Superfluid solitons in ^{4}He films

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It is shown that in two-dimensional superfluid ^{4}He films the one-dimensional solitons do not represent stable states in general. The system, however, admits of nonlinear excitations which are localized two-dimensional "lumps" consisting of superfluid condensate and which share many properties common with solitons. These objects should be experimentally observable in monolayer superfluid ^{4}He films at very low temperature. For thick films where the finite thickness effect dominates over the surface term in the dispersion and changes its sign and solitons are stable with respect to transverse perturbations and may represent the dominant mode of nonlinear long-wave excitations.

Two-dimensional ^{4}He films has attracted much attention of both experimentalists and theorists during the recent years. A good deal of understanding of these systems has been achieved through the study of third-sound phenomenon originally proposed by Atkins.[1] Third-sound excitations have been investigated with great accuracy by Rutledge *et al.*[2] These authors studied monolayer superfluid ^{4}He films and found positive dispersion for the surface modes, together with a roton branch that becomes excited above 0.6 K. The correct dispersion relation and temperature dependence of these modes was then derived in the linear regime by using a two-dimensional formulation of Landau's quantum hydrodynamics. Several experiments[3] on third-sound propagation, however, have revealed finite-amplitude effects that cannot be explained in terms of the linearized theory. In these experiments[3] there are indications of having undistorted-wave propagation in thin films at very low temperature. Recently, a theoretical formalism for the possible nonlinear excitations in monolayer superfluid ^{4}He films has been developed by Huberman[4] on the basis of a conjectured nonlinear superfluid density equation. It was found that the nonlinear effects can lead to existence of gapless solitons made up of superfluid condensate. Starting from the phenomenological Hamiltonian suggested by Rutledge *et al.*, Biswas and Warke[5] derived systematically a nonlinear superfluid density equation and theoretically confirmed the prediction by Huberman about the existence of superfluid solitons. A detailed experimental study of the type of possible nonlinear excitations in superfluid ^{4}He films is, however, yet to be made. The theory of these possible nonlinear excitations discussed so far has been restricted to one spatial dimension. Considering its theoretical and experimental interest and the fact that the problem is in fact a two-dimensional one we study in this Brief Report: (i) the physical relevance of the one-dimensional solitons in the two-dimensional system and (ii) the possibility of having two-dimensional nonlinear excitations in a thin superfluid ^{4}He film. We start from the following phenomenological equation of motion for the monolayer superfluid motion as proposed by Rutlege *et al.*,[2]

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi-\frac{A\psi}{(a+|\psi|^2)^3}-\mu\psi-B\psi\nabla^2|\psi|^2 \;, \tag{1}$$

where $\psi(\vec{x},t)$ is the condensate wave function, m is the mass of the helium atom, A and a ($A=14$ K and $a=1.2$ atomic layers) are constants of van der Waal's interaction, and μ and B are the chemical potential and surface tension constants, respectively. The vector $\vec{x}=(x,y)$ is a two-dimensional vector in the monolayer superfluid helium film. The superfluid surface density is $\rho(\vec{x},t)=|\psi(\vec{x},t)|^2$. With the chosen form of

$$\psi(\vec{x},t)=[\rho(\vec{x},t)]^{1/2}\exp[i\phi(\vec{x},t)] \;,$$

where ρ and ϕ are real functions one easily derives equations of motion for ρ and ϕ from Eq. (1),

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\rho\vec{v}=0 \;, \tag{2}$$

$$\frac{\hbar}{m}\frac{\partial\phi}{\partial t}=\frac{\hbar^2}{2m^2}\left[\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}-(\vec{\nabla}\phi)^2\right]+\frac{A}{m(a+\rho)^3}+\frac{\mu}{m}+\frac{B}{m}\nabla^2\rho \;. \tag{3}$$

We now define the superfluid velocity $\vec{v}=(\hbar/m)\vec{\nabla}\phi=(v_x,v_y)$. From (3) we derive

$$\frac{\partial\vec{v}}{\partial t}+\frac{1}{2}\vec{\nabla}v^2=\frac{\hbar^2}{2m^2}\vec{\nabla}\left[\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right]-\frac{3A}{m}\vec{\nabla}\rho\frac{1}{(a+\rho)^4}+\frac{B}{m}\vec{\nabla}\nabla^2\rho \;. \tag{4}$$

Using Eqs. (2) and (4) we intend to study the evolution of a localized initial disturbance consisting primarily of nearly-one-dimensional long waves of small amplitude. Let us orient the horizontal coordinate system such that the x direction is the principal direction of wave propagation.

We make the following coordinate transformation[6]:

$$X=x+C_3t,\quad t\to t \tag{5}$$

and use the method of multiple scaling.[7]

We make the scaling transformation

$$\bar{x}=\epsilon^{1/2}X,\quad \bar{y}=\epsilon y,\quad \bar{t}=\epsilon^{3/2}t\ . \tag{6}$$

The solution we are looking for is corresponding to a uniform superfluid helium density ρ_0 at rest on the average superimposed on it a characteristic collective density oscillation mode propagating through the two-dimensional liquid. The long-wavelength approximations of (2) and (4) can be carried out by expanding ρ and $\vec{v}$ in powers of the small parameter ϵ consistent with the above restrictions. We write

$$\begin{aligned} \rho &= \rho_0+\epsilon\rho_1(\bar{x},\bar{y},\bar{t})+\epsilon^2\rho_2(\bar{x},\bar{y},\bar{t})+\cdots\ ,\\ v_x &= \epsilon u_1(\bar{x},\bar{y},\bar{t})+\epsilon^2 u_2(\bar{x},\bar{y},\bar{t})+\cdots\ ,\\ v_y &= \epsilon^{1/2}[\epsilon v_1(\bar{x},\bar{y},\bar{t})+\epsilon^2 v_2(\bar{x},\bar{y},\bar{t})+\cdots]\ . \end{aligned} \tag{7}$$

The expansion of v_y is made consistent with $\partial v_y/\partial x=\partial v_x/\partial y$. Using Eqs. (5)–(7) in (2) and (4) and equating to zero the coefficients of the three lowest terms $\epsilon^{3/2}$, ϵ^2, and $\epsilon^{5/2}$ we have

$$\begin{aligned} &C_3\frac{\partial\rho_1}{\partial\bar{x}}+\rho_0\frac{\partial u_1}{\partial\bar{x}}=0\ ,\\ &C_3\frac{\partial u_1}{\partial\bar{x}}+\frac{3A}{m}\frac{1}{(a+\rho_0)^4}\frac{\partial\rho_1}{\partial\bar{x}}=0\ , \end{aligned} \tag{8}$$

$$C_3\frac{\partial v_1}{\partial\bar{x}}+\frac{3A}{m}\frac{1}{(a+\rho_0)^4}\frac{\partial\rho_1}{\partial\bar{y}}=0\ , \tag{9}$$

$$\frac{\partial\rho_1}{\partial\bar{t}}+C_3\frac{\partial\rho_2}{\partial\bar{x}}+\rho_0\frac{\partial u_2}{\partial\bar{x}}+\frac{\partial}{\partial\bar{x}}(\rho_1u_1)+\rho_0\frac{\partial v_1}{\partial\bar{y}}=0\ , \tag{10a}$$

$$\frac{\partial u_1}{\partial\bar{t}}+u_1\frac{\partial u_1}{\partial\bar{x}}+C_3\frac{\partial u_2}{\partial\bar{x}}+\frac{3A}{m(a+\rho_0)^4}\frac{\partial\rho_2}{\partial\bar{x}}-\frac{\hbar^2}{4m^2\rho_0}\frac{\partial^3\rho_1}{\partial\bar{x}^3}-\frac{12A}{m(a+\rho_0)^5}\rho_1\frac{\partial\rho_1}{\partial\bar{x}}-\frac{B}{m}\frac{\partial^3\rho_1}{\partial\bar{x}^3}=0\ . \tag{10b}$$

Using the boundary conditions from Eqs. (8) that both ρ_1 and u_1 go to zero as $\bar{x}$ tends to infinity we get

$$\begin{aligned} &u_1=-C_3\frac{\rho_1}{\rho_0},\quad u_1=-\frac{3A}{mC_3}\frac{\rho_1}{(a+\rho_0)^4}\ ,\\ &C_3^2=\frac{3A\rho_0}{m(a+\rho_0)^4}\ . \end{aligned} \tag{11}$$

Thus $C_3=\pm C_0$ $(C_0>0)$, C_0 being the velocity of an ordinary dispersionless third-sound mode. Multiplying (10a) by C_3 and (10b) by ρ_0 and subtracting and then using (11) and (9), we get the following equation for ρ_1:

$$\frac{\partial}{\partial\bar{x}}\left[\frac{\partial\rho_1}{\partial\bar{t}}+\frac{C_3(\rho_0-3a)}{2\rho_0(a+\rho_0)}\rho_1\frac{\partial\rho_1}{\partial\bar{x}}+\frac{\hbar^2+4mB\rho_0}{8m^2C_3}\frac{\partial^3\rho_1}{\partial\bar{x}^3}\right]-\frac{C_3}{2}\frac{\partial^2\rho_1}{\partial\bar{y}^2}=0\ . \tag{12}$$

Equation (12) is a variation of the two-dimensional generalization of the Korteweg–de Vries[8] (KdV) equation, first introduced by Kadomtsev and Petviashvili[9] who studied the stability of one-dimensional solitons with respect to transverse perturbations. From their work it is clear that the solitons satisfying (12) are unstable with respect to transverse perturbations. The solitons in this case therefore cannot be viewed as the asymptotic $(t\rightarrow\infty)$ states towards which the solution evolves, as they are in the one-dimensional problem. It is however interesting to note, as pointed out by Ablowitz and Segur,[10] that the two-dimensional KdV equation, in this case, admits of "lump" solutions which share many of the important properties of solitons:

(i) Each is a permanent wave whose speed, relative to the linearized speed C_3 can be made proportional to its amplitude.

(ii) Solitons are localized waves with exponential tails in one dimension; lumps are localized waves with algebraic tails in two dimensions.

(iii) Two solitons regain their original amplitudes and speed after a collision, the final effect of a collison is a phase shift of each soliton. Two lumps regain their original amplitudes and speed after a collision and suffer no phase shift.

For definiteness, let us assume that $\rho_0<3a\simeq3.6$ atomic layers. We then make the following substitutions in Eq. (12)

$$\begin{aligned} &\rho_1=-\gamma\bar{\rho}_1,\quad \gamma=\frac{2\rho_0(a+\rho_0)}{3a-\rho_0}\ ,\\ &\frac{1}{k_0^2}=\frac{\hbar^2+4mB\rho_0}{8m^2C_3^2},\quad \xi=k_0\bar{x},\quad \eta=\sqrt{2}k_0\bar{y}\ ,\\ &\tau=C_3k_0\bar{t}\ . \end{aligned} \tag{13}$$

Then the equation transforms to

$$\frac{\partial}{\partial\xi}\left[\frac{\partial\bar{\rho}_1}{\partial\tau}+\bar{\rho}_1\frac{\partial\bar{\rho}_1}{\partial\xi}+\frac{\partial^3\bar{\rho}_1}{\partial\xi^3}\right]-\frac{\partial^2\bar{\rho}_1}{\partial\eta^2}=0\ . \tag{14}$$

As discussed by Ablowitz and Segur[10] this equation has multiple-lump solutions. The one-lump solution can be written as

$$\bar{\rho}_1=12\frac{\partial^2}{\partial\xi^2}\ln\left[(\xi'+p\eta')^2+q^2\eta'^2+\frac{3}{q^2}\right]\ , \tag{15}$$

where

$$\xi'=\xi-(p^2+q^2)\tau,\quad \eta'=\eta+2p\tau\ . \tag{16}$$

Thus we have

$$\bar{\rho}_1=\frac{24[-(\xi'+p\eta')^2+q^2\eta'^2+3/q^2]}{[(\xi'+p\eta')^2+q^2\eta'^2+3/q^2]^2}\ , \tag{17}$$

where p and q are dimensionless parameters depending on initial perturbation. Thus we have a permanent lump solution decaying as $(1/x^2,1/y^2)$ for $|x|,|y|\rightarrow\infty$ and moving with velocity $v_x=(p^2+q^2)C_3$, and $v_y=-2pC_3/\sqrt{2}$. For $p=0$, the speed is $v_x=q^2C_3$ which is proportional to the amplitude of the lump, i.e.,

$$v_x=C_3\frac{|\rho_0-3a|}{16\rho_0(a+\rho_0)}|\rho_1|_{\text{ampl}}\ . \tag{18}$$

The above discussion also applies for $\rho_0>3a$. For $\rho_0\approx3a$ the problem is essentially linear. It is thus expected that in the two-dimensional superfluid helium at low temperatures (below 0.4 K) one should be able to detect experimentally two-dimensional lumps with the properties described above.[11]

For thicker films the situation can be quite different. As remarked by Huberman,[4] in a thick film crossover to threedimensional behavior may take place. In that case an equation for ρ_1 can be written similar to Eq. (12) with the difference that the dispersion introduced by finite thickness can overcome the surface term $(C_3/k_0^2)\partial^3\rho_1/\partial\bar{x}^3$ and may become negative. Then the equation has one-dimensional soliton solutions which are stable with respect to transverse perturbations and hence can represent asymptotic states.

Thus we conclude that in monolayer superfluid ^{4}He films the two-dimensional localized waves (lumps) that decay algebraically in all horizontal directions and interact like solitons should be detectable at very low temperatures. For thick films, however, one-dimensional solitons are expected to be the corresponding dominant nonlinear modes of excitation at very low temperature.

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