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Conductance plateaus in the quantized Hall effect

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An equilibrium thermodynamic model for the quantized Hall effect is studied. It is shown that plateaus in the Hall conductivity σ_H are expected to occur even in the absence of impurities or electron interactions. An independent-electron system is discussed and generalized to a Hartree system of interacting electrons.

The quantized Hall effect is observable by virtue of the finite range of magnetic field over which the Hall conductivity σ_H remains constant.[1,2] In the typical geometry a small current I_y flows in a two-dimensional layer of electrons held perpendicular to a magnetic field $\vec{B}=B\hat{z}$. The ratio of the current I_y to the observed voltage across the width of the sample V_x is found to be given by

$$\sigma_H=\frac{I_y}{V_x}=\frac{je^2}{h},\quad j=1,2,\ldots, \tag{1}$$

to the order of one part in 10^7.

It is relatively easy to derive theoretical expressions for these quantized values of σ_H at magnetic fields such that the Fermi energy lies exactly between Landau-level energies. At magnetic fields slightly removed from these values, however, one expects to have partially filled Landau levels, and the prediction of a quantized value for σ_H is not so easy to obtain. Laughlin,[3] for example, assumes that impurities will give rise to bands of localized states, so that the Fermi energy can be pinned between the Landau levels for a finite range of the magnetic field. Luryi and Kazarinov,[4] on the other hand, use continuum percolation theory to argue that fluctuations in the self-consistent potential will cause the sample to break up into regions of localized states with occupation numbers 0 or 1. For a Hall voltage larger than a critical voltage V_{cr} a small fraction of states will become extended. The Fermi energy is then pinned to the extended states over a region of the magnetic field proportional to the fractional area occupied by the extended states.

The purpose of the present paper is to demonstrate that plateaus should be expected to occur even in the absence of impurities or electron interactions. We show this first by discussion of an independent-electron model, which is then generalized to a Hartree model of interacting electrons. Because the quantized Hall effect describes a dissipationless system it is possible to obtain useful results by treating the device using equilibrium thermodynamics with a constant electric current as a constraint.

We examine first an independent-electron model, and consider a slab of dimensions L_x and L_y, with an applied magnetic field $\vec{B}=B\hat{z}$ and a confining potential $V(x)$. A small current I_y is forced through the slab in the y direction. In the Landau gauge, in which $\vec{A}=(0,Bx,0)$, and with the imposition of periodic boundary conditions in the y direction we can write the wave functions as $\psi_k=e^{iky}\Phi_k$, where $k=2\pi l/L_y$, for $l=0,\pm1,\ldots$. With the definitions $\omega_c=eB/mc$ and $x_k=\hbar k/m\omega_c$, the Schrödinger equation becomes

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega_c^2(x-x_k)^2+V(x)\right]\Phi_k(x)=\epsilon_k\Phi_k(x)\ . \tag{2}$$

The current density $\vec{j}_k$ carried by the state k is given by

$$\vec{j}_k=\frac{ie\hbar}{2m}[(\vec{\nabla}\psi_k^*)\psi_k-\psi_k^*\vec{\nabla}\psi_k]-\frac{e^2}{mc}\vec{A}|\psi_k|^2\ . \tag{3}$$

We assume $\Phi_k(x)$ to be real, making $j_{k,x}$ zero, and the y component of the current density of state k to be independent of y and hence given by $j_k\equiv j_{k,y}=-e\omega_c(x-x_k)\Phi_k^2$. With Φ_k real, we also have

$$\epsilon_k=\int\Phi_k\left[\frac{p_x^2}{2m}+\frac{1}{2}m\omega_c^2(x-x_k)^2+V(x)\right]\Phi_k d^2r \tag{4}$$

so that

$$\frac{\partial\epsilon_k}{\partial x_k}=-\int\Phi_k m\omega_c^2(x-x_k)\Phi_k d^2r=\frac{m\omega_c}{e}\int j_k d^2r \tag{5}$$

since, by the variational principle, we need not differentiate Φ_k in (4) with respect to x_k.

In two dimensions the current density is in units of current per unit length, so the total current i_k, carried by state k, is thus $i_k=\int j_k dx$. Using (5) we can write

$$\frac{\partial\epsilon_k}{\partial x_k}=\frac{m\omega_c}{e}\int j_k d^2r=\frac{m\omega_c}{e}\int i_k dy=\frac{m\omega_c}{e}L_y i_k \tag{6}$$

since i_k is a constant.

The total current I_n carried by the nth Landau level is given by

$$I_n=\sum_k i_k^{(n)}=\sum_k\frac{e}{m\omega_c L_y}\frac{\partial\epsilon_k^{(n)}}{\partial x_k}\ .$$

Since $x_k=\hbar k/m\omega_c$ we can sum over x_k instead of k, and by letting L_y become arbitrarily large we can make $|x_k-x_{k-1}|$ arbitrarily small, allowing replacement of the sum over x_k by an integral, so that

$$I_n=\frac{m\omega_c L_y}{2\pi\hbar}\int\frac{e}{m\omega_c L_y}\frac{\partial\epsilon^{(n)}}{\partial x}dx=\frac{e}{h}\int\frac{\partial\epsilon^{(n)}}{\partial x}dx \tag{7}$$

with the integral extending over all occupied states. Then, as noted by Halperin,[5]

$$I_n=\frac{e}{h}\Delta\epsilon_n\equiv\frac{e}{h}[\epsilon^{(n)}]_{\text{left side}}^{\text{right side}}\ , \tag{8}$$

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with $\Delta\epsilon_n$ denoting the Fermi energy difference between the right side and the left side of the nth Landau level.

Let us see how relation (8) will give us plateaus in σ_H. For simplicity, we consider the two lowest Landau levels of a system. On the right side $\partial\epsilon^{(n)}/\partial x > 0$, so the states at that edge carry positive current, while on the left side $\partial\epsilon^{(n)}/\partial x < 0$, which gives rise to a negative current. We start with the flat part of the lower band filled up, and we study the behavior of the Hall conductivity as we add electrons to the system while maintaining the total current I_y. Initially, there must then be a Fermi energy difference $\Delta\epsilon_1 = \epsilon_F$ (right side) $- \epsilon_F$ (left side) $\equiv \epsilon_R - \epsilon_L = hI_y/e$ to support the current, according to Eq. (8). Since the Hall voltage measured across the sample is exactly $e^{-1}\Delta\epsilon_1$ we immediately conclude that $\sigma_H = e^2/h$. As more electrons are added to the system, they will distribute themselves in the empty states at the edges in such a way that the Fermi energy difference $\Delta\epsilon_1 = (h/e)I_y$ is maintained, in order to keep the current constant; the Hall conductivity thus remains constant at $\sigma_H = e^2/h$. In a large sample this occupies only a vanishingly small range of magnetic field. The more important effects occur as the Fermi energy at the right end of the lower Landau level, $\epsilon_R^{(1)}$, reaches $\frac{3}{2}\hbar\omega_c$. The states of lowest energy in the second Landau level are those at the center of the sample, and these will be the first to be filled as the electron density is increased. Because these states carry no current, $\partial\epsilon^{(2)}/\partial x$ being zero here, the Fermi energy difference between the edges is kept constant at $\Delta\epsilon_F = \Delta\epsilon_1 = (h/e)I_y$, and so σ_H is kept constant at $\sigma_H = e^2/h$. As we have filled the flat part of the upper level and approach the edges where $\partial\epsilon^{(2)}/\partial x \neq 0$, both levels will now be filled at both edges until the Fermi energies of the two levels have been equalized at each edge; then $\epsilon_R^{(1)} - \epsilon_L^{(1)} = \epsilon_R^{(2)} - \epsilon_L^{(2)}$ and both Landau levels thus carry an equal amount of current. Since we are keeping the total current I_y constant, each Landau level must carry a current of $\frac{1}{2}I_y$; then the Fermi energy difference between the edges has dropped to $\frac{1}{2}(h/e)I_y$, and so the Hall conductivity is now $\sigma_H = 2e^2/h$. When we continue to add electrons to the system they will be distributed so that at each edge the Fermi energies of both levels are the same, thus maintaining the difference between the two edges, and holding σ_H constant at $2e^2/h$.

The claim that the Fermi energy is the same at each edge for all Landau levels might need some justification. The local Fermi energy of a certain Landau level is given by the energy at the point where the chemical potential $\mu(x)$ equals the energy of that level. We assume that $\mu(x)$ does not vary over the distance Δx between the two points at a single edge where the Landau levels assume the same energy ϵ (when $\epsilon \neq \frac{3}{2}\hbar\omega_c$), since $\Delta x \ll L_x$. It then follows[6] that $\epsilon_R^{(1)} = \epsilon_R^{(2)}$ and $\epsilon_L^{(1)} = \epsilon_L^{(2)}$. (This is true for the self-consistent calculations.)

We can easily generalize our thought experiment to a multilevel system. When the flat part of the nth level has been filled, each level carries a current I_y/n; then the Fermi energy difference between the two edges has dropped to $(h/e)(I_y/n)$, and the Hall conductivity increases to $\sigma_H = e^2/hn$. If we define the Landau length l_B as $(\hbar c/eB)^{1/2}$, and then let the quantity l_B^2/L_xL_y become very small, so that the fraction of states that are edge states becomes negligible, the transition regions where the Hall conductivity increases an amount e^2/h will become negligible compared to the regions of constant σ_H: a plot of σ_H as a function of the filling factor ν will have the form of the dashed line in Fig. 1.

It might be argued that the independent-electron model is not a realistic representation of a Hall measurement in that it does not reflect the macroscopic Maxwell equations that the electron gas must obey. In order to verify that our conclusions remain valid in the presence of interactions, the calculation was extended to a Hartree gas of interacting electrons. For the sake of simplicity a model was constructed in which the electrostatic potential was independent of the z coordinate. This was achieved by considering an infinitely high stack of jellium slabs separated by insulating sheets. Each slab has dimensions L_x and L_y and a thickness L_z and carries a current I_y in the y direction. A magnetic field $\vec{B} = B\hat{z}$ is applied. We assume that L_z is much less than the scale of variation of the charge density in the x direction, so that the charge density can be taken as uniform in the z direction.

In two dimensions, the conductivity $\sigma_{yx} = j_y/E_x$ is the same as the conductance $Y_{yx} = I_y/V_x$. For the jellium slabs, however, the conductivity of each slab depends on the thickness L_z, whereas the conductance does not. Hence the comparable quantity to the two-dimensional conductivity for our system is the (Hall) conductance per slab.

Again, using the Landau gauge and periodic boundary conditions in the y direction and letting f_k^n denote the average occupancy of state k in the nth Landau level, one easily derives the following set of Hartree equations:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_c^2(x-x_k)^2 + \phi(x)\right]\psi_k^n(x) = \epsilon_k^n\psi_k^n(x) \quad , \tag{9}$$

$$\phi(x) = 2\pi e^2\int_{-\infty}^{\infty}|x-x'|[n_+(x') - n(x')]\,dx' \quad , \tag{10}$$

where $n_+(x)$ is the jellium charge density, given by

$$n_+(x) = \begin{cases} 0, & |x| > L_x/2 \\ n_+, & |x| \leq L_x/2 \end{cases} \tag{11}$$

and

$$n(x) = \sum_{k,n} f_k^n|\psi_k^n(x)|^2 \quad . \tag{12}$$

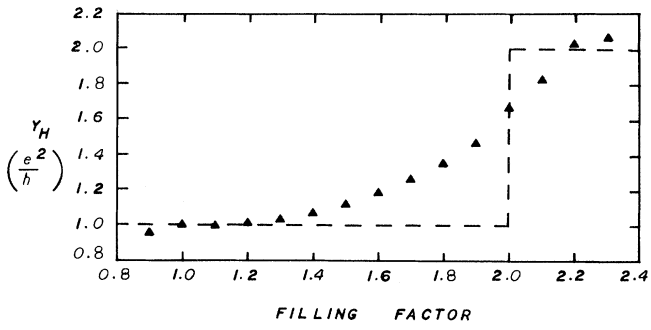

FIG. 1. Hall conductance Y_H in units of e^2/h vs filling factor. The dashed line represents the independent-electron model, and the filled triangles are for the numerical calculations of the Hartree model.

To obtain an expression for the occupancy f_k^n, we use the fact that, since the current flow is dissipationless, the system may be treated as in thermodynamic equilibrium. One may thus minimize the free energy F, which is expressed as

$$F = U + k_B T \sum_{k,n} [(1-f_k^n)\ln(1-f_k^n) + f_k^n \ln(f_k^n)] \ , \tag{13}$$

where U is the total (Hartree) energy of the system and k_B is Boltzmann's constant. In addition to the customary constraint

$$\sum_{k,n} f_k^n = N = n_+ L_x L_y L_z \tag{14}$$

we impose the additional condition

$$\sum_{k,n} f_k^n i_k^n = I_y \ . \tag{15}$$

That is, the total number of electrons N and the total current I_y are fixed. [Equation (14) also defines $n_+(x)$ for charge neutrality.] The minimization is a standard exercise in Lagrange multipliers, and the result is

$$f_k^n = \frac{1}{\exp\left(\dfrac{\epsilon_k^n - \zeta - \xi i_k^n}{k_B T}\right) + 1} \tag{16}$$

with the constants ζ and ξ determined by Eqs. (14) and (15).

The set of Eqs. (9)–(16) was solved self-consistently with the parameters (in units of $e = m = \hbar = 1$) $L_x = 7.0$, $L_y = 40\pi$, $L_z = 1.0$, $\omega_c = 1.0$, $k_B T = 0.005$, $I_y = 0.05$. Thus the distance $|x_k - x_{k-1}| = 0.05$ and the typical spread in the x direction of a wave function in the lowest Landau level is 1.0. For a filling factor of $\nu = 1$, the system then holds 141 electrons. The chosen value of L_z is low enough to ensure that all motion in the z direction is frozen in the lowest state.

In the numerical computations the criterion for self-consistency was chosen such as to make the maximum difference between the nth and the $(n+1)$th iterations of the Coulomb potential energy less than 10^{-3} (which is some tenths of a percent of the Fermi energy difference required for $\sigma_H = 1$). The Hall conductance Y_H was then calculated as the quotient of the total current I_y and the chemical potential difference between the two edges (in the x direction) for filling factors ν ranging from 0.9 to 2.3 in steps of 0.1.

We have calculated the current carried by each Landau level as a function of the Fermi energy difference of the level as compiled from all calculations. These values follow a linear relationship with a slope of I_n vs $\Delta\epsilon_n$ equal to e/h to within 1.2%, showing the validity of Eq. (8).

In Fig. 1 the Hall conductance in units of e^2/h is plotted against ν. There are two distinct plateaus, one at $1 \leq \nu \leq 1.3$, and the other at $2.2 \leq \nu \leq 2.3$. The Hall conductance assumes the value 1 to closer than 4% at the first plateau, and the value 2 to closer than 6% at the second plateau. These results must be considered satisfactory since each wave function was expanded in only its four lowest Fourier-Hermite components.

The maximum derivative of Y_H as a function of ν occurs at $\nu = 2$, indicating that the transition between $Y_H = 1$ and $Y_H = 2$ indeed takes place at $\nu = 2$. The smoothness of the transition as obtained from the plot can be attributed to two things. Firstly, a size effect appears to be present; when recalculating the point $\nu = 1.9$ for a system with identical parameters, except for L_x, which was increased to 14 (534 electrons), the value of Y_H decreased to 1.370 from its previous value of 1.470. Unfortunately, due to the immense computing time required to make such a large system converge, we could not recalculate any more points. Secondly, in the experiments of Paalanen *et al.*,[2] where extremely wide plateaus were observed, the quotient of $\hbar\omega_c$ and $k_B T$ was about five times higher than in our calculations, so at least part of the smoothing could be due to high thermal energy.

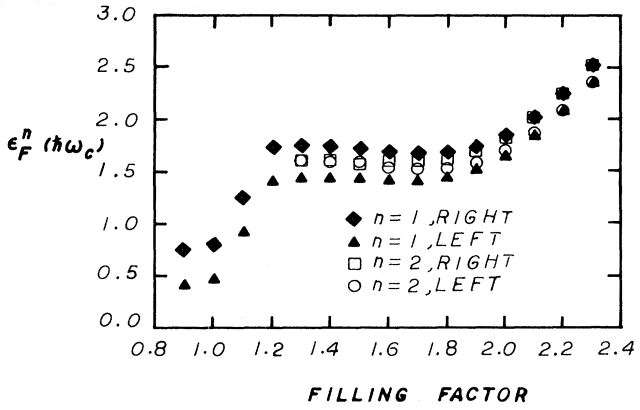

FIG. 2. Fermi energies $\epsilon_F^{(n)}$ at each edge for the two lowest Landau levels ($n = 1, 2$) vs filling factor.

In Fig. 2, the Fermi energies at the edges of the Landau levels for the two lowest Landau levels are plotted vs ν. We see that as the second level starts to be filled up the Fermi energies at its edges rapidly approach those of the lowest Landau level.

We have presented a theoretical model of the quantized Hall effect based on a thermodynamical equilibrium approach in which a constant-current constraint is imposed. This model produces a quantization of the Hall conductivity in steps of e^2/h in any confining potential that is only a function of the transverse dimension. The model should also be appropriate in the presence of bands of localized states between the principal Landau levels, so long as the fundamental relation (8) holds for these levels. Calculations for interacting electrons in a layered structure are in accordance with this model. The exact quantization of the Hall conductivity is attributed to the properties of the edge states. A significant discrepancy between this model and the experimental results lies in the fact that in the experiments the transition between two adjacent plateaus appears to be located halfway between integral filling factors.[2] This might be explained by introducing localized states between the Landau levels.[3,5] It is also a possibility that the electron-electron interaction in two dimensions may change the transition points from where they are in our calculations with an effective one-dimensional Coulomb interaction.

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[1]K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).
[2]M. A. Paalanen, D. C. Tsui, and A. C. Gossard, Phys. Rev. B 25, 5566 (1982).
[3]R. B. Laughlin, Phys. Rev. B 23, 5632 (1981).
[4]S. Luryi and R. F. Kazarinov, Phys. Rev. B 27, 1386 (1983).
[5]B. I. Halperin, Phys. Rev. B 25, 2185 (1982).
[6]This question is the subject of some recent work. For a Corbino geometry with random impurities in a confining box potential or harmonic-oscillator potential one would expect the Fermi energies to be different for different Landau levels, and that would lead to deviations from the perfect quantization of the Hall conductivity. However, the predicted deviations are several orders of magnitude larger than the present experimentally determined accuracy of the quantization of the Hall conductivity. [See G. Giuliani, J. J. Quinn, and S. C. Yang (unpublished).]