Localization effects near the percolation threshold

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The scaling theory of localization is used to explore the percolation-localization crossover near the percolation threshold. When intercluster tunneling is neglected, electron states are always localized before the percolation threshold is reached. The detailed behavior depends on the relation between the correlation lengths for percolation (ξ_p) and localization (ξ_l). At nonzero temperature the inelastic scattering length ξ_i is also important. If intercluster tunneling is taken into account a sharp metal-insulator transition at p_c is possible if a generalized Harris criterion is satisfied.

I. INTRODUCTION

Various studies of the relation between percolation theory and Anderson localization have been made.[1] It is generally agreed, and recent studies[2,3] have confirmed, that for a system in which the potential is either zero or infinite, or one in which bonds are either present or missing, there will be some degree of disorder for which classical percolation is possible but for which the states in the corresponding quantum problem are localized.

In this paper we use the scaling theory of localization to explore this region in which classical electrons are close to the percolation threshold. The anomalous dimension of the infinite cluster at the percolation threshold dominates the problem at short distances, as long as intercluster tunneling and hopping are unimportant. In Sec. II we explore the interplay between the characteristic lengths of the percolation problem and of Anderson localization. In Sec. III nonzero-temperature effects are taken into account by including a third length, the inelastic scattering length. Finally we argue in Sec. IV that when intercluster tunneling is present, a transition is possible even at (or near) the percolation threshold p_c. In that situation the metal concentration is characterized by long-range correlations (LRC), and a generalized Harris criterion is needed in order to determine whether the transition is sharp or not.

II. SCALING THEORY OF PERCOLATION-LOCALIZATION CROSSOVER

In classical percolation theory, close to the percolation limit p_c, there is a characteristic length ξ_p proportional to $|p-p_c|^{-\nu_p}$. Over length scales smaller than ξ_p the system behaves as it does at the percolation limit, while over length scales larger than ξ_p it behaves like a modified bulk material: an insulator for $p<p_c$, and a conductor for $p>p_c$. In particular on the conducting side of the transition ($p>p_c$) there is a bulk electrical conductivity proportional to $(p-p_c)^{t_p}$, or $\xi_p^{-t_p/\nu_p}$, over length scales greater than ξ_p. A review of these scaling aspects of percolation theory and information on estimates of the value of the exponents ν_p and t_p can be found in Ref. 4.

To obtain a zero-temperature quantum theory of the conductivity this might be combined with the conductance scaling theory of Abrahams *et al.*[5] In this theory $g(L)$, which is the conductance divided by $e^2/2\pi\hbar$ for a d-dimensional cube of volume L^d, satisfies a scaling equation of the form

$$\frac{d\ln g}{d\ln L}=\beta(g)\ , \quad (1)$$

where $\beta(g)$ only attains its classical value $d-2$ in the limit of large values of g. It approaches this limit from below, with a leading correction proportional to $-g^{-1}$, and for small values of g the exponential localization of the wave functions is manifested by the asymptotic approach of β to $\ln(g/g_0)$.

Since the system is homogeneous for length scales L larger than ξ_p the theory should apply as usual, with β unaltered from its form for a bulk homogeneous material. For $L<\xi_p$ we know that the classical theory gives a conductivity that scales as L^{-t_p/ν_p} (the conductance scales as L^{-t_p/ν_p+d-2}) and so we expect the β function of the zero-temperature quantum theory to approach $-t_p/\nu_p+d-2$ from below for large g. According to the "nodes and links" model[6] this is equal to $-1/\nu_p$ for $d\leq 6$. The self-similar model for the backbone of the infinite cluster[7] also gives a negative value. In any case it is undoubtedly negative even if these theories are not accurate. The result is that at the classical percolation limit p_c, where ξ_p is infinite, the conductance always scales down to the region where $\beta(g)$ is close to $\ln(g/g_0)$ and so exponential locali-

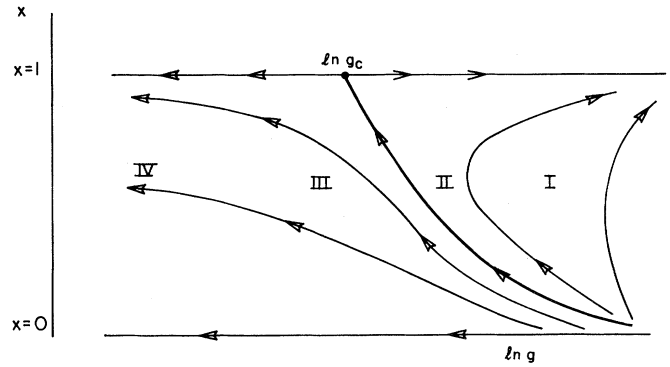

FIG. 1. Schematic flow diagram in the x-($\ln g$) plane.

zation occurs. At the percolation limit, the situation is similar to the situation in homogeneous one-dimensional systems, or more accurately like that in an effectively $d_{\rm eff}$-dimensional system with $d_{\rm eff}=d-t_p/\nu_p \leq 2$ (for $d \leq 6$).

Figure 1 shows a schematic sketch of a possible diagram for $\ln g(L)$ as a function of the parameter $x=L/(\xi_p+L)$ for $p \geq p_c$. For $x \lesssim \frac{1}{2}$ the flows are dominated by the behavior at the percolation limit, and so they are always towards negative values of lng, while for $x \gtrsim \frac{1}{2}$ the homogeneous bulk behavior is expected, which in three dimensions implies flow to negative lng for $g<g_c$ and to positive lng for $g>g_c$. In two dimensions the flow is always to negative lng since g_c is infinite. The figure can be divided into four general regions separated by these flow lines. In region I, which is the good metallic region, localization effects are unimportant, and the conductance (up to factors which depends on ξ) is proportional to x^{-t_p/ν_p+1} for small x, and to $(1-x)^{-1}$ for x close to unity. In this region the flow curves have all the same shape. In region II, the poor metallic region, localization effects reduce the conductance so that it is close to $\ln g_c$ for $x=\frac{1}{2}$, but for larger values of x the bulk metallic behavior dominates. In this region the resultant bulk conductivity σ gives a correlation length $\xi_l'=2\pi\hbar\sigma/e^2$ which is larger than ξ_p. The distinction between regions I and II is not sharp, but there is a sharp distinction between regions II and III. In region III, which lies just to the left of the critical line, the behavior for $x \lesssim \frac{1}{2}$ is similar to that in region II, but for larger values of x the scaling is towards exponential localization, with a localization length ξ_l which is greater than ξ_p. Finally in region IV localization occurs for length scales less than ξ_p.

A similar discussion could be carried out for the finite clusters with $p<p_c$. There is a localization region in which $\xi_l \ll \xi_p$, and a classical region in which ξ_l is of the order of the average cluster size.[8]

Notice that experimentally if the temperature is sufficiently low ($\xi_i \gtrsim \xi_p$; see Sec. III) the conductivity cannot follow the classical $\sigma(p)$ curve once $g(\xi_p) \lesssim e^2/\hbar$. If $\sigma(p)$ departs from the classical curve only at $g(\xi_p) \simeq e^2/\hbar$ this means that indeed intercluster tunneling and hopping are not important and the effects discussed above predominate.

Notice also that the condition $g(\xi_p) \simeq e^2/\hbar$ can be rewritten as

$$\sigma(l_{\rm el})l_{\rm el}\left[\frac{\xi_p}{l_{\rm el}}\right]^{1-t_p/\nu_p} \simeq \frac{e^2}{\hbar}\,, \tag{2}$$

where $l_{\rm el}$, the elastic mean free path, plays the role of a microscopic length scale. Using the relations $\sigma(l_{\rm el}) \simeq ne^2D$, $D \simeq v_F l_{\rm el}$, $n \simeq k_F^3/\epsilon_F$, we obtain the condition for a crossover to a quantum behavior,

$$(k_F l_{\rm el})^2 \simeq \left[\frac{\xi_p}{l_{\rm el}}\right]^{1-t_p/\nu_p}. \tag{3}$$

This is a generalization of the condition $k_F l_{\rm el} \simeq 1$ for homogeneous system (near percolation $\xi_p \gg l_{\rm el}$). In some situations the quantity $k_F l_{\rm el}$ may be interpreted as the ratio between the wavelength characterizing the motion in the random potential (with LRC) and the de Broglie wavelength of the electron. The crossover exponent from percolation to common (short-range) Anderson localization $\phi=t_p-\nu$ which was found previously by Shapiro,[2] follows directly from Eq. (2).

III. NONZERO TEMPERATURES

At nonzero temperatures, electrons are subject to inelastic collisions after they have traveled some characteristic distance ξ_i (measured in a straight line between points). Thus it is only for length scales less than ξ_i that the zero-temperature quantum theory of localization is relevant, while for greater length scales the electron diffuses classically, without any quantum interference effects. This idea can be used to analyze the situation in the four different regions described in Sec. II, in the case of a three-dimensional system.

In region I the zero-temperature behavior is essentially classical, so the inelastic collisions make only an unimportant modification of the behavior. In regions II and III the occurrence of inelastic scattering modifies the behavior of the conductivity in a way which is familiar from the study of localization in homogeneous systems. If $\xi_i<\xi_p$ we are in a weak localization regime, since $\xi_l>\xi_p$. We suppose the β function has the form

$$\beta(g) \equiv \frac{d\ln g}{d\ln L} \approx -k-A/g\,, \tag{4}$$

where $k=t_p/\nu_p-1$ (for a general d this is $t_p/\nu_p+2-d=2-d_{\rm eff}$). Integration from a value g_0 at some microscopic length λ_0 gives

$$g(L) \approx (\lambda_0/L)^k(A/k+g_0)-A/k\,, \tag{5}$$

while the classical value $g_p(L)$ is obtained by setting $A=0$ in this expression. Therefore, the relative change in resistance produced by localization effects on a length scale ξ_i is

$$[g_p(\xi_i)-g(\xi_i)]/g_p(\xi_i) \approx (A/kg_0)[(\xi_i/\lambda_0)^k-1]\,, \tag{6}$$

so there is a relative increment in the resistance proportional to ξ_i^k, which is similar to the usual weak localization behavior for $d_{\rm eff}=2-k$. The diffusion constant $D(\xi_i)$ over the distance is, on the one hand, related to the inelastic scattering time by $D(\xi_i)=\xi_i^2/\tau_i$. On the other hand, it is proportional to $\sigma(\xi_i)/n(\xi_i)$, where $n(\xi_i)$, the density of electrons belonging to the infinite cluster on scale ξ_i, is proportional to the density of the infinite cluster on this scale,[4,9] $n(\xi_i)\simeq\xi_i^{-\beta_p/\nu_p}$. Since $\sigma(\xi_p)\simeq\xi_i^{-k-1}$ it follows that $D(\xi_i)\simeq\xi_i^{-k-1+\beta_p/\nu_p}\equiv\xi_i^{-\theta}$ (see Ref. 9). Thus we have $\tau_i\simeq\xi_i^{2+\theta}$. The final result for the increment $\delta\rho$ in resistance due to localization in this region is (for $d=3$)

$$\delta\rho/\rho\propto\tau_i^{(t_p/\nu_p-1)/(2+t_p/\nu_p-\beta_p/\nu_p)}\simeq\tau_i^{0.3}\ . \tag{7}$$

In region IV the localization length ξ_l is less than ξ_p, so the localization can be weak, if $\xi_i\ll\xi_l$, or strong with $\xi_i\approx\xi_l$. In the weak localization case the situation is much the same as it is for regions II and III. In the strong localization case the electrons diffuse a distance ξ_l by hops in time τ_i,[10] so the conductivity on this length scale is proportional to $n(\xi_l)\xi_l^2/\tau_i$. Beyond this length scale the motion is classical, so the bulk conductivity σ, which is approximately the conductivity at the percolation length ξ_p, can be found from the scaling relation

$$\begin{aligned}\sigma\approx\sigma(\xi_p)&=(\xi_p/\xi_l)^{-t_p/\nu_p}\sigma(\xi_l)\\ &\propto\xi_l^{2-\beta_p/\nu_p}\left[\frac{\xi_l}{\xi_p}\right]^{t_p/\nu_p}\frac{1}{\tau_i}=\frac{\xi_p^{-t_p/\nu_p}\xi_l^{2+\theta}}{\tau_i}\ .\end{aligned} \tag{8}$$

For a detailed comparison of the formulas in this section with experimental results it is necessary to know how the inelastic scattering time τ_i depends on temperature near the percolation threshold. In some cases it may be dominated by long-wavelength phonons which are little affected by the inhomogeneities, in which case a T^{-4} dependence of τ_i is to be expected.[11] In other cases electron-electron scattering may dominate, in which case the present theoretical results[11,12] must be modified to allow for the structure of the infinite cluster at the percolation threshold. This question is left for further study.

IV. EXTENDED HARRIS CRITERION

So far we have neglected intercluster tunneling effects. We now consider a situation in which our system consists of, for example, a mixture of poor conducting and poor insulating materials, that is, the conductance is not restricted to the infinite cluster of the conducting component. In this case even at or near p_c we may have a metal-insulator transition (as a function of, for example, the microscopic conductivity of the insulator). This transition is characterized by LRC of the inhomogeneities, with a characteristic correlation length ξ_p. We now develop an extended Harris criterion[13] to establish the relevance of such LRC. (A similar problem in the context of magnetic systems was discussed by Weinrib and Halperin.[14])

The statistical fluctuations of the transition point are proportional to $V^{-1}(\sum_{i,j}\Delta n_i\,\Delta n_j)^{1/2}$, where Δn_i is the fluctuation in metal concentration of a microscopic volume of size $V=\xi_l^d$ ($\xi_l\ll\xi_p$). One can write this as $\propto\xi_l^{-d/2+\gamma_p/2\nu_p}$. The statistical fluctuations would be irrelevant (i.e., there would be a uniform sharp transition) if they are smaller than the distance from the transition point. The condition for this is

$$\xi_l^{-1/\nu_l}>\xi_l^{\gamma_p/2\nu_p-d/2} \tag{9}$$

or

$$\nu_l>\frac{2\nu_p}{d\nu_p-\gamma_p}\ , \tag{10}$$

where ν_l is the exponent related to the LRC localization fixed point. The crossover exponent from the regime governed by the LRC to that of the short-range correlations is

$$\phi=2-(d-\gamma_p/\nu_p)\nu_l\ . \tag{11}$$

One can now study various crossover scenarios, from the regime of LRC to the regime where the common Harris criterion[13] is relevant. We emphasize that for $p\neq p_c$ the true asymptotic behavior always corresponds to short-range correlations. However, near p_c one can have a large regime where the behavior is dominated by LRC. We also remark that if the inhomogeneity is sufficiently strong it may be inadequate to use the Harris criterion at all.

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[1]G. Toulouse, C. R. Acad. Sci. Ser. B 280, 629 (1975); J. Friedel, J. Phys. (Paris) Lett. 37, L9 (1976); Y. Gefen and M. Ya. Azbel, J. Phys. C 15, L655 (1982).

[2]D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. Pis'ma Red. 32, 248 (1980) [JETP Lett. 32, 229 (1981)]; B. Shapiro, Phys. Lett. 48, 823 (1982); Percolation Structures and Processes, Annals

of the Israel Physical Society, edited by G. Deutscher, J. Adler, and R. Zallen (in press).

[3]Y. Shapir, A. Aharony, and A. B. Harris, Phys. Rev. Lett. 49, 486 (1982); A. B. Harris, *ibid.* 49, 296 (1982).

[4]D. Stauffer, Phys. Rep. 54, 1 (1979).

[5]E. Abrahams, P. W. Anderson, D. G. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).

[6]A. S. Skal and B. I. Shklovskii, Fiz. Tekh. Poluprovodn. 8, 1582 (1974) [Sov. Phys.—Semicond. 8, 1029 (1975)]; P. G. de Gennes, J. Phys. (Paris) Lett. 37, L1 (1976).

[7]Y. Gefen, A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, Phys. Rev. Lett. 47, 1771 (1981).

[8]After this work was completed, we learned that scaling for $T=0$ at p_c was recently discussed by R. Rammal and G. Toulouse [J. Phys. (Paris) Lett. 44, L13 (1983)].

[9]Y. Gefen, A. Aharony, and S. Alexander, Phys. Rev. Lett. 50, 77 (1983).

[10]This problem for one-dimensional systems was discussed by D. J. Thouless, Phys. Rev. Lett. 39, 1167 (1977).

[11]A. Schmid, Z. Phys. 271, 251 (1974).

[12]P. W. Anderson, E. Abrahams, and T. V. Ramakrishnan, Phys. Rev. Lett. 43, 718 (1979).

[13]A. B. Harris, J. Phys. C 7, 1671 (1974).

[14]A. Weinrib and B. I. Halperin, Phys. Rev. B 27, 413 (1983).