Crossover from first-order and near-spinodal first-order to continuous transitions in the three- and four-state Potts model

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A calculation in renormalized perturbation theory, to one-loop order, in $d=6-\epsilon$ dimensions, shows that quadratic-symmetry breaking (QSB) in a continuum-field p-state Potts model yields a further distinction between the usual first-order and a "near-spinodal first-order" transition with a metastable state: Whereas the latter can cross over to a continuous transition at a tricritical point with arbitrarily small but finite QSB, the former one requires a QSB *above a threshold value*. Specific results for the three- and four-state Potts model suggest that this could be used to distinguish experimentally between the two first-order transitions.

In recent work on the p-state Potts model,[1] Pytte found the very interesting result that, depending on the value of p, there appear to be two distinct types of first-order transitions.[2] One is the usual first-order (FO) transition that is thought to be associated with a renormalization-group fixed-point runaway that takes place when $p>\frac{10}{3}$. The other, which appears for $2<p<\frac{10}{3}$, in spite of an accessible and stable fixed point, occurs near a spinodal point of a metastable state in which the small-order parameter ($Q=0$) is not an absolute minimum. We refer in the following to this second type of transition as a "near-spinodal first-order" (NS-FO) transition, to distinguish it from the usual one. Both transitions are supposed to be characterized by a finite discontinuity of the order parameter at the transition. Technically, however, since there is an available fixed point for the NS-FO transition, a low-order renormalization-group (RG) perturbation expansion can be carried out which, at least for $p\gtrsim 2$, also turns out to be an expansion in powers of the fluctuating field $Q(\vec{x})$.[2] This is similar to the familiar expansion for a second-order transition,[3] and consequently, for small positive $(p-2)$, the NS-FO looks like an almost continuous first-order transition.

More recently we showed that, even for a given p, the Potts model can have both a NS-FO and a FO transition. This is the case for all $p>\frac{10}{3}$ due to the presence of a new accessible and stable, nonsymmetric fixed point for $2.2\lesssim p<\infty$.[4]

Since a metastable state is not directly accessible experimentally, at least in static phenomena, one may first ask the question: How does one distinguish practically between the equilibrium properties of NS-FO and FO transitions? The extent to which these are really two different transitions remains to be investigated and one of the purposes of this paper is to motivate this search. This can be of considerable interest in view of the important physical realizations especially of the three- and four-state Potts model. These are known to describe various magnetic and structural phase transitions.[5]

The purpose of this paper is to suggest that experimentally accessible anisotropies can be used to distinguish between NS-FO and FO transitions. We can show that quadratic-symmetry breaking (QSB) in a continuum-field version of the p-state Potts model that can be traced back to an anisotropic coupling between the components of the Potts vectors in the discrete model, leads to a clear theoretical distinction of a NS-FO and a usual FO transition. Our demonstration consists of two parts. One is a mean-field argument on the free energy, used before by Blankschtein and Aharony[6(a)] for the three-state Potts model, that helps to search for a crossover between a first- and a second-order transition at a tricritical point. In the second part we resort to a RG calculation that accounts for fluctuations in order to demonstrate that the crossover from the NS-FO transition takes place for arbitrarily small QSB, while the usual FO transition requires a finite QSB *above a threshold value*. This is the mechanism that enables one to distinguish between the two transitions. Technically, the latter is necessary in order to suppress the runaway and restore a stable and accessible fixed point.[7]

Our calculations are based on the effective Hamiltonian[4,8,9]

$$\mathscr{H}=\int_k \left[\tfrac{1}{2}[k^2A^2+tA^2+\tilde{m}^2A^2(2)]+\frac{1}{3!}\kappa^{\epsilon/2}u_0\sum_{\mu,\nu,\eta}D_{\mu\nu\eta}A_\mu A_\nu A_\eta \right.$$
$$\left. +\tfrac{1}{2}\kappa^{\epsilon/2}v_0\sum_{\mu,q,r}D_{\mu qr}A_\mu A_q A_r+\frac{1}{3!}\kappa^{\epsilon/2}w_0\sum_{q,r,s}D_{qrs}A_qA_rA_s+\text{quartic terms}\right], \tag{1}$$

which is a generalization of that by Priest and Lubensky,[10] where the integrations are over all momenta $\vec{k}$ in $d=6-\epsilon$ dimensions. The $p-1$ fields $A_\alpha(k)$, $\alpha=1,\ldots,p-1$ are split into m "longitudinal" and $(n-m)$ "transverse" components, with $n=p-1$ and $(\mu,\nu,\eta)<m$, while (q,r,s) indicate transverse components and $A^2=A^2(1)+A^2(2)$,

$$A^2(1)=\sum_{\mu=1}^{m} A_\mu^2(k)\ ,\quad A^2(2)=\sum_{q=1}^{n-m} A_q^2(k)\ , \tag{2}$$

and the tensorial coefficients are[4]

$$D_{\alpha\beta\gamma}=[(p-\alpha)(p-\alpha+1)]^{-1/2}\times\begin{cases}-1 \ \text{ if } \alpha<\beta=\gamma\\ (p-\alpha-1) \text{ if } \alpha=\beta=\gamma\\ 0 \ \text{ otherwise}\ .\end{cases} \tag{3}$$

The arbitrary momentum-scale parameter κ is introduced to have dimensionless trilinear bare couplings u_0, v_0, and w_0, which are not assumed to be the same. The reduced temperature is proportional to $t=T-T_c(\widetilde{m}^2)$, and the square of the noncritical mass, $\widetilde{m}^2$, is a measure of QSB that is related, to lowest order, to the strength g of the more conventional symmetry-breaking term,[11]

$$\mathscr{H}_g=-\tfrac{1}{2}g\int_k\left[(n-m)\sum_{\mu=1}^{m}A_\mu^2-m\sum_{q=1}^{n-m}A_q^2\right]\ , \tag{4}$$

by $\widetilde{m}^2=g+O(u_0,v_0,w_0)$. For simplicity, we assume a single longitudinal component $m=1$, suitable for uniaxial ordering when $g>0$. In the presence of a magnetic field $h_1(k)$ in the direction of $A_1(k)$ there is an additional term $-A_1(k)h_1(k)$ in Eq. (1).

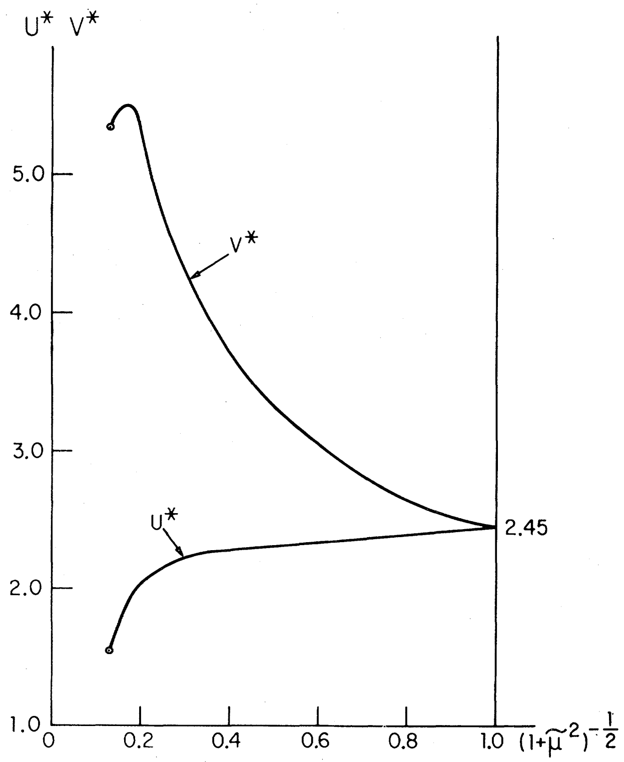

FIG. 1. Dependence of the dimensionless renormalized fixed-point couplings u^* and v^*, in units of $\epsilon^{1/2}$, for the three-state Potts model with the dimensionless mass parameter $\widetilde{\mu}=\widetilde{m}/\kappa$, as a measure of quadratic-symmetry breaking.

The mean-field free energy that follows from Eq. (1) leaving aside fluctuation terms can be written as

$$F(p=3)=\tfrac{1}{2}rA^2-\tfrac{1}{2}g(A_1^2-A_2^2)+u\left[A_1^3-3A_1A_2^2+3\left[1-\frac{v}{u}\right]A_1A_2^2\right]+\text{quartic terms}-A_1h_1 \tag{5}$$

for the three-state model, where κ is absorbed in the trilinear couplings, and $r=T-T_c(0)$. Except for the term in $(1-v/u)$, Eq. (5) is essentially the same as in Ref. 6(a). For $g=h_1=0$ and $(1-v/u)<0$, because of the trilinear term, Eq. (5) favors a first-order transition with $A_1\neq0$.[4] As long as $(1-v/u)$ remains nonpositive, as confirmed by the RG (Fig. 1) there is a competition between the linear term which favors $A_2=0$ and the trilinear term that prefers $A_2\neq0$, whenever $h_1>0$ for $g=0$ and one should expect a crossover between a first- and a second-order transition at a tricritical point that will be enhanced by $g>0$. The result is qualitatively similar to Fig. 1(d) of Ref. 6(a).

The mean-field free energy of the four-state Potts model becomes

$$F(p=4)=\tfrac{1}{2}rA^2-\tfrac{1}{2}g[2A_1^2-(A_2^2+A_3^2)]+u\left[A_1^3-3\frac{v}{u}A_1(A_2^2+A_3^2)+\frac{w}{u}A_2(A_2^2-A_3^2)\right]+\text{quartic terms}-A_1h_1\ . \tag{6}$$

There are now four distinct phases: a disordered phase (I) in which $A_1\neq0$, $A_2=0=A_3$, with $A_1\rightarrow0$ as $h_1\rightarrow0$, and three ordered phases, (II) $A_1\neq0$, $A_2=0=A_3$, but $A_1\not\rightarrow0$ as $h_1\rightarrow0$, (III) $A_1\neq0$, $A_2\neq0$, $A_3=0$, and (IV) $A_1\neq0$, $A_2\neq0$, $A_3\neq0$.[12] If $A_2\sim A_3$, which is reasonable to assume since the ordering is uniaxial along A_1, then there is a competition between the trilinear term that favors $A_2\sim A_3\neq0$ and the linear term for $h_1>0$ that prefers $A_2=A_3=0$. One should then expect a crossover to a second-order transition at a tricritical point on the surface I—IV of first-order transitions, as shown on Fig. 2.

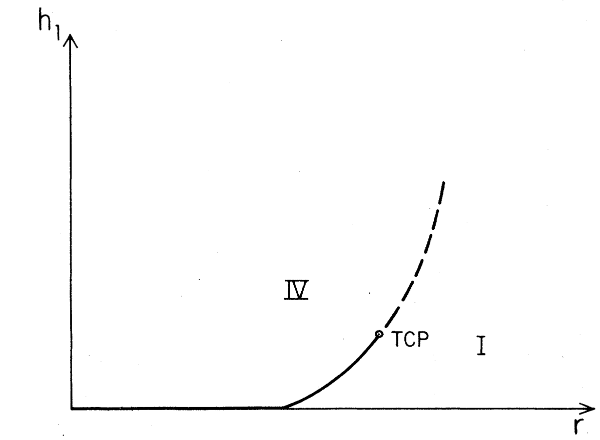

FIG. 2. Schematic part of the phase diagram for the four-state Potts model, illustrating the crossover from the surface I—IV of first-order transitions (solid line) to the critical line (dashed line) at the tricritical point (TCP), for $g=0$.

The RG calculation described below yields three sets of nontrivial fixed-point lines for the scaled couplings:

$$\hat{u}^*(\tilde{\mu})=u^*/(1+\tilde{\mu}^2)^{1/2}\ ,$$
$$\hat{v}^*(\tilde{\mu})=v^*/(1+\tilde{\mu}^2)^{1/2}\ , \qquad (7)$$
$$\hat{w}^*(\tilde{\mu})=w^*/(1+\tilde{\mu}^2)^{1/2}\ ,$$

as functions of the dimensionless noncritical mass

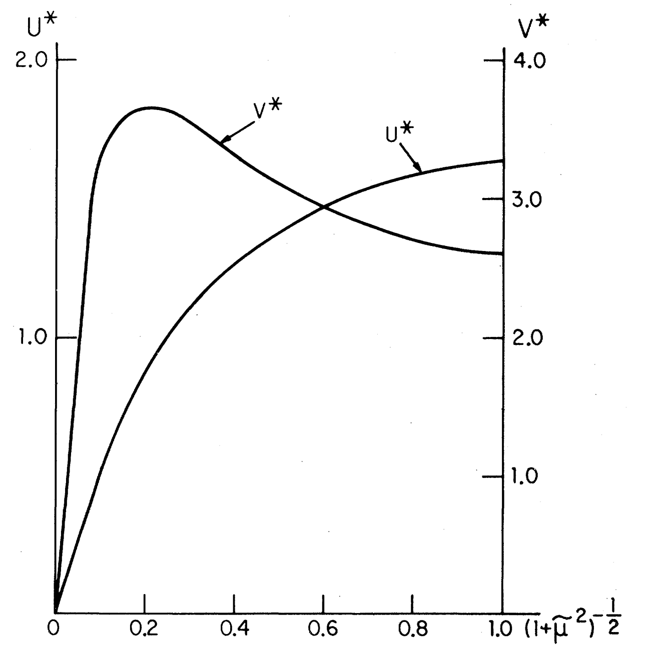

FIG. 3. Dependence with quadratic-symmetry breaking of the dimensionless renormalized fixed-point couplings u^* and v^*, in units of $\epsilon^{1/2}$, for the near-spinodal first-order transition in the four-state Potts model.

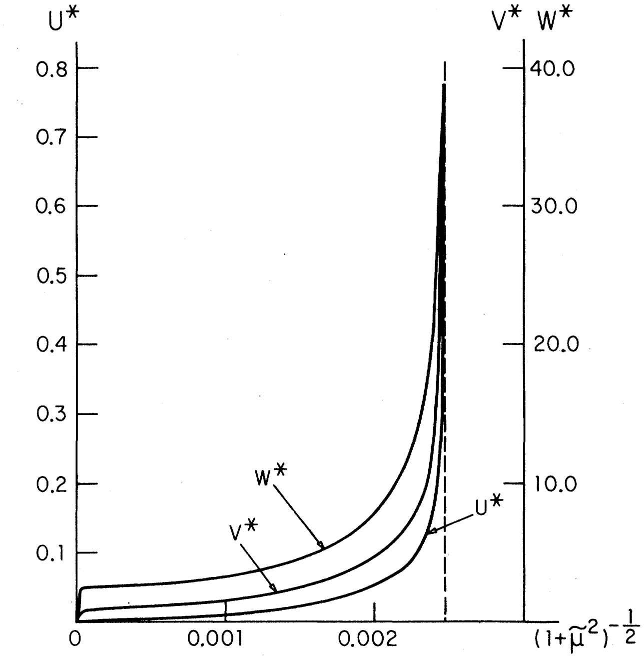

FIG. 4. Dependence with quadratic-symmetry breaking beyond threshold (dashed line) of the fixed-point couplings u^*, v^*, and w^*, in units of $\epsilon^{1/2}$, for the usual first-order transition in the four-state Potts model.

$\tilde{\mu}=\tilde{m}/\kappa$ which plays the role of a crossover parameter. These are given by (i) $u^*\neq 0$, $v^*\neq 0$, and $w^*=0$, shown in Fig. 3 for small but arbitrary, *including zero,* QSB, which is consistent with the mean-field prediction and it corresponds to a NS-FO transition. Next, (ii) $u^*\neq 0$, $v^*\neq 0$, and $w^*\geq 0$ (the equality when $\tilde{\mu}\simeq\infty$) for QSB *larger* than a threshold value shown in Fig. 4; otherwise there is a fixed-point runaway that yields the usual FO transition in the four-state model. Finally, (iii) there is a fixed-point line given by

$$w^{*2}/(1+\tilde{\mu}^2)=2(p-1)\epsilon/(13-3p)=6\epsilon \quad \text{for } p=4 \qquad (8)$$

when $\tilde{\mu}$ is arbitrary but finite, and $u^*=0=v^*$. The mass $\tilde{\mu}$ is related to g through $\tilde{\mu}^2\simeq g/\kappa^2$, for small and finite κ. The asymptotic crossover behavior follows from $\kappa\to 0$,[3] for small but finite g, whereas a threshold in $\tilde{\mu}$ implies one in g. We interpret the crossover to $\tilde{\mu}\simeq\infty$ in the first two fixed points as the crossover to the critical lines. Although

$$\hat{u}^*(\infty)=\hat{v}^*(\infty)=\hat{w}^*(\infty)=0$$

in both, u^* and v^* remain finite and this is responsible for *non*classical exponents on the critical lines, as shown below.

Next, we turn briefly to the RG calculation.[4] The zeros of the Wilson β functions,

$$\beta_u = -\frac{\epsilon}{2}u - \tfrac{3}{4}(p-2)^2c^2u^3 + a(\tilde{\mu}^2)[\tfrac{1}{4}(p-2)uv^2 + v^3] , \tag{9}$$

$$\beta_v = -\frac{\epsilon}{2}v + \tfrac{1}{12}(p-2)^2c^2vu^2 - a(\tilde{\mu}^2)\{\tfrac{1}{3}[2-\tfrac{1}{4}(p-2)]v^3 + \tfrac{5}{6}p(p-3)vw^2 - (p-2)uv^2\} , \tag{10}$$

$$\beta_w = -\frac{\epsilon}{2}w - a(\tilde{\mu}^2)[\tfrac{5}{2}wv^2 - \tfrac{1}{4}(13-3p)pw^3] \tag{11}$$

[here $a(\tilde{\mu}^2) = c^2(1+\tilde{\mu}^2)^{-1}$], yield the fixed points discussed above for $p=3$ and 4. We also find (i) that there are fixed-point lines that start at $\tilde{\mu}=0$ and end at *finite* $\tilde{\mu}$, as shown here for $p=3$, whenever $2.62 < p < 3.27$, and this means that in this case a vanishingly small g is necessary to see the crossover at all; (ii) there are lines of stable and accessible fixed points (generalizing FP1 in I) of the kind shown in Fig. 4 that require a threshold g for $\frac{10}{3} \le p < \frac{13}{3}$; (iii) QSB *always* implies a break in trilinear symmetry.

The exponents η and ν on the critical line—related by $5\eta = \nu^{-1} - 2$ (Ref. 4)— turn out to be given by

$$\eta \simeq \tfrac{137}{36}\epsilon \quad \text{for } p=3$$
$$\eta = \epsilon \ \text{ and } \ \eta \simeq \tfrac{39}{9}\epsilon \quad \text{for } p=4 . \tag{12}$$

The first one differs from the usual $\eta = \epsilon$,[2,10] but that is without QSB on the spinodal point, and there is no reason *a priori* for the exponent on the critical line to be the same.

Although the value of the exponents may not serve for comparison with experiments in $d=3$, the fact that there are two for $p=4$ is a consequence of trilinear-symmetry breaking due to a break in quadratic symmetry and this is a definite qualitative feature of the continuum Potts model in $d=6=\epsilon$ dimensions that may apply to $d=3$, and which cannot be accounted for by the simpler RG calculation in $d=4-\epsilon'$. Since an alignment of a quadratic anisotropy (induced by an anisotropic stress, say) along an easy axis is expected to yield a tricritical point,[6] it should be possible to test whether the crossover in the three-state model takes place with small but finite g. This would be consistent with the RG calculation in $d=6-\epsilon$ dimensions in that the model has a NS-FO and not a FO transition. It would be interesting to see in physical realizations of the four-state model if there is only a crossover from a FO transition, with a threshold g, or if there is *also* a crossover with arbitrarily small g that would imply a NS-FO transition presumably at a lower temperature than the FO transition.

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