Wave propagation and localization in a long-range correlated random potential

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We examine the effect of long-range spatially correlated disorder on the Anderson localization transition in $d=2+\epsilon$ dimensions. This is described as a phase transition in an appropriate nonlinear σ model. We consider a model of scalar waves in a medium with an inhomogeneous index of refraction characterized by scattering strength γ^2 and spatial correlations of range a decaying (i) exponentially $\gamma_1^2 a^{-d} e^{-x/a}$ and (ii) by power laws $\gamma_2^2(a^2+x^2)^{-m}$ $(m>0)$. A replica-field-theory representation is utilized in the calculation of the one- and two-particle Green's functions. In addition to the usual diffusive Goldstone mode of the field theory arising from energy conservation, the nonlinear σ model is shown to possess a discrete spectrum of low-lying nondiffusive modes associated with approximate wave-vector $(\vec{k})$ conservation in the geometric optics limit $ka \gg 1$. For waves it is shown that all states are localized for $d \leq 2$ with diverging localization lengths in the low-frequency limit and that the mobility edge in $d=2+\epsilon$ separating high-frequency, localized states from low-frequency, extended states is characterized by the same critical exponents as for spatially uncorrelated disorder provided $m > \epsilon$. The problem of electron localization in a long-range correlated random potential is also described within the same universality class.

I. INTRODUCTION

The phenomenon of Anderson localization[1] is a fundamental property of waves in a disordered medium. The nature of this transition from extended to localized states has been studied extensively for elementary excitations in systems in which the disorder is spatially uncorrelated. In this paper, we consider the transport properties of waves in a medium with long-range correlated disorder. For exponentially decaying correlations, we examine the nature of normal modes of the disordered system as the range of correlations is made long compared to the wavelength of the excitation. The infinite-range case of power-law-decaying correlations is also discussed in detail. This latter case may be of importance for phonons in an elastic medium possessing vacancies, dislocations, or other topological defects in the crystalline structure.[2] The long-range strain field associated with a quenched distribution of such defects gives rise to a scattering potential for phonons with power-law-decaying corelations.[3] Another example is that of electronic conduction in solids possessing a quenched distribution of defects with power-law-decaying impurity potentials. Also, in a polar semiconductor power law, correlations may be realized by means of structural disorder. Here, the local charge imbalances associated with the deviation of atoms from their crystalline position produce long-range correlated random electric fields familiar from the problem of the Urbach optical-absorption edge.[4,5]

We show from first principles using the methods of field theory and renormalization that the localization transition for waves in a long-range correlated random potential is in the same universality class (see *Note added in proof*) as the spatially uncorrelated case. All states are localized in two dimensions and below. The crossover in $d \leq 2$ from localized to extended states as the range of correlations is made longer, and hence the potential smoother, is found to occur only in the limit of a completely flat potential. It is shown that in the presence of arbitrarily weak disorder, increasing the smoothness of the potential results simply in a corresponding increase of the localization lengths. Above two dimensions the mobility edge separating low-frequency extended states from high-frequency localized states is shown to be characterized by critical exponents in accord with the theories of Wegner[6] and Abrahams *et al.*[7] of electronic conduction in a short-range correlated random potential. As the range of correlations is made longer, this mobility edge moves to higher frequencies.

II. THE MODEL

We consider the general problem of wave propagation in a medium with a correlated spatially varying index of refraction. For simplicity we consider a scalar wave equation in the continuum-field-theory limit. For concreteness we choose a model for elastic waves described by a displacement field ϕ, in a medium consisting of atoms with spatially varying random masses $m(x)$ and constant "spring stiffness" V_0,

$$m(x)\frac{\partial^2\phi}{\partial t^2}-V_0\nabla^2\phi=0 . \tag{2.1}$$

The random masses here fluctuate about a mean value m_0,

$$m(x)=m_0+m'(x) , \tag{2.2a}$$

and are characterized by spatial correlations which we write as

$$\langle m'(x)m'(y)\rangle = B(x-y)\ , \tag{2.2b}$$

$$\langle m'(x)\rangle = 0\ . \tag{2.2c}$$

We assert that the critical properties of the present scalar model near the localization transition are identical to those of a general vector field propagating in an isotropic medium with inhomogeneous index of refraction.

As in the case of spatially uncorrelated disorder we probe the nature of normal modes of this system by introducing a spatially localized zero-momentum phonon wave packet and examining its subsequent time evolution as it scatters from random mass fluctuations in its environment. It is convenient to define an average diffusion coefficient in terms of the long-time behavior of the vibrational energy $E(x,t)$,

$$D \equiv \lim_{t\to\infty} \frac{1}{t} \frac{\int d^dx\, x^2 \langle E(x,t)\rangle_{\text{ensemble}}}{\int d^dx\, \langle E(x,t)\rangle_{\text{ensemble}}}\ . \tag{2.3a}$$

Here, the energy density takes the form

$$E(x,t) = \tfrac{1}{2} m(x)\dot{\phi}^2 + \tfrac{1}{2} V_0 |\vec{\nabla}\phi|^2\ . \tag{2.3b}$$

The angular brackets denote an average over all possible realizations of the random-mass field compatible with conditions (2.2). By standard methods[8] the diffusivity can be expressed in terms of a product of the retarded and advanced phonon Green's functions in the weak scattering limit,

$$B(x-y)/m_0^2 \ll 1\ , \tag{2.4a}$$

$$D \simeq \lim_{\eta\to 0} \left[\eta \frac{\int_{-\infty}^{\infty} d\omega\, \omega^2 \int d^dx\, x^2 \langle G(x,0;\omega_+)G(x,0;\omega_-)\rangle_{\text{ensemble}}}{\int_{-\infty}^{\infty} d\omega\, \omega^2 \int d^dx \langle G(x,0;\omega_+)G(x,0;\omega_-)\rangle_{\text{ensemble}}} \right]\ . \tag{2.4b}$$

Here, $\omega_\pm \equiv \omega \pm i\eta$ denotes the addition of an infinitesimal imaginary part to the phonon frequency in the retarded and advanced Green's functions, respectively. This latter form suggests a spectral decomposition of the diffusivity into contributions $D(\omega)$ from normal modes of a given frequency ω,

$$D \equiv \frac{\int_{-\infty}^{\infty} d\omega\, D(\omega)E(\omega)}{\int_{-\infty}^{\infty} d\omega\, E(\omega)}\ . \tag{2.5}$$

Here, $E(\omega)$ is the spectral density of energy excited in the medium by the initially injected phonon wave packet. The behavior of $D(\omega)$ on sufficiently long length scales, in the renormalization-group sense, determines whether normal modes of a given frequency ω are extended or localized. If states of frequency ω are localized, then $D(\omega)$ will renormalize to zero, whereas extended states of frequency ω gives rise to a finite value of the diffusivity at that frequency.

III. FIELD-THEORY FORMULATION

In order to calculate the diffusivity $D(\omega)$ we utilize the replica-field representation[8-10] of the two-particle phonon Green's function,

$$G(x,y;\omega_+)G(x,y;\omega_-) = \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int D\phi\, \phi^{1+}(x)\phi^{1+}(y)\phi^{1-}(x)\phi^{1-}(y) \times e^{-(L^+ + L^-)}\ , \tag{3.1a}$$

where

$$L^\pm = \tfrac{1}{2}\int d^dx\, \phi^\alpha[\omega_\pm^2 m(x) + V_0\nabla^2]\phi^\alpha,\quad \alpha = 1,\dots,n_\pm \tag{3.1b}$$

Here we have introduced two sets of replica indices denoted by superscripts $\pm$ for the retarded and advanced Green's functions, respectively. Accordingly, the contours of integration are chosen for the $\pm$ replicas as shown in Fig. 1 so that the infinitesimal imaginary parts $\pm i\eta$ ensure convergence of the functional integrals. The symbol $D\phi$ denotes functional integration over all of the replica fields.

The conditions (2.2) may be implemented by the probability distribution for the fluctuating part of the mass density,

$$P[m'(x)] = \text{const}\exp\left[-\tfrac{1}{2}\int d^dx\, d^dy\, m'(x)B_{\text{op}}^{-1}(x-y)m'(y)\right]\ , \tag{3.2a}$$

where B_{op}^{-1} refers to an operator inverse of $B(x-y)$ with respect to its two coordinate-space arguments,

$$\int d^dz\, B(x-z)B_{\text{op}}^{-1}(z-y) = \delta^d(x-y)\ . \tag{3.2b}$$

Averaging over all possible realizations of the random-mass field we obtain

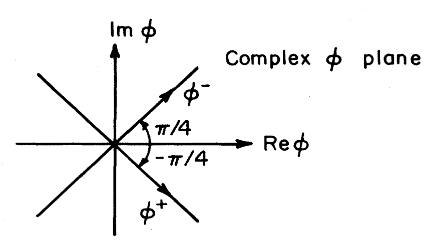

FIG. 1. Contour of integration for the retarded (+) and advanced (−) replica fields.

$$\langle |G(x,y;\omega_+)|^2\rangle_{\text{ensemble}} = \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int D\phi\, \phi^{1+}(x)\phi^{1+}(y)\phi^{1-}(x)\phi^{1-}(y) \times e^{-(L_0^+ + L_0^- + L_{\text{int}})}, \tag{3.3a}$$

where $L_0^{\pm}$ are obtained from $L^{\pm}$ by replacing $m(x)$ by its mean value m_0 and

$$L_{\text{int}} = \frac{-\omega^4}{8}\int d^dx\, d^dy\, \phi^\alpha(x)\phi^\alpha(x)B(x-y)\phi^\beta(y)\phi^\beta(y)\,. \tag{3.3b}$$

It is convenient to reduce this quartic replica-field coupling, generated by averaging, through the introduction of a set of intermediate bilocal fields

$$Q^{\alpha\beta}(x,y),\quad \alpha,\beta=1,\ldots,(n_+ + n_-)\,.$$

The interacting part of the field theory (3.3b) can then be expressed as a functional integral over this set of fields as follows:

$$e^{-L_{\text{int}}} = \text{const}\int DQ \exp\left[-\tfrac{1}{2}\int d^dx\, d^dy[B^{-1}(x-y)Q^{\alpha\beta}(x,y)Q^{\alpha\beta}(x,y) + \omega^2 Q^{\alpha\beta}(x,y)\phi^\alpha(x)\phi^\beta(y)]\right]. \tag{3.4a}$$

Here, and in subsequent discussions, B^{-1} refers to the reciprocal of the correlation function,

$$B^{-1}(x-y) \equiv \frac{1}{B(x-y)}\,. \tag{3.4b}$$

In terms of the intermediate fields, the generating functional for the averaged Green's functions may be expressed as a quadratic form on the ϕ fields,

$$Z(J) \equiv \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int D\phi\, dQ \exp\left[-\tfrac{1}{2}\langle\phi, A(Q)\phi\rangle - \tfrac{1}{2}\int d^dx\, d^dy[B^{-1}(x-y)Q^{\alpha\beta}Q^{\alpha\beta} + 2J^{\alpha\beta}(x,y)Q^{\alpha\beta}(x,y)]\right], \tag{3.5a}$$

where

$$\langle\phi, A(Q)\phi\rangle = \int d^dx\, d^dy\, \phi^\alpha(x)A^{\alpha\beta}(Q)\phi^\beta(y) \tag{3.5b}$$

and

$$\underline{A}(Q) \equiv \begin{bmatrix} (m_0\omega_+^2 + V_0\nabla^2)\delta^d(x-y)\underline{I} & \underline{0} \\ \underline{0} & (m_0\omega_-^2 + V_0\nabla^2)\delta^d(x-y)\underline{I} \end{bmatrix} + \omega^2 \begin{bmatrix} Q^{++}(xy) & Q^{+-}(xy) \\ Q^{-+}(xy) & Q^{--}(xy) \end{bmatrix}. \tag{3.5c}$$

Here $\underline{I}$ denotes the $n_\pm \times n_\pm$ identity matrix in replica space. Differentiation with respect to the source $J^{\alpha\beta}(x,y)$ before and after integration over the ϕ fields yields the following relations between the averaged Green's functions and the Q-correlation functions:

$$\langle Q^{\pm\pm}(x,y)\rangle = -\tfrac{1}{2}\omega^2 B(x-y)\langle G(x,y;\omega_\pm)\rangle_{\text{ensemble}}\,, \tag{3.6a}$$

$$\langle Q^{+-}(x_1,y_1)Q^{+-}(x_2,y_2)\rangle = \tfrac{1}{4}\omega^4 B(x_1-y_1)B(x_2-y_2)\langle G(x_1,x_2;\omega_+)G(y_1,y_2;\omega_-)\rangle_{\text{ensemble}}\,. \tag{3.6b}$$

These are quantitites relevant to the calculation of the average density of states and energy diffusivity, respectively.

IV. SPONTANEOUS SYMMETRY BREAKING AND THE AVERAGED ONE-PARTICLE GREEN'S FUNCTION

Straightforward integration over the replica ϕ fields in (3.5) yields a field theory in the bilocal fields Q of the form

$$Z(0) = \lim_{\substack{n_+\to 0\\ n_-\to 0}} \int DQ\, e^{-L[Q]}\,, \tag{4.1a}$$

where

$$L[Q] \equiv \tfrac{1}{2}\ln\det\underline{A}(Q) + \tfrac{1}{2}\int d^dx\, d^dy\, Q^{\alpha\beta}(x,y)Q^{\alpha\beta}(x,y)B^{-1}(x-y)\,. \tag{4.1b}$$

Evaluation of the expectation value $\langle Q\rangle$ with respect to the above action in a saddle-point approximation yields the generalization to the case of spatially correlated disorder of the coherent-potential approximation[11] (CPA) for the one-particle Green's function. Expanding the Lagrangian $L[Q]$ about a point Q_0 and requiring that there be no linear terms in the small fluctuation $\hat{Q} = Q - Q_0$ gives the condition that Q_0 be the required saddle point of the field theory. A straightforward calculation[12] yields

$$Q_0^{++}(x,y) = -\tfrac{1}{2}\omega^2 B(x-y)G_0(x,y;\omega_+)\,, \tag{4.2a}$$

$$Q_0^{--}(x,y) = -\tfrac{1}{2}\omega^2 B(x-y)G_0(x,y;\omega_-)\,, \tag{4.2b}$$

$$Q_0^{+-} = Q_0^{-+} = 0\,, \tag{4.2c}$$

where

$$G_0(x,y;\omega_\pm) \equiv \{[(m_0 + Q_0^{\pm\pm})\omega_\pm^2 + V_0\nabla^2]^{-1}\}_{xy}\,. \tag{4.2d}$$

Condition (4.2c) corresponds to our choice of a replica diagonal saddle point. Since averaging restores translational invariance, the CPA Green's function G_0 depends only on the difference of its coordinate space arguments. Defining the Fourier transform

$$Q^{\pm}(k) \equiv \int d^d(x-y) e^{ik(x-y)} Q_0^{\pm\pm}(x-y) , \quad (4.3)$$

condition (4.2d) on the one-particle CPA Green's function may be rewritten as

$$G_0^+(k) = \frac{1}{[m_0 + Q^+(k)]\omega_+^2 - V_0 k^2} , \quad (4.4a)$$

$$Q^+(k) = -\tfrac{1}{2}\omega^2 \int d^d q \, \widetilde{B}(k-q) G_0^+(q) . \quad (4.4b)$$

In the last equation we have introduced the Fourier transform of the correlation function,

$$\widetilde{B}(k) \equiv \int d^d x \, e^{ik\cdot x} B(x) . \quad (4.4c)$$

As noted by Schäfer and Wegner,[10] the Lagrangian $L[Q]$ possesses an internal global symmetry among the replicas. In the limit as $\eta \to 0$, the symmetry between the retarded and advanced replica fields may be expressed as

$$\underline{Q} \to \underline{U}\underline{Q}\underline{U}^T . \quad (4.5a)$$

This leaves $L[Q]$ invariant for the set of all "pseudo-orthogonal" matrices

$$\underline{U} = \begin{bmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{bmatrix} . \quad (4.5b)$$

The saddle-point solution (4.2) corresponds to a state of the field Q in which this symmetry is spontaneously broken. This broken symmetry is manifested in the imaginary part of Q_0 for frequencies at which the density of states and hence $\mathrm{Im}Q^+(k)$ is nonzero,

$$Q_0(k) = \mathrm{Re}Q^+(k) \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{I} \end{bmatrix} + i\,\mathrm{Im}Q^+(k) \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & -\underline{I} \end{bmatrix} . \quad (4.6)$$

As usual,[8-10] by applying the symmetry operation U to this solution we may generate a continuous manifold of distinct saddle points of the Langrangian $L[Q]$.

The CPA equations (4.4) must, in general, be solved self-consistently since the self-energy (4.4b) of the Green's function involves the full propagator itself. We present approximate solutions to these equations for some particular choices of the correlation function $B(x)$ in the weak scattering limit (2.4b). Since the average mean free path for waves propagating in the disordered medium is related to the imaginary part of the self-energy, we concentrate on the evaluation of this quantity,

$$\mathrm{Im}Q^+(k) = -\tfrac{1}{2}\omega^2 \int d^d q \, \widetilde{B}(q-k) \mathrm{Im}G_0^+(q) . \quad (4.7)$$

In the weak scattering limit, $\mathrm{Im}Q^+(k)$ is correspondingly small and so the imaginary part of the propagator (4.4a) takes the form of a δ function,

$$\mathrm{Im}G_0^+(q) \to -\pi\delta(\omega^2 - q^2) \text{ with } m_0 = V_0 = 1 . \quad (4.8)$$

In this limit, the correlation function appearing in (4.7) may be evaluated for $|\vec{q}| = \omega$ and pulled out of the integral,

$$\mathrm{Im}Q^+(k) \simeq \frac{\pi\omega^2}{2\pi m_0} \langle \widetilde{B}(\vec{k} - \omega\hat{q}) \rangle_{\hat{q}} N(\omega^2) . \quad (4.9a)$$

Here we have introduced an angular average of the correlation function over the directions of the unit vector $\hat{q}$,

$$\langle \widetilde{B}(\vec{k} - \omega\hat{q}) \rangle_{\hat{q}} \equiv \frac{\int d\Omega_{\hat{q}} \widetilde{B}(\vec{k} - \omega\hat{q})}{\int d\Omega_{\hat{q}}} , \quad (4.9b)$$

and

$$N(\omega^2) \equiv \frac{-m_0}{\pi} \int d^d q \, \mathrm{Im}G_0^+(q) \quad (4.9c)$$

is the phonon density of states which has an asymptotic behavior $\sim \omega^{d-2}$ in the low-frequency limit.[8]

We examine the dependence of the phonon mean free path

$$\Lambda(\omega) \equiv \left[\frac{\omega^2}{V_0} \mathrm{Im}Q^+(\omega) \right]^{-1/2} \quad (4.10)$$

on the length scale of correlations in d dimensions by means of the normalized, exponentially decaying correlation function

$$B(x) = \frac{\gamma^2}{a^d} e^{-x/a} . \quad (4.11)$$

Here γ^2 and a determine the scattering strength and range of the correlations, respectively. It is shown in Appendix A that in the limit of correlation length a very large compared to the phonon wavelength, the angular average in (4.9a) has the asymptotic behavior

$$\langle B \rangle \equiv [\langle \widetilde{B}(\vec{k} - \omega\hat{q}) \rangle_{\hat{q}}]_{k=\omega} \sim \frac{\gamma^2}{(\omega a)^{d-1}}, \quad \omega a \gg 1 . \quad (4.12a)$$

It follows that the mean free path diverges in the same limit as

$$\Lambda(\omega) \sim \left[\frac{m_0 V_0}{N(\omega^2)} \right]^{1/2} \frac{1}{\gamma\omega^2} (\omega a)^{(d-1)/2} . \quad (4.12b)$$

We also consider the behavior of the self-energy (4.9a) in the limit of very-long-wavelength phonons. It is also shown in Appendix A that for any exponential correlation of the type (4.11) or any power-law-decaying correlation of the form

$$B(x) = \frac{\gamma^2}{(x^2 + a^2)^m}, \quad 2m > d \quad (4.13)$$

the low-frequency ($\omega \to 0$) asymptotic behavior of the self-energy is the same as for uncorrelated disorder,

$$\mathrm{Im}Q^+(\omega) \sim \omega^d, \quad 2m > d . \quad (4.14a)$$

For longer-range correlations ($0 < 2m \le d$), however, this becomes

$$\mathrm{Im}Q^+(\omega) \sim \begin{cases} -\omega^d \ln\omega, & 2m = d \\ \omega^{2m}, & 1 < 2m < d \\ -\omega \ln\omega, & 2m = 1 \\ \omega^{m/(1-m)}, & m < \frac{1}{2} . \end{cases} \quad (4.14b)$$

The stronger scattering in the low-frequency limit apparent in this latter case is associated with the nonintegrability of correlation function (4.13) for $2m \le d$. Although the overall potential is smoother, the long-range nature of the scattering centers gives rise to scattering which is stronger than the Rayleigh type, Eq. (4.14a).

Finally in the limit of an ensemble of completely flat potentials $B(x)=\gamma^2$, the CPA equation (4.4) reduces to a simple quadratic equation with the solution

$$G_0^{\pm}(k)=\frac{1}{\gamma^2\omega^4}\{(m_0\omega^2-V_0k^2) \mp[(V_0k^2-m_0\omega^2)^2-2\gamma^2\omega^4]^{1/2}\} . \quad (4.15)$$

We now proceed to examine the effect of these various spatial correlations on the average energy diffusivity.

V. GOLDSTONE MODES AND THE AVERAGE DIFFUSIVITY

In a field theory with a spontaneously broken symmetry there is an associated Goldstone mode. In the present section we demonstrate the existence of a Goldstone mode associated with the symmetry (4.5) and discuss its relation to the average energy diffusivity.

We consider the expansion of the Lagrangian (4.1b) to second order in the fluctuations about the saddle point Q_0. Retaining only those fluctuations $\hat{Q}^{+-}$ transverse to the direction of spontaneous symmetry breaking which enter expression (2.4b) for the diffusivity [see also (3.6b)], we may write

$$L[\hat{Q}]=\frac{1}{2}\int d^dx_1\, d^dx_2\, d^dx_3\, d^dx_4\, \hat{Q}^{+-}(x_1,x_2) \times C(x_1,x_2,x_3,x_4)\hat{Q}^{-+}(x_3,x_4) , \quad (5.1a)$$

where

$$C(x_1,x_2,x_3,x_4)\equiv B^{-1}(x_1-x_2)\delta^d(x_1-x_3)\delta^d(x_2-x_4) -\frac{1}{2}\omega^4G_0^+(x_1-x_3)G_0^-(x_2-x_4) . \quad (5.1b)$$

By transforming to "center of mass" and relative coordinates,

$$R_1=\frac{x_1+x_2}{2},\quad R_2=\frac{x_3+x_4}{2},\quad s_1=x_1-x_2,\quad s_2=x_3-x_4 , \quad (5.2)$$

and introducing the Fourier transform

$$C(K_1,K_2,k_1,k_2)\equiv \int d^dR_1d^dR_2d^ds_1d^ds_2 \times e^{i(K_1\cdot R_1+K_2\cdot R_2+k_1\cdot s_1+k_2\cdot s_2)} \times C(R_1,R_2,s_1,s_2) \quad (5.3a)$$

$$\equiv C(K_1,k_1,k_2)\delta^d(K_1-K_2) , \quad (5.3b)$$

the Lagrangian for quadratic fluctuations may be expressed as

$$L[\hat{Q}]=\frac{1}{2}\int d^dK\, d^dk_1d^dk_2\hat{Q}^{+-}(Kk_1) \times C(Kk_1k_2)\hat{Q}^{-+}(Kk_2) , \quad (5.4a)$$

where

$$C(Kk_1k_2)\equiv\tilde{B}^{-1}(k_1-k_2)-\frac{1}{2}\omega^4\delta^d(k_1-k_2)G_0^+(k_1-\tfrac{1}{2}K) \times G_0^-(k_1+\tfrac{1}{2}K) \quad (5.4b)$$

and

$$\int d^dk_3\tilde{B}(k_1-k_3)\tilde{B}^{-1}(k_3-k_2)=\delta^d(k_1-k_2) . \quad (5.4c)$$

The diagonal nature of this operator in K expresses simply the translational symmetry of the theory resulting from averaging over the disorder.

A Goldstone mode in the field theory corresponds to a zero eigenvalue of the operator

$$C(0,k_1,k_2)=\delta^d(k_1-k_2)[B^{-1}(i\nabla_{k_1})-\tfrac{1}{2}\omega^4\,|\,G_0^+(k_1)\,|^2] . \quad (5.5a)$$

In order to obtain this differential form of the operator we have replaced the coordinate-space argument of the reciprocal correlation function by $i\nabla_k$,

$$B^{-1}(i\nabla_k)\equiv\frac{1}{B(x\to i\nabla_k)} . \quad (5.5b)$$

It is straightforward to verify, with the use of CPA equation (4.4), that there is in fact one zero eigenvalue in the limit as $\eta\to 0$ for which the associated eigenvector is $\mathrm{Im}Q^+(k_2)$,

$$\int d^dk_2C(0,k_1,k_2)\mathrm{Im}Q^+(k_2)=-\eta m_0\omega^3\,|\,G_0^+(k_1)\,|^2 \to 0 \;\text{ as }\; \eta\to 0 . \quad (5.6)$$

Since the diffusion coefficient is related to the operator C by the expression

$$D=\lim_{\eta\to 0}\left[\frac{\eta\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\int d^dk_1d^dk_2\frac{d^2}{dK^2}C^{-1}(K,k_1,k_2)\,|_{K=0}}{\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\int d^dk_1d^dk_2C^{-1}(0,k_1,k_2)}\right] , \quad (5.7)$$

the existence of the Goldstone mode guarantees that the diffusivity is nonzero.

For the special case of a correlation function

$$B(x)=\frac{\gamma^2}{a^2+x^2} , \quad (5.8a)$$

the operator $C\equiv C(0k_1k_2)$ takes the form of a Schrödinger operator,

$$\gamma^2C=-\nabla_k^2+a^2-\frac{\omega^4\gamma^2}{2}\,|\,G_0^+(k)\,|^2 . \quad (5.8b)$$

We may use the relation

$$|G_0^+(k)|^2 = \frac{-\mathrm{Im}G_0^+(k)}{\omega^2 \mathrm{Im}Q^+(k)} \tag{5.9}$$

to rewrite this operator in the weak scattering limit (4.8) as

$$\gamma^2 C = -\nabla_k^2 + V(k) , \tag{5.10a}$$

where

$$V(k) \equiv a^2 - g\delta(k-\omega) \quad \text{with } m_0 = V_0 = 1 , \tag{5.10b}$$

and g is a frequency-dependent coupling constant. In $d=3$, for instance, it is shown in Appendix B that

$$\frac{1}{g\omega} \equiv e^{-\omega a}\frac{\sinh(\omega a)}{\omega a} . \tag{5.10c}$$

This is a δ-shell potential used sometimes in low-energy nuclear scattering theory.[13] It is shown in Appendix B that there is precisely one $l=0$ bound state for which the corresponding eigenvalue λ_0 of the operator C is zero. This establishes the spherically symmetric Goldstone mode $\mathrm{Im}Q^+(k)$ as the ground state of this operator. Also for sufficiently high frequencies,

$$g\omega > 2l+1 , \tag{5.11}$$

the δ-shell potential admits bound states up to and including angular momentum l. In the geometrical optics limit $\omega a \gg 1$ condition (5.11) is simply $\omega a > \frac{1}{2}(2l+1)$. In this limit the disorder is extremely smooth on the scale of the phonon wavelength and the scattering is correspondingly weak. As a result the higher angular-momentum eigenvalues λ_l of the operator C approach the Goldstone mode $\lambda_0 = 0$,

$$\lambda_l \sim \frac{2l(l+1)}{\gamma^2\omega^2}, \quad \omega a \gg 1 . \tag{5.12}$$

This behavior is illustrative of the general case of correlations $B(x)$. For shorter-range correlations of the form (4.11), the "kinetic energy" of the associated operator C, rather than the usual form $-\nabla_k^2$, will be exponentially large. By analogy with the spectrum of a quantum particle in a box, we expect the separation between bound states to be correspondingly large. In the opposite limit of correlations of longer range than (5.8a), the kinetic energy of C will be less dominant and so the separation between bound states smaller. Nevertheless, in the presence of arbitrarily smooth disorder there is a gap between the Goldstone mode λ_0 and the first excited state λ_1. It is only in the limit of an ensemble of completely flat mass densities $B(x)=\gamma^2$ that (5.5) has no kinetic-energy term,

$$\gamma^2 C(0,k_1,k_2) = \delta^d(k_1-k_2)[1-\tfrac{1}{2}\gamma^2\omega^4 |G_0^+(k_1)|^2] . \tag{5.13a}$$

It follows from CPA equations (4.4) and (4.15), that for $(k^2-\omega^2)^2 < 2\omega^4\gamma^2$,

$$|G_0^+(k)|^2 = \frac{2}{\gamma^2\omega^4} , \tag{5.13b}$$

so that the continuous spectrum of (5.13a) includes an infinite set of Goldstone modes (see Fig. 2). An immediate consequence of this is that the energy diffusivity is infinite, as can be easily verified by direct evaluation of (5.7). Since in any particular realization of the mass density there is no disorder, all phonon modes freely propagate.

For the case of decaying correlations such as (5.8a) we may replace C^{-1} appearing in the diffusivity (5.7) by its projection onto the Goldstone subspace,

$$C^{-1}(K,k_1,k_2) \to \frac{1}{\lambda_0(K)} v_0^*(K,k_1) v_0(K,k_2) . \tag{5.14}$$

Here, $v_0(K,k_2)$ is the eigenvector of (5.4b) which as $K \to 0$ corresponds to the Goldstone mode $\mathrm{Im}Q^+(k_2)$ and the associated eigenvalue may be expanded for small K as

$$\lambda_0(K) = \lambda_0 + K^2\lambda_0''(0) + \cdots , \tag{5.15a}$$

where

$$\lambda_0''(K) \equiv \frac{1}{2}\frac{\partial^2}{\partial K^2}\lambda_0(K) . \tag{5.15b}$$

The diffusivity can then be expressed as

$$D = -\lim_{\eta\to 0}\left[\frac{2\eta\int_{-\infty}^{\infty} d\omega\, \omega^2 N^2(\omega^2)(\lambda_0''(0)/\lambda_0^2)}{\int_{-\infty}^{\infty} d\omega\, \omega^2 N^2(\omega^2)/\lambda_0}\right] , \tag{5.16}$$

where $N(\omega^2)$ is the density of states (4.9c).

With the use of (5.6) we may write

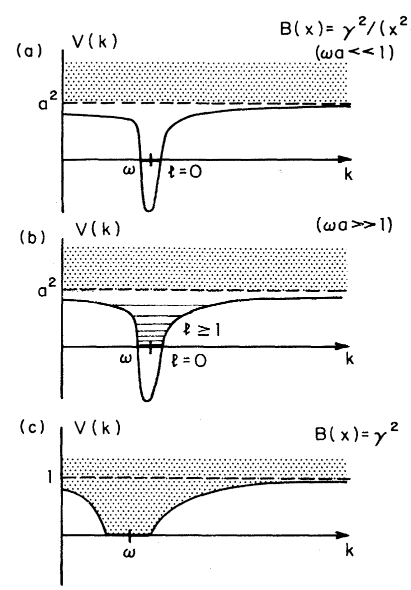

FIG. 2. Spectrum of the operator $C(0k_1k_2)$. For cases (a) and (b) of correlations $B(x)=\gamma^2/(x^2+a^2)$, the Goldstone mode ($l=0$) is separated from higher modes ($l \geq 1$) by a gap. The continuous spectrum begins at a^2. In the long-wavelength limit (a), there is only a single bound state, whereas in the geometrical optics limit (b) the higher angular-momentum bound states approach the Goldstone mode. For case (c), $B(x)=\gamma^2$, the spectrum is that of a particle in a classical potential. There is no gap between the continuous spectrum and the infinite set of Goldstone modes in the range $(k^2-\omega^2)^2 < 2\omega^4\gamma^2$.

$$\lambda_0 = -\eta m_0 \omega^3 \int d^dk \, | G_0^+(k) |^2 \mathrm{Im} Q^+(k) = \frac{-\eta}{\pi} m_0 \omega N(\omega^2) \, . \tag{5.17}$$

The coefficient $\lambda_0''(0)$ may be obtained by applying lowest-order Rayleigh-Schrödinger perturbation theory to the operator $C(Kk_1k_2)$, treating the terms arising from an expansion in small K as a perturbation to the operator $C(0k_1k_2)$. This yields, to leading order in the scattering strength γ^2 of the mass disorder,

$$\lambda_0''(0) = \frac{-\omega^4}{16d} \int d^dk [\mathrm{Im} Q^+(k)]^2 \times \left[\nabla_k^2 \, | G_0^+(k) |^2 - 4 \frac{\partial}{\partial k_i} G_0^+(k) \frac{\partial}{\partial k_i} G_0^-(k) \right] . \tag{5.18}$$

By substitution of (5.17) into (5.16) and comparison with (2.5) it is apparent that $\lambda_0''(0)$ is the conductance $\Sigma(\omega)$ associated with phonons of frequency ω,

$$\Sigma(\omega) \equiv \rho(\omega) D(\omega) \tag{5.19a}$$

$$= \frac{4\pi}{m_0} \lambda_0''(0) \, . \tag{5.19b}$$

Here we have introduced the density of states normalized in the variable ω,

$$\rho(\omega) \equiv 2\omega N(\omega^2) \, . \tag{5.20}$$

VI. SPECTRAL CONTENT OF NORMAL MODES

The eigenfunctions $v_l(0,k_1)$ of the operator $C(0,k_1,k_2)$ have a simple physical interpretation which we now describe. The Goldstone mode $v_0(0,k) = \mathrm{Im} Q^+(k)$ is associated with an exact pseudo-orthogonal symmetry (4.5) of the replica-field Lagrangian. Physically, this symmetry expresses energy conservation for excitations in the disordered medium and leads to the existence of an isotropic diffusion mode in the two-particle Green's function. As the disorder is made smoother and smoother a geometric optics limit is achieved in which there is also approximate momentum conservation over short length scales. The field theory realizes this approximate conservation law by means of an approximate translational symmetry in the interaction Lagrangian (3.3b). Namely, the replica fields $\phi^\alpha(x)$ and $\phi^\beta(y)$ can be independently translated on length scales over which $B(x-y)$ remains relatively flat. In the limit $B(x-y)=\gamma^2$ this symmetry becomes exact, leading to an infinite number of Goldstone modes [Fig. 2(c)]. The low-lying eigenvalues ($l \geq 1$) of the operator $C(0k_1k_2)$ for $\omega a \gg 1$ are an expression of this approximate symmetry and the propagation of plane-wave-like excitations in the disordered medium over short distances.

If we introduce a transmitter of frequency ω into the disordered medium at a location $x=0$, the response far away may be characterized by a coherence function[14]

$$\Gamma(x,x') \equiv \langle \phi(x,t) \phi^*(x',t) \rangle_{\text{ensemble}} \tag{6.1a}$$

$$= \langle G(x,0;\omega_+) G(x',0;\omega_-) \rangle_{\text{ensemble}} \, . \tag{6.1b}$$

This may be related to the present replica-field theory by means of (3.6b),

$$\Gamma(R,s) = \frac{4}{\omega^4} B^{-1}(s) B^{-1}(0) \langle Q^{+-}(R,s) Q^{+-}(0,0) \rangle \, , \tag{6.2a}$$

where

$$R = \tfrac{1}{2}(x+x'), \quad s = x - x' \, . \tag{6.2b}$$

Introducing the Fourier transform as in (5.3),

$$\Gamma(K,q) = \mathrm{const} \frac{1}{\omega^4} \int d^dk \, \tilde{B}^{-1}(q-k) \times \langle Q^{+-}(K,k) Q^{+-}(0,0) \rangle \, , \tag{6.3}$$

and expanding

$$Q^{+-}(K,k) = \sum_l Q_l(K) v_l(0,k) \tag{6.4}$$

in terms of the eigenfunctions of $C(0k_1k_2)$, the coherence for small K (large R) may be expressed as

$$\Gamma(K,q) = \mathrm{const} \frac{1}{\omega^4} \sum_l \frac{v_l \psi_l(q)}{\lambda_l + K^2 \lambda_l'' + \cdots} \, , \tag{6.5a}$$

where

$$v_l \equiv \int d^dk \, v_l(0,k) \tag{6.5b}$$

and

$$\psi_l(q) \equiv \int d^dk \, \tilde{B}^{-1}(q-k) v_l(0,k) \, . \tag{6.5c}$$

λ_l'' is the expansion coefficient of the lth eigenvalue $\lambda_l(K)$ of the operator $C(K,k_1,k_2)$.

At large distances R from the transmitter, only the spherically symmetric ($l=0$) component has an appreciable amplitude since $\lambda_0=0$. Higher angular-momentum components are exponentially damped on a length scale determine by $(\lambda_l/\lambda_l'')^{-1/2}$. At higher frequencies,

$$\lambda_l^{-1/2} \sim \frac{\gamma\omega}{\sqrt{2l(l+1)}}, \quad \omega a \gg 1 \, . \tag{6.6}$$

Clearly $\psi_l(q)$ represents the spectral content of the lth angular-momentum component of the transmitted disturbance. For the isotopic diffusion mode ($l=0$),

$$\psi_0(q) = \int d^dk \, \tilde{B}^{-1}(k-q) \mathrm{Im} Q^+(k) = -\tfrac{1}{2}\omega^2 \mathrm{Im} G_0^+(q) \, , \tag{6.7}$$

indicating that sufficiently far away from an arbitrary transmitter of frequency ω, the set of wave vectors excited in the medium is isotropic and highly peaked at $|\vec{q}| = \omega$ in the weak scattering limit.

VII. NONLINEAR σ MODEL

The nature of normal modes of frequency ω is determined by the manner in which the conductance $\Sigma(\omega)$ renormalizes as we integrate over short-wavelength fluctuations of the field theory and rescale to longer lengths. For frequencies at which $\Sigma(\omega)$ renormalizes to zero we deduce that normal modes of frequency ω are localized. Extended states of frequency ω are characterized by corresponding $\Sigma(\omega)$ which renormalizes to a nonzero value in the long-distance limit. Expanding the field $\hat{Q}^{+-}$ in (5.4) in terms of the eigenfunctions $v_l(Kk_1)$ of $C(Kk_1k_2)$ [see (6.4)], the quadratic fluctuations about the saddle point may be written

$$L[Q]=\sum_l \int d^dK\,\lambda_l(K)Q_l(K)Q_l(-K)\ . \qquad (7.1)$$

Since there is a gap between the Goldstone mode ($l=0$) and other low-lying modes of $C(0k_1k_2)$ for arbitrarily smooth disorder, the critical behavior of the field theory is determined by retaining the $l=0$ term alone. The fluctuations about the saddle-point manifold which we shall consider tending to restore the spontaneously broken symmetry (4.6) may be written as

$$\underline{Q}(R,k)=i\,\mathrm{Im}Q^{+}(k)\underline{U}(R)\begin{bmatrix}\underline{I} & \underline{0}\\ \underline{0} & -\underline{I}\end{bmatrix}\underline{U}^{T}(R)\ . \qquad (7.2)$$

Comparison with the eigenfunction expansion (6.4) and using $v_0(0,k)=\mathrm{Im}Q^{+}(k)$ yields a nonlinear σ model identical to that for uncorrelated disorder,[8-10]

$$L[Q]=\Sigma(\omega)\int d^dK\,K^2Q_0^{\alpha\beta}(K)Q_0^{\alpha\beta}(-K)\ , \qquad (7.3a)$$

where

$$\underline{Q}_0(R)=i\underline{U}(R)\begin{bmatrix}\underline{I} & \underline{0}\\ \underline{0} & -\underline{I}\end{bmatrix}\underline{U}^{T}(R)\ . \qquad (7.3b)$$

Here we have made use of the low-momentum expansion (5.15a) of $\lambda_0(K)$ and relation (5.19) between expansion coefficient $\lambda_0''(0)$ and the conductance $\Sigma(\omega)$. As in the uncorrelated case,[8] all normal modes are localized in two dimensions and below. In $d=2+\epsilon$ there is a mobility edge ω_* separating low-frequency extended states from high-frequency localized states which is characterized by a localization length ξ diverging as $|\omega-\omega_*|^{-1/\epsilon}$. The divergence of the localization length as $\omega\to 0$ for $d\le 2$ is governed by the corresponding asymptotic behavior of the bare conductance $\Sigma(\omega)$. This is evaluated in Appendix A using the solutions (4.14) of the CPA equation for correlations of the form (4.13). The result is

$$\Sigma(\omega)\sim\begin{cases}\dfrac{1}{\omega^2}, & 2m>d\\[2mm] -\dfrac{1}{\omega^2\ln\omega}, & 2m=d\\[2mm] \omega^{d-2}\dfrac{1}{\omega^{2m}}, & 1<2m<d\\[2mm] -\omega^{d-2}\dfrac{1}{\omega\ln\omega}, & 2m=1\\[2mm] \omega^{d-2}\dfrac{1}{\omega^{m/(1-m)}}, & 0<2m<1\ .\end{cases} \qquad (7.4)$$

This leads to the following localization lengths in $d=1$ and 2: For $d=1$,

$$\xi\sim\begin{cases}\dfrac{1}{\omega^2}, & \dfrac{1}{2}<m\\[2mm] -\dfrac{1}{\omega^2\ln\omega}, & m=\frac{1}{2}\\[2mm] \dfrac{1}{\omega^{1/(1-m)}}, & 0<m<\frac{1}{2}\end{cases} \qquad (7.5)$$

and for $d=2$,

$$\xi\sim\begin{cases}e^{1/\omega^2} & m>1\\ e^{-1/\omega^2\ln\omega}, & m=1\\ e^{1/\omega^{2m}}, & \frac{1}{2}<m<1\\ e^{-1/\omega\ln\omega}, & m=\frac{1}{2}\\ e^{1/\omega^{m/(1-m)}}, & 0<m<\frac{1}{2}\ .\end{cases} \qquad (7.6)$$

Similarly, the problem of electron localization in a long-range correlated random potential is described within the same universality class. For the case of a Schrödinger equation

$$[-\nabla^2+V(x)]\psi(x)=E\psi(x) \qquad (7.7a)$$

with a random potential satisfying the conditions

$$\langle V(x)V(y)\rangle=B(x-y)\ , \qquad (7.7b)$$

$$\langle V(x)\rangle=0\ , \qquad (7.7c)$$

the quantity analogous to the energy diffusivity (2.3) is the diffusion coefficient for the locally conserved electron probability density,

$$D\equiv\lim_{t\to\infty}\frac{\dfrac{1}{t}\int d^dx\,x^2\langle\,|\psi(x,t)|^2\rangle_{\text{ensemble}}}{\int d^dx\langle\,|\psi(x,t)|^2\rangle_{\text{ensemble}}}\ . \qquad (7.8)$$

Here $\psi(x,t)$ is the electron wave function. The spectral decomposition of the diffusivity into contributions $D(E)$ from electron eigenstates of energy E [compare with (2.5)] now takes the form

$$D\equiv\frac{\int_{-\infty}^{\infty}dE\,D(E)\rho(E)}{\int_{-\infty}^{\infty}dE\,\rho(E)}\ , \qquad (7.9)$$

where $\rho(E)$ is the electron density of states at energy E. The derivation of a nonlinear σ model (7.3) follows in an analogous manner to that presented for the wave equation (2.1). The coupling constant $\Sigma(\omega)$ is now replaced by the dc electrical conductance

$$\Sigma(E)\equiv D(E)\rho(E)\ . \qquad (7.10)$$

Again, there is a gap between the Goldstone mode and higher angular-momentum modes of the nonlinear σ model, for arbitrarily long-range correlations $B(x-y)$, leading to a lower critical dimension of two. All electron eigenstates are localized for $d\le 2$. For $d=2+\epsilon$ there are two mobility edges $\pm E_*$ characterized by localization lengths diverging as $\xi \sim |E \pm E_*|^{-1/\epsilon}$. As in the case of uncorrelated disorder, the conductance vanishes linearly at the mobility edges,

$$\Sigma(E \simeq \pm E_*) \sim (E \pm E_*)^t, \quad t=1 \; . \tag{7.11a}$$

For the case of a system of noninteracting electrons in a long-range correlated random potential filled to a Fermi level E_F, the response to an external electric field of frequency ν is characterized by the ac conductance $\Sigma(E_F,\nu)$. The critical exponents (see *Note added in proof*) are the same as those described in previous theories.[15–17] We have

$$\Sigma(E_F=E_*, \ \nu) \sim \nu^{(d-2)/d} \quad \text{as } \nu \to 0 \; . \tag{7.11b}$$

Also the static electrical polarizability diverges from the insulator side of the transition,

$$\alpha(E_F \simeq \pm E_*, \ \nu=0) \sim |E_F \pm E_*|^{-2/\epsilon} \; . \tag{7.11c}$$

VIII. SUMMARY AND DISCUSSION

We have shown that the Anderson localization transition for waves in a disordered medium with spatial correlations as long as $B(x)=\gamma^2/(x^2+a^2)^m$ $(m>0)$ can be described as a phase transition in an appropriate nonlinear σ model in $2+\epsilon$ dimensions. Owing to the existence of a gap between the Goldstone mode of the field theory and higher modes, this nonlinear σ model is in the same universality class as that of uncorrelated disorder. Since the Goldstone mode and higher modes can be interpreted as the bound states of a quantum particle in a suitable potential well with a generalized kinetic energy, we argue that for arbitrarily smooth disorder, the low-lying spectrum of the nonlinear σ model is discrete and hence leads to the same critical behavior as discussed in the theories of Wegner[6] and Abrahams *et al.*[7] The effect of long-range correlations in the geometrical optics limit $\omega a \gg 1$ is to produce a corresponding increase in the mean free path or for $d \leq 2$ an increase in the localization lengths.

This is in contrast to the critical behavior of a Heisenberg ferromagnet with power-law-decaying exchange interactions. For example, in the case of spins with interactions falling off with spatial separation as $R^{-(d+\sigma)}$ and $(0<\sigma<2)$, the associated nonlinear σ model has a weak singularity of the form K^σ leading to the existence of a phase transition between the paramagnetic and ferromagnetic phases for all dimensions $d>\sigma$.[18] In the present case of eigenvalue $\lambda_0(K)$ of the operator $C(K,k_1,k_2)$ is analytic near $K=0$ leading to the usual K^2 term in the nonlinear σ model (7.3). The only way that a lowering of the lower critical dimension could arise in the present model is by the existence of a continuous spectrum of $C(0,k_1,k_2)$ touching the Goldstone mode $\lambda_0=0$. This, however, occurs only in the limit $B(x)=\gamma^2$ of perfect order within each realization of the mass field $m(x)$.

Note added in proof. In the nonlinear σ model (7.1) we have only considered terms quadratic in the fluctuations Q_l, i.e., terms of the form $(\nabla Q_l)^2$ in the Lagrangian. It is possible that in the case of long-range correlations $B(x)=\gamma^2(x^2+a^2)^{-m}$ the higher-order terms in ∇Q_l resulting from the expansion around the saddle point may no longer be irrelevant in the renormalization-group sense. Rough arguments indicate that this may occur in $d=2+\epsilon$ dimensions when $m \lesssim \epsilon$. This case is being investigated further. The case of ferromagnets with long-range correlated disorder in the exchange interactions near four dimensions has been investigated by Weinrib and Halperin [Phys. Rev. B 27, 413 (1982)], who show that the critical dimension and critical exponents are changed by the long-range correlations. The magnetic problem near $d=2$ with long-range correlated disorder is similar to the present problem and is also under current investigation. We wish to thank Professor B. I. Halperin for a useful discussion and for pointing out that higher-order terms may not be irrelevant in the present problem for these extreme power-law correlations.

ACKNOWLEDGMENTS

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE MEAN FREE PATH AND THE CONDUCTANCE

The behavior of the mean free path, the energy diffusivity, and hence the localizations lengths in the asymptotic regimes $\omega \to 0$ and $\omega a \gg 1$ are governed by the solutions of CPA equation (4.4) in these same limits. In the weak scattering limit these follow from the evaluation of the angular average $\langle B \rangle \equiv [\langle \tilde{B}(\vec{k}-\omega\hat{q}) \rangle_{\hat{q}}]_{k=\omega}$. First we consider the case of exponentially decaying correlations (4.11). In $d=1$, this average is trivial and yields

$$[\mathrm{Im}Q^+(k)]_{k=\omega} = \frac{\gamma^2\omega}{2}\left[1+\frac{1}{1+(2\omega a)^2}\right] \to \gamma^2\omega, \quad \omega a \gg 1 \; . \tag{A1}$$

For $d=2$ we may write

$$\langle B \rangle = \tfrac{1}{2}\int_0^\infty dx \, x J_0^2(\omega x) B(x) \tag{A2a}$$

$$= \frac{\gamma^2}{2}\frac{\partial}{\partial a}\int_0^\infty dx \, e^{-x/a} J_0^2(\omega x) \; , \tag{A2b}$$

where J_0 is a Bessel function. The integral here may be expressed as a complete elliptic integral $K(s)$ of the first kind,[19]

$$\langle B \rangle = \frac{\gamma^2}{\pi}\frac{\partial}{\partial s}[sK(s)]\frac{\partial s}{\partial y} \; , \tag{A2c}$$

where

$$s \equiv y/(1+y^2)^{1/2}, \quad y \equiv 2\omega a \; . \tag{A2d}$$

In the limit of the correlation length much longer than the phonon wavelength $\omega a \gg 1$, we may use the asymptotic form of the elliptic integral[20]

$$K(s) \sim \tfrac{1}{2}\ln\frac{16}{1-s} \quad \text{as } s \to 1 \tag{A2e}$$

leading to the required asymptotic behavior

$$\langle B \rangle \sim \frac{\gamma^2}{2\pi(\omega a)}, \quad \omega a \gg 1 \ . \tag{A2f}$$

Finally for $d=3$, the angular average may be expressed in terms of the spherical Bessel functions j_0,

$$\langle B \rangle = \frac{\gamma^2}{2\pi^2 a^3}\int_0^\infty dx\, x^2 j_0^2(\omega x) e^{-x/a} \tag{A3a}$$

$$= \frac{\gamma^2}{4\pi^2}\frac{1}{1+(a\omega)^2} \ . \tag{A3b}$$

We consider also the low-frequency ($\omega \to 0$) limit. From (4.9a),

$$\mathrm{Im}Q^+(\omega) \sim \omega^d \langle B \rangle \ . \tag{A4}$$

For any integrable $\int d^d x\, B(x) < \infty$ correlation function $\langle B \rangle$ approaches a finite constant as $\omega \to 0$. For power-law correlations of the form (4.13) in the nonintegrable range $1 < 2m \le d$,

$$\langle B \rangle \sim \begin{cases} -\ln(\omega a), & 2m = d \\ \dfrac{1}{\omega^{d-2m}}, & 1 < 2m < d \end{cases} \ . \tag{A5}$$

If $2m \le 1$, the δ-function approximation (4.8) to the CPA Green's function is now longer self-consistent. Long-range power laws of this form result in a singularity in the angular average of the Fourier-transformed correlation function

$$\langle \tilde{B}(\vec{k}-\vec{q}) \rangle_{\hat{q}} \sim \begin{cases} -\ln|k-q|, & 2m = 1 \\ \dfrac{1}{|k-q|^{1-2m}}, & 2m < 1 \ . \end{cases} \tag{A6}$$

This leads to a divergent self-energy if the approximation (4.8) is used. More generally, we may write

$$\mathrm{Im}Q^+(k) \sim \omega^4 \int_0^\infty dq \frac{q^{d-1}}{(kq)^{(d-1)/2}|k-q|^{1-2m}} \times \frac{\mathrm{Im}Q^+(q)}{(\omega^2-q^2)^2+\omega^4[\mathrm{Im}Q^+(q)]^2} \ . \tag{A7}$$

It follows that

$$[\mathrm{Im}Q^+(\omega)]^{2-2m} \sim \begin{cases} \omega^{2m}, & 2m < 1 \\ -\omega\ln\omega, & 2m = 1 \end{cases} \tag{A8}$$

yielding the required result for the low-frequency asymptotic behavior of the self-energy (4.14b) for $2m \le 1$.

The low-frequency behavior of the conductance (5.18) is governed by the term

$$\Sigma(\omega) \sim \frac{\omega^4}{4}\int d^d k [\mathrm{Im}Q^+(k)]^2 \frac{\partial}{\partial k_i} G_0^+(k) \frac{\partial}{\partial k_i} G_0^-(k) \ . \tag{A9a}$$

With the use of

$$\frac{\partial}{\partial k_i} G_0^+(k) = -[G_0^+(k)]^2 \left[-2k_i + \omega^2 \frac{\partial}{\partial k_i} Q^+(k) \right] , \tag{A9b}$$

the most singular contribution to the conductance as $\omega \to 0$ may be written (see also Ref. 8, Appendix B)

$$\Sigma(\omega) \sim \int d^d k\, k^2 [\mathrm{Im}G_0^+(k)]^2 \tag{A9c}$$

$$\sim \frac{\omega^{d-2}}{\mathrm{Im}Q^+(\omega)} \ . \tag{A9d}$$

With the use of the asymptotic behavior of the self-energy $\mathrm{Im}Q^+(\omega)$ from (A4), (A5), and (A8) we obtain the required result (7.4).

APPENDIX B: SPECTRUM OF THE NONLINEAR σ MODEL

We consider the spectrum of the Schrödinger equation

$$[-\nabla_k^2 + V(k)]v_l = \gamma^2 \lambda_l v_l \tag{B1a}$$

for the δ-shell potential

$$V(k) \equiv a^2 - \frac{\gamma^2 \delta(k^2-\omega^2)}{N(\omega^2)\langle B \rangle} \ . \tag{B1b}$$

For $d=3$,

$$N(\omega^2)\langle B \rangle = \frac{\gamma^2 \omega}{\pi}\int_0^\infty dx \frac{x^2 j_0^2(\omega x)}{x^2+a^2} \tag{B2a}$$

$$= \frac{\gamma^2}{2} e^{-\omega a}\frac{\sinh(\omega a)}{\omega a} \ . \tag{B2b}$$

This establishes the result (5.10c). The potential may be rewritten in the form

$$V(k) = a^2 - g\delta(k-\omega) \ , \tag{B3a}$$

where

$$\frac{1}{-g\omega} = \omega a j_0(i\omega a) h_0(i\omega a) \ . \tag{B3b}$$

Here, the spherical Bessel and Hankel functions have a pure imaginary argument. It is possible to show[13] that the condition for a bound state of angular momentum l is given by

$$-\frac{1}{g\omega} = \beta j_l(i\beta) h_l(i\beta) \ , \tag{B4a}$$

where β is related to the eigenvalue $\gamma^2\lambda_l$ of the operator (B1a) by

$$\gamma^2 \lambda_l = a^2 - \beta^2/\omega^2 \ . \tag{B4b}$$

Clearly there is precisely one solution for $l=0$ ($\beta = \omega a$) for which the corresponding eigenvalue $\lambda_0 = 0$. For sufficiently large values of ωa the equation (B4a) admits higher angular-momentum bound states. In the geometric optics limit $\omega a \gg 1$, these excited states ($l \ge 1$) approach the ground state $l=0$. Using the asymptotic (large-β) forms of the Bessel function,[20] Eq. (B4a) becomes

$$-\frac{1}{2\omega a} \sim -\frac{1}{2\beta}\left[1-\frac{l(l+1)}{\beta^2}\right] . \tag{B5a}$$

It follows that

$$\beta \sim \omega a - \frac{l(l+1)}{\omega a} . \tag{B5b}$$

Substitution into (B4b) yields the asymptotic behavior of the lth bound state

$$\lambda_l \sim \frac{2l(l+1)}{\gamma^2\omega^2}, \quad \omega a \gg 1 . \tag{B5c}$$

[1] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[2] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, New York, 1970).
[3] P. Carruthers, Rev. Mod. Phys. 33, 92 (1961).
[4] H. Sumi, J. Phys. Soc. Jpn. 32, Suppl. No. 3, 616 (1972).
[5] J. D. Dow and D. Redfield, Phys. Rev. B 5, 594 (1972).
[6] F. Wegner, *Anderson Localization,* edited by Y. Nagaoka and H. Fukuyama (Springer, New York, 1982).
[7] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
[8] S. John, H. Sompolinsky, and M. J. Stephen, Phys. Rev. B 27, 5592 (1983).
[9] A. J. McKane and M. Stone, Ann. Phys. (N.Y.) 131, 36 (1981).
[10] L. Schafer and F. Wegner, Z. Phys. B 38, 113 (1980).
[11] R. J. Elliott, J. A. Krumhansl, and P. L. Leath, Rev. Mod. Phys. 46, 465 (1974).
[12] This is done in complete analogy with the case of uncorrelated disorder discussed in Ref. 8.
[13] K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966).
[14] V. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, New York, 1961).
[15] R. Oppermann and F. Wegner, Z. Phys. B 34, 327 (1979).
[16] B. Shapiro and E. Abrahams, Phys. Rev. B 24, 4889 (1981).
[17] S. Hikami, Phys. Rev. B 24, 2671 (1981).
[18] E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976).
[19] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[20] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington, D.C. 1964).