2\Theta_2+\frac{\omega^2}{c_t^2}\frac{k_1\alpha_t^2(k_{\|})}{\Delta(k_{\|} \mid \omega)}\Lambda_2\right] , \tag{C13}$$

$$c_{22}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=\frac{1}{k_{\|}^2\alpha_t(k_{\|})}\left[k_1\Theta_2+\frac{\omega^2}{c_t^2}\frac{k_2\alpha_t^2(k_{\|})}{\Delta(k_{\|} \mid \omega)}\Lambda_2\right] , \tag{C14}$$

$$c_{23}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=iq_2\left[1-2\frac{c_t^2}{c_l^2}\right]+i\frac{\beta(k_{\|})\Lambda_2}{\Delta(k_{\|} \mid \omega)} , \tag{C15}$$

$$c_{31}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=iq_1+ik_1\frac{\omega^2}{c_t^2}\frac{\beta(k_{\|})}{\Delta(k_{\|} \mid \omega)} , \tag{C16}$$

$$c_{32}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=iq_2+ik_2\frac{\omega^2}{c_t^2}\frac{\beta(k_{\|})}{\Delta(k_{\|} \mid \omega)} , \tag{C17}$$

$$c_{33}(\vec{q}_{\|};\vec{k}_{\|} \mid \omega)=-\frac{\omega^4}{c_t^4}\frac{\alpha_l(k_{\|})}{\Delta(k_{\|} \mid \omega)} . \tag{C18}$$

Here we have introduced the following definitions:

$$\beta(k_{||};\omega)=2k_{||}^2-\frac{\omega^2}{c_t^2}-2\alpha_l(k_{||})\alpha_t(k_{||}) , \tag{C19}$$

$$\Delta(k_{||};\omega)=4k_{||}^2\alpha_l(k_{||})\alpha_t(k_{||})-\left[2k_{||}^2-\frac{\omega^2}{c_t^2}\right]^2 , \tag{C20}$$

$$\Theta_1(\vec{q}_{||};\vec{k}_{||}\mid\omega)=k_2(\vec{q}_{||}\cdot\vec{k}_{||})+k_1(\vec{q}_{||}\times\vec{k}_{||})_3-k_2\frac{\omega^2}{c_t^2} , \tag{C21}$$

$$\Theta_2(\vec{q}_{||};\vec{k}_{||}\mid\omega)=k_1(\vec{q}_{||}\cdot\vec{k}_{||})-k_2(\vec{q}_{||}\times\vec{k}_{||})_3-k_1\frac{\omega^2}{c_t^2} , \tag{C22}$$

$$\Lambda_1(\vec{q}_{||};\vec{k}_{||}\mid\omega)=2k_1(\vec{q}_{||}\cdot\vec{k}_{||})+2q_1k_{||}^2\left[1-2\frac{c_t^2}{c_l^2}\right]-k_1\frac{\omega^2}{c_t^2} , \tag{C23}$$

$$\Lambda_2(\vec{q}_{||};\vec{k}_{||}\mid\omega)=2k_2(\vec{q}_{||}\cdot\vec{k}_{||})+2q_2k_{||}^2\left[1-2\frac{c_t^2}{c_l^2}\right]-k_2\frac{\omega^2}{c_t^2} . \tag{C24}$$

We have also defined the wave vectors $\alpha_l(k_{||};\omega)$ and $\alpha_t(k_{||};\omega)$ [for brevity denoted in the above equations as $\alpha_l(k_{||})$ and $\alpha_t(k_{||})$, respectively] according to $(\eta\rightarrow 0+)$:

$$\alpha_{l,t}(k_{||};\omega)=\left[k_{||}^2-\frac{(\omega+i\eta)^2}{c_{l,t}^2}\right]^{1/2} , \tag{C25}$$

and where we take the branch of the square root such that

$$\mathrm{Re}\,\alpha_{l,t}>0 . \tag{C26}$$

The physical meaning of the definitions given by Eqs. (C25) and (C26) is made clear with the aid of the results given in Eqs. (4.9)—(4.11) for the solution to the equation of motion for the average displacement field (in the absence of external sources). For $k_{||}>\omega/c_{l,t}$ we have that $\alpha_{l,t}^{-1}$ gives the inverse decay length (into the medium) of a displacement field that is confined to the surface region. For $k_{||}<\omega/c_{l,t}$, $2\pi\,|\,\alpha_{l,t}^{-1}\,|$ gives the wavelength (perpendicular to the nominal surface) for outgoing bulk waves.

It is of interest to note that, for a given wave vector $\vec{k}_{||}$, the function $\Delta(k_{||}\mid\omega)$ [defined by Eq. (C20)] vanishes at the frequency $\omega=c_Rk_{||}$ of a Rayleigh wave propagating on a flat, stress-free surface.[3] (Here c_R is the speed of such a wave.) Note that this zero of $\Delta(k_{||}\mid\omega)$ gives rise to a pole in the coefficients $c_{\lambda\beta}(\vec{q}_{||};\vec{k}_{||}\mid\omega)$.[7]

APPENDIX D: A NOTE ON THE BOUNDARY CONDITIONS FOR THE FLAT-SURFACE GREEN'S FUNCTIONS

The "flat-surface" Green's functions $D_{\mu\nu}(\vec{x};\vec{x}'\mid\omega)$ introduced in Sec. II and Appendix A satisfy the stress-free boundary conditions given by Eq. (A7), namely,

$$\left.\sum_{\mu,\nu}c_{\alpha3\mu\nu}\frac{\partial}{\partial x_\nu}D_{\mu\beta}(\vec{x};\vec{x}'\mid\omega)\right|_{x_3=0}=0 . \tag{D1}$$

In the case of an isotropic medium, it is convenient to consider the boundary conditions for the Fourier transforms $D_{\mu\nu}(\vec{k}_{||}\omega\mid x_3x_3')$ defined by Eq. (3.26a). Substituting Eq. (3.1) in Eq. (D1) and carrying out the sums over μ and ν we readily obtain, for $\alpha=1,2,3$, respectively, the following boundary conditions for $D_{\mu\nu}(\vec{k}_{||}\omega\mid x_3x_3')$:

$$\left[\frac{d}{dx_3}D_{1\beta}(\vec{k}_{||}\omega\mid x_3x_3')+ik_1D_{3\beta}(\vec{k}_{||}\omega\mid x_3x_3')\right]_{x_3=0}=0 , \tag{D2}$$

$$\left[\frac{d}{dx_3}D_{2\beta}(\vec{k}_{||}\omega\mid x_3x_3')+ik_2D_{3\beta}(\vec{k}_{||}\omega\mid x_3x_3')\right]_{x_3=0}=0 , \tag{D3}$$

and

$$\left[ik_1(c_l^2-2c_t^2)D_{1\beta}(\vec{k}_{\|}\omega\,|\,x_3x_3')+ik_2(c_l^2-2c_t^2)D_{2\beta}(\vec{k}_{\|}\omega\,|\,x_3x_3')+c_l^2\frac{d}{dx_3}D_{3\beta}(\vec{k}_{\|}\omega\,|\,x_3x_3') \right]_{x_3=0}=0\;. \tag{D4}$$

We note that the stress-free boundary conditions given by Eqs. (D2)—(D4) apply whenever the source point x_3' lies *inside* the medium ($x_3'>0$). However, in the effective boundary conditions given by Eq. (3.35) we need the boundary values of the Green's functions $D_{\mu\nu}(\vec{k}_{\|}\omega\,|\,x_3x_3')$ and their derivatives when the source point $x_3'=0$. We note that this fact is to be traced back to Eq. (2.18), in which the fluctuating component of the displacement field is given in terms of the values of the average displacement field at the surface $x_3'=0$. When the source point x_3' lies on the surface $x_3'=0$, we must investigate whether Eqs. (D2)—(D4) still apply. (We remark that the present discussion is relevant to obtaining the results given in Appendix C.)

The problem just posed can be conveniently addressed to by using the results of Appendix A of the paper by Maradudin and Mills.[2] These authors obtained explicit results for the "rotated" Green's functions $d_{\mu\nu}(k_{\|}\omega\,|\,x_3x_3')$ related to $D_{\mu\nu}(\vec{k}_{\|}\omega\,|\,x_3x_3')$ by the equation

$$D_{\mu\nu}(\vec{k}_{\|}\omega\,|\,x_3x_3')=\sum_{\alpha,\beta}[\overleftrightarrow{S}^{-1}(\vec{k}_{\|})]_{\mu\alpha}d_{\alpha\beta}(k_{\|}\omega\,|\,x_3x_3')\times S_{\beta\nu}(\vec{k}_{\|})\;, \tag{D5}$$

where the matrix $\overleftrightarrow{S}(\vec{k}_{\|})$ is given by

$$\overleftrightarrow{S}(\vec{k}_{\|})=\frac{1}{k_{\|}}\begin{bmatrix} k_1 & k_2 & 0 \\ -k_2 & k_1 & 0 \\ 0 & 0 & k_{\|} \end{bmatrix}\;. \tag{D6}$$

Note that the matrix $\overleftrightarrow{S}(\vec{k}_{\|})$ rotates the coordinate axes about the $\hat{x}_3$ direction in such a way that the $\hat{x}_1$ axis is aligned with the wave vector $\vec{k}_{\|}$.

From Eqs. (A33) of Ref. 2 we can obtain the following results for the nonzero components of $d_{\mu\nu}(k_{\|}\omega\,|\,x_3x_3')$ evaluated at $x_3'=0$:

$$d_{11}(k_{\|}\omega\,|\,x_30)=\frac{1}{c_t^2}\frac{\alpha_t}{\Delta(k_{\|}\,|\,\omega)}\left[-2k_{\|}^2e^{-\alpha_lx_3}+\left[2k_{\|}^2-\frac{\omega^2}{c_t^2}\right]e^{-\alpha_tx_3}\right]\;, \tag{D7}$$

$$d_{13}(k_{\|}\omega\,|\,x_30)=\frac{i}{c_t^2}\frac{k_{\|}}{\Delta(k_{\|}\,|\,\omega)}\left[-\left[2k_{\|}^2-\frac{\omega^2}{c_t^2}\right]e^{-\alpha_lx_3}+2\alpha_l\alpha_te^{-\alpha_tx_3}\right]\;, \tag{D8}$$

$$d_{22}(k_{\|}\omega\,|\,x_30)=-\frac{1}{c_t^2\alpha_t}e^{-\alpha_tx_3}\;, \tag{D9}$$

$$d_{31}(k_{\|}\omega\,|\,x_30)=\frac{i}{c_t^2}\frac{k_{\|}}{\Delta(k_{\|}\,|\,\omega)}\left[-2\alpha_l\alpha_te^{-\alpha_lx_3}+\left[2k_{\|}^2-\frac{\omega^2}{c_t^2}\right]e^{-\alpha_tx_3}\right]\;, \tag{D10}$$

$$d_{33}(k_{\|}\omega\,|\,x_30)=\frac{1}{c_t^2}\frac{\alpha_l}{\Delta(k_{\|}\,|\,\omega)}\left[\left[2k_{\|}^2-\frac{\omega^2}{c_t^2}\right]e^{-\alpha_lx_3}-2k_{\|}^2e^{-\alpha_tx_3}\right]\;, \tag{D11}$$

where $\alpha_l(k_{\|}\,|\,\omega)$, $\alpha_t(k_{\|}\,|\,\omega)$ (denoted above simply as α_l,α_t), and $\Delta(k_{\|}\,|\,\omega)$ are defined by Eqs. (C25) and (C20), respectively.

Substituting the results given by Eqs. (D7)—(D11) in Eq. (D5) and then setting $x_3=0$ gives us the values of $D_{\mu\nu}(\vec{k}_{\|}\omega\,|\,x_30)$ at $x_3=0$. Similarly, differentiating Eqs. (D5) and (D7)—(D11) we obtain the values of the derivatives of $D_{\mu\nu}(\vec{k}_{\|}\omega\,|\,x_30)$ with respect to x_3 evaluated at $x_3=0$. For brevity, the results thus obtained are not given here (they are implicit in the results given in Appendix C). It is, however, instructive to consider explicitly the validity of Eqs. (D2)—(D4) for $x_3'=0$. Making use of the relevant results among those obtained in the manner just indicated we can show that for $x_3'=0$ the boundary conditions given by Eqs. (D2)—(D4) are replaced, respectively, by the following equations:

$$\left[\frac{d}{dx_3}D_{1\beta}(\vec{k}_{\|}\omega\,|\,x_30)+ik_1D_{3\beta}(\vec{k}_{\|}\omega\,|\,x_30)\right]_{x_3=0}=\frac{1}{c_t^2}\delta_{\beta1}\;, \tag{D12}$$

$$\left[\frac{d}{dx_3}D_{2\beta}(\vec{k}_{\|}\omega\,|\,x_30)+ik_2D_{3\beta}(\vec{k}_{\|}\omega\,|\,x_30)\right]_{x_3=0}=\frac{1}{c_t^2}\delta_{\beta2}\;, \tag{D13}$$

and

$$\left[ik_1(c_l^2-2c_t^2)D_{1\beta}(\vec{k}_{\parallel}\omega\,|\,x_30)+ik_2(c_l^2-2c_t^2)D_{2\beta}(\vec{k}_{\parallel}\omega\,|\,x_30)+c_l^2\frac{d}{dx_3}D_{3\beta}(\vec{k}_{\parallel}\omega\,|\,x_30) \right]_{x_3=0}=\delta_{\beta 3} . \quad \text{(D14)}$$

We emphasize that in the above discussion the limits $x_3' \to 0$, $x_3 \to 0$ were taken in that order. In fact, the "non-stress-free boundary conditions" given by Eqs. (D12)–(D14) originate from the fact that the limits $x_3' \to 0$, $x_3 \to 0$ do not commute for certain of the functions $d_{\mu\nu}(k_{\parallel}\omega\,|\,x_3x_3')$ and/or their derivatives.

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[1]F. G. Bass, Izv. Vyssh. Uchebn. Zaved. Radiofiz. IV, 476 (1961).

[2]A. A. Maradudin and D. L. Mills, Ann. Phys. (N.Y.) 100, 262 (1976).

[3]A. G. Eguiluz and A. A. Maradudin, following paper, Phys. Rev. B 28, 728 (1983).

[4]We note that in Eq. (2.23) we have made the additional assumption that the correlation function W depends only on the *magnitude* of the vector $\vec{x}_{\parallel}-\vec{x}_{\parallel}'$. This assumption has the effect that *isotropy* is also restored to the problem. A simple consequence of this property is encountered in Sec. IV: The frequency $\omega_R(q_{\parallel})$ of a Rayleigh wave propagating along the rough surface of a semi-infinite isotropic medium is a function of the *magnitude* of the wave vector $\vec{q}_{\parallel}$.

[5]See, for example, A. A. Maradudin, E. W. Montroll, G. W. Weiss, and I. P. Ipatova, *Theory of Lattice Dynamics in the Harmonic Approximation* (Academic, New York, 1971), p. 575.

[6]Lord Rayleigh, Philos. Mag. 14, 70 (1907); *Theory of Sound,* 2nd ed. (Dover, New York, 1945), Vol. II, p. 89.

[7]The appearance of the function $\Delta(k_{\parallel}\,|\,\omega)$ in the denominator of the results for $c_{\alpha\beta}(\vec{q}_{\parallel};\vec{k}_{\parallel}\,|\,\omega)$ given by Eqs. (C10)–(C18) can be traced back to the results for $D_{\mu\nu}(\vec{k}_{\parallel}\omega\,|\,x_30)$ (and derivatives) evaluated at $x_3=0$, obtained in Appendix D.