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Finite-size scaling for directed self-avoiding walks

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A directed self-avoiding walk model is solved exactly for a finite cylinder geometry where the cylinder's axis is parallel to the directed axis of the walk. The finite-size scaling behavior of the longitudinal and transverse correlation lengths is examined. Although the infinite system has a continuous transition, one of the scaling variables for the finite system varies exponentially with the cylinder circumference, which is reminiscent of finite-size effects at first-order transitions in scalar-spin systems.

Finite-size scaling theory[1,2] is a topic of appreciable theoretical interest (see Ref. 3 for a review). Precise estimates of critical exponents for various two-dimensional lattice models have been obtained by applying this theory to transfer-matrix calculations (see Ref. 4 for a review). The finite-size behavior at normal critical or continuous transition points can be derived in a simplified way[5,6] by regarding the inverse lengths $1/L_i$ as extra scaling fields. For example, for the order parameter m of a spatially anisotropic two-dimensional system on an $L_\parallel \times L_\perp$ lattice, one obtains

$$m(T;L_\parallel,L_\perp) \approx m_\infty(T) M\left(\frac{L_\parallel}{\xi_{\parallel,\infty}(T)},\frac{L_\perp}{\xi_{\perp,\infty}(T)}\right) \quad (1)$$

as the reduced temperature $t \equiv (T_c - T)/T_c \to 0^+$ at ordering field $H \equiv 0$: Here m_∞, $\xi_{\parallel,\infty}$, and $\xi_{\perp,\infty}$ are the bulk order parameter and correlation lengths, which vary as t^β, $t^{-\nu_\parallel}$, and $t^{-\nu_\perp}$, respectively, near the bulk critical temperature $T = T_c$. We shall refer to (1) as a *strong* scaling hypothesis. More elaborate renormalization-group analysis[7] suggests that for, say, the case $L_\parallel = L_\perp \equiv L$, L should probably be regarded not as the inverse of a standard scaling field but as a renormalization-group flow parameter. The leading scaling relation (1) has been confirmed in many cases,[3] although there are difficulties[8] with the systematic analysis of correction terms.

Recently, a scaling theory of finite-size effects at *first-order* transitions has been developed.[9,10] In two dimensions, it prescribes that the two arguments of relation (1) be replaced by the single scaling variable combination $L_\parallel L_\perp / l_H^2$, where

$$l_H(T) \equiv |k_G T / m_\infty(T) H|^{1/2} \quad (2)$$

is a "phase persistence length" whose divergence at $H \to 0$ for $T < T_c$ fixed reflects the occurrence of long-range order in the correlation functions that distinguish the two, coexisting, oppositely magnetized phases at the first-order transition.[9] In addition, it is found[10] that in the limit $L_\parallel >> L_\perp$ (where the lattice becomes a "strip"), a new length, $l_\parallel(L_\perp,T)$, also enters the finite-size scaling forms. Fluctuations break the ferromagnetic strip at $H = 0$ into *segments* of oppositely magnetized domains: $l_\parallel$ represents the characteristic longitudinal extent of one of these single-phase domains. As such, it is proportional to the finite-strip longitudinal correlation length at the bulk first-order transition and varies[10,11] as

$$l_\parallel(L_\perp,T) \sim e^{L_\perp \sigma(T)} \quad , \quad (3)$$

where $\sigma(T) \equiv \Sigma(T)/k_B T$ is the reduced surface tension of the transverse domain walls. Thus, a new argument $L_\parallel / l_\parallel(L_\perp)$, which depends exponentially on $L_\perp$, must be included in the finite-size scaling relation for the first-order transition, which may then be written

$$m(T,H;L_\parallel,L_\perp) \approx m_\infty(T) \tilde{M}\left(\frac{L_\parallel L_\perp}{[l_H(T)]^2},\frac{L_\parallel}{l_\parallel(L_\perp,T)}\right) \quad (4)$$

as $H \to 0$ at $T < T_c$ fixed. For further details and a discussion of the difficulties associated with the derivation of finite-size scaling for first-order transitions from renormalization-group analyses,[9,12,13] the reader should consult Ref. 10.

By adopting a more general notation in which $\xi_{\parallel,\infty}$ and $\xi_{\perp,\infty}$ are taken to represent the bulk correlation lengths in the case of a continuous transition but the phase persistence lengths in the case of a first-order transition, one can subsume both scaling relations (1) and (4) under the single relation

$$m(T,H;L_\parallel,L_\perp) \approx m_\infty(T) X\left(\frac{L_\parallel}{\xi_{\parallel,\infty}},\frac{L_\perp}{\xi_{\perp,\infty}},\frac{L_\parallel}{l_\parallel(L_\perp)}\right) \quad . \quad (5)$$

We shall refer to (5) as a *weak* scaling hypothesis: Note, in particular, that it generalizes the usual finite-size scaling relation (1) to include the length $l_\parallel(L_\perp) \sim e^{L_\perp \sigma}$. For continuous bulk transitions, this new length is not normally expected, but it does in fact arise in the model that will now be discussed.

Our model is a directed self-avoiding walk on a finite square lattice of $L_\parallel \times L_\perp$ sites[14-18]: This problem was first studied (for infinite lattices) by Fisher and Sykes,[14] and for finite lattices by Redner.[15] It is a self-avoiding walk subject to the additional constraint that steps along one "directed" axis (taken here to be the x axis) may be in only one direction (here, $+\hat{x}$). We consider periodic boundary conditions in the y direction, so that the walk is actually on a cylinder of length $L_\parallel$ and circumference $L_\perp$. (Here and henceforth we assume the lattice constant a to be unity.) Assign weights $z_\parallel$, z_+, and z_-, respectively, to every horizontal $(+\hat{x})$, upward $(+\hat{y})$, and downward $(-\hat{y})$ step, and let $G^{(N)}_{n_+ n_-}(L_\parallel,L_\perp)$ denote the number of N-step walks which start at the origin $(x=0, y=0)$ and have n_+ upward and n_- downward steps.

Consider first the case $L_\parallel = \infty$. Then the generating function for these walks is[14,18,19]

$$G(z_{\parallel},z_{+},z_{-};\infty,L_{\perp}) \equiv \sum_{N,n_{+},n_{-}} G^{(N)}_{n_{+}n_{-}}(\infty,L_{\perp}) z_{\parallel}^{N-n_{+}-n_{-}} z_{+}^{n_{+}} z_{-}^{n_{-}} = H[1+z_{\parallel}H+(z_{\parallel}H)^2+(z_{\parallel}H)^3+\cdots] = H/(1-z_{\parallel}H) \ , \tag{6}$$

where H denotes the generating function for a walk parallel to the y axis:

$$\begin{aligned} H(z_{+},z_{-};L_{\perp}) &= 1+(z_{+}+z_{+}^2+\cdots+z_{+}^{L_{\perp}-1}) \\ &\quad +(z_{-}+z_{-}^2+\cdots+z_{-}^{L_{\perp}-1}) \\ &= 1+\frac{z_{+}-z_{+}^{L_{\perp}}}{1-z_{+}}+\frac{z_{-}-z_{-}^{L_{\perp}}}{1-z_{-}} \ . \end{aligned} \tag{7}$$

The total number of N-step walks,[14] $c_N(\infty,L_{\perp})$, is then generated from the "susceptibility"

$$\begin{aligned} \chi(z;\infty,L_{\perp}) &\equiv G(z,z,z;\infty,L_{\perp}) = \sum_{N=0}^{\infty} c_N(\infty,L_{\perp}) z^N \\ &= (1+z-2z^{L_{\perp}})/(1-2z-z^2+2z^{L_{\perp}+1}) \ . \end{aligned} \tag{8}$$

(See Refs. 20 and 21 for discussions of the analogy between the self-avoiding walk problem and ferromagnetic models exhibiting continuous transitions.) For the infinite lattice ($L_{\perp}=\infty$), the singularity of χ nearest the origin is a simple pole at

$$z=z_c=\sqrt{2}-1 \ , \tag{9}$$

which represents a bulk critical point near which $\chi(z;\infty,\infty)$ behaves as $(z_c-z)^{-\gamma}$ with $\gamma=1$. The mean horizontal displacement of N-step walks is defined by

$$\langle R_{\parallel}^{(N)}(\infty,L_{\perp})\rangle \equiv c_N^{-1} \sum_{n_{+},n_{-}} (N-n_{+}-n_{-}) G^{(N)}_{n_{+}n_{-}} \ , \tag{10}$$

in terms of which one then defines the longitudinal correlation length as

$$\begin{aligned} \xi_{\parallel}(z;\infty,L_{\perp}) &\equiv \left[\sum_{N=0}^{\infty} c_N \langle R_{\parallel}^{(N)}\rangle z^N\right] \Bigg/ \left[\sum_{N=0}^{\infty} c_N z^N\right] \\ &= z\left[\frac{\partial \ln G}{\partial z_{\parallel}}\right]_{z_{\parallel}=z_{+}=z_{-}=z} \ . \end{aligned} \tag{11}$$

Thus, from (6) and (8) one obtains the simple result

$$\xi_{\parallel}(z;\infty,L_{\perp}) = z\chi(z;\infty,L_{\perp}) \ . \tag{12}$$

Defining $t\equiv(z_c-z)/z_c$, one then has, in the bulk system,

$$\xi_{\parallel}(z;\infty,\infty) \approx \tfrac{1}{2} t^{-\nu_{\parallel}} \text{ with } \nu_{\parallel}=1 \ . \tag{13}$$

The (squared) transverse correlation length is similarly defined in terms of the mean-square transverse displacement of N-step walks,

$$\langle [R_{\perp}^{(N)}(\infty,L_{\perp})]^2\rangle \equiv c_N^{-1} \sum_{n_{+},n_{-}} (n_{+}-n_{-})^2 G^{(N)}_{n_{+}n_{-}} \ , \tag{14}$$

as

$$\begin{aligned} \xi_{\perp}^2(z;\infty,L_{\perp}) &\equiv \left[\sum_{N=0}^{\infty} c_N \langle (R_{\perp}^{(N)})^2\rangle z^N\right] \Bigg/ \left[\sum_{N=0}^{\infty} c_N z^N\right] \\ &= \hat{D}_{\perp} G(z_{\parallel},z_{+},z_{-},;\infty,L_{\perp}) \ , \end{aligned} \tag{15}$$

where

$$\hat{D}_{\perp} f \equiv (1/f)\left[z^2\left(\frac{\partial}{\partial z_{+}}-\frac{\partial}{\partial z_{-}}\right)^2+2z\frac{\partial}{\partial z_{+}}\right] f\Big|_{z_{\parallel}=z_{+}=z_{-}=z} \ . \tag{16}$$

[Note that according to the convention adopted here, a walk with a total of, say, $2L_{\perp}-1$ upward steps and no downward steps is regarded as having a transverse displacement of $2L_{\perp}-1$ from its origin (not 1, as measured from the shortest path back to the origin). This convention is mathematically convenient and, in the limit $L_{\perp}\to\infty$, reduces to the usual bulk definition of transverse displacement.] Thus one readily finds

$$\xi_{\perp}^2(z;\infty,L_{\perp}) = [\xi_{\parallel}(z;\infty,L_{\perp})+1]\hat{D}_{\perp}H(z_{+},z_{-};L_{\perp}) \ , \tag{17}$$

where

$$\hat{D}_{\perp}H(z_{+},z_{-};L_{\perp}) = 2z/(1-z)^2+O(L_{\perp}^2 z^{L_{\perp}}) \ . \tag{18}$$

Therefore, in the bulk system, one has

$$\xi_{\perp}^2(z;\infty,\infty) \approx \frac{1}{2z_c} t^{-2\nu_{\perp}} \text{ with } \nu_{\perp}=\tfrac{1}{2} \ . \tag{19}$$

Now let us examine the behavior of these results (12) and (17) for the correlation lengths when $L_{\perp}$ is finite. In that case, (8) implies that $\xi_{\parallel}$ and $\xi_{\perp}$ still diverge (again with exponents $\nu_{\parallel}=1$, $\nu_{\perp}=\frac{1}{2}$), but at a shifted critical point

$$z=z_0(L_{\perp})=z_c[1+2^{-1/2}z_c^{L_{\perp}}+O(L_{\perp}z_c^{2L_{\perp}})] \ . \tag{20}$$

It follows that *at* $z=z_c$, $\xi_{\parallel}$ diverges exponentially with $L_{\perp}$:

$$\xi_{\parallel}(z_c;\infty,L_{\perp}) \approx 2^{-1/2} z_c^{-L_{\perp}} \ . \tag{21}$$

Therefore, as the finite system approaches bulk criticality ($z\to z_c^{-}, L_{\perp}\to\infty$), this new length scale appears in addition to the usual diverging length scales set by the system dimensions $L_{\parallel,\perp}$ and the bulk correlation lengths $\xi_{\parallel,\perp}(z;\infty,\infty)$. Accordingly, in order to cast the scaling form of $\xi_{\parallel}$ near bulk criticality in the same form as the scaling hypothesis (5), it proves convenient to set $c\equiv-\ln z_c>0$ and define an exponentially divergent length

$$l_{\parallel}(L_{\perp}) \equiv e^{cL_{\perp}} = z_c^{-L_{\perp}} \sim \xi_{\parallel}(z_c;\infty,L_{\perp}) \ . \tag{22}$$

From (12) and (8), the exact result for $\xi_{\parallel}$ can then be written asymptotically as

$$\xi_{\parallel}(z;\infty,L_{\perp}) \approx \frac{1}{2t} W[2tl_{\parallel}(L_{\perp})] \ , \tag{23}$$

where

$$W(v) \equiv v/\sqrt{2}+v) \ . \tag{24}$$

By virtue of (13), this is equivalent to

$$\xi_{\parallel}(z;\infty,L_{\perp}) \approx \xi_{\parallel}(z;\infty,\infty) W\left[\frac{l_{\parallel}(L_{\perp})}{\xi_{\parallel}(z;\infty,\infty)}\right] \ . \tag{25}$$

One can now recognize (25) as a special form of the weak scaling hypothesis (5), where for this problem the scaling function $X(x,y,z)$ is reduced to a single-variable function $W(x/z)$ at $L_{\parallel}\to\infty$. However, (25) does *not* have the form of the strong scaling hypothesis (1) that one might expect to hold for this system with a continuous bulk transition. Instead, the scaling form (25) involves the additional length $l_{\parallel}(L_{\perp})$ which varies asymptotically with $L_{\perp}$ in a manner similar to the longitudinal correlation length (3) for first-order systems. [Clearly, by virtue of (17) and (18), the same conclusions apply to $\xi_{\perp}$.] Notice that the *strong* scaling hypothesis for the correlation lengths in the case $L_{\parallel}=\infty$ can be rewritten

$$\xi_i(t;\infty,L_{\perp})\approx L_{\perp}^{\nu_i/\nu_{\perp}}Y(L_{\perp}t^{\nu_{\perp}}),\quad i=\parallel \text{ or } \perp\ , \tag{26}$$

whereas (22) and (23) imply that for this directed self-avoiding walk model,

$$\xi_{\parallel}(z;\infty,L_{\perp})\approx e^{cL_{\perp}}\tilde{Y}(e^{cL_{\perp}}t^{\nu_{\parallel}})\ , \tag{27}$$

with a similar result for $\xi_{\perp}$. This anomalous dependence on $L_{\perp}$ is clearly a consequence of the extreme asymptotic anisotropy of the model, although it is worth noting that for another anisotropic problem, namely, directed percolation (see Ref. 22 for a review), no discrepancies were observed when the strong scaling hypotheses (26) were employed in numerical calculations.[22,23]

The behavior for $L_{\parallel}$ finite, which we now outline briefly, is similar. The generating function (6) becomes

$$\begin{aligned}G(z_{\parallel},z_+,z_-;L_{\parallel},L_{\perp})&=H[1+z_{\parallel}H+(z_{\parallel}H)^2+\cdots+(z_{\parallel}H)^{L_{\parallel}-1}]\\&=H[1-(z_{\parallel}H)^{L_{\parallel}}]/(1-z_{\parallel}H)\ ,\end{aligned} \tag{28}$$

whence $\xi_{\parallel}$ defined as in (11) is found to be

$$\xi_{\parallel}(z;L_{\parallel},L_{\perp})=\xi_{\parallel}(z;\infty,L_{\perp})-\frac{L_{\parallel}(zH_0)^{L_{\parallel}}}{[1-(zH_0)]^{L_{\parallel}}}\ , \tag{29}$$

where

$$H_0\equiv H(z,z;L_{\perp})=(1+z-2z^{L_{\perp}})/(1-z)\ . \tag{30}$$

Its asymptotic form near bulk criticality is

$$\xi_{\parallel}(z;L_{\parallel},L_{\perp})\approx\frac{1}{2t}\tilde{W}(2tL_{\parallel},2tl_{\parallel}) \tag{31}$$

with scaling function

$$\tilde{W}(u,v)\equiv W(v)-u/(e^{u/W(v)}-1)\ , \tag{32}$$

which again has the form of the weak but not the strong scaling hypothesis. The transverse correlation length is still given by (17) (but with $L_{\parallel}<\infty$), so that in the scaling region, one has

$$\xi_{\perp}^2(z;L_{\parallel},L_{\perp})\approx(1/z_c)\xi_{\parallel}(z;L_{\parallel},L_{\perp})\ . \tag{33}$$

In summary, this simple soluble model demonstrates that the conventional finite-size scaling relation (1), which breaks down at first-order transitions,[10] should also be used with caution in models with continuous transitions which exhibit anisotropic divergence of critical correlations.

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