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Exact solution of a one-dimensional XY model in a random field

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We present an exact solution of a one-dimensional XY model in a random magnetic field in the limit of strong-field pinning. The structure factor exhibits Lorentzian-squared behavior at nonzero temperatures. The scaling behavior of the correlation length as a function of randomness is different from that obtained in the weak-field pinning regime. We also study the dynamical behavior of the system using a simple relaxational model. The magnetization decays anomalously with time as $t^{7/6}e^{-ct^{1/3}}$.

Much attention has been focused in recent years on the behavior of spin systems with quenched random fields. Domain arguments developed by Imry and Ma[1] predict that the lower critical dimensionality for such systems is two for the Ising case and four for the vector spin case. This result for the vector spin model has been substantiated by renormalization-group calculations[2] and a "Mermin-Wagner" style proof.[3] The Ising situation remains less clear, however.[4] Besides predicting a shift in the lower critical dimensionality, the domain arguments also give the size of the domains below the lower critical dimensionality. If we consider a general model where the random fields are dilute with concentration p and strength h, then the domain size L_0 will be given by

$$L_0 \sim \left(\frac{J}{h\sqrt{p}}\right)^{2/(2-d)} \tag{1a}$$

for the Ising case and

$$L_0 \sim \left(\frac{J}{h\sqrt{p}}\right)^{2/(4-d)} \tag{1b}$$

for vector spins. Here J is the exchange energy. The density of sites with random fields within a domain is then given by the product pL_0^d. However, in deriving (1) one assumes that this density is large so that the field energy is a $\sqrt{N}$, where N is the number of sites with random fields in the domain. Thus two regimes emerge: a "weak-pinning" regime where $pL_0^d \gg 1$, and a "strong-pinning" regime where $pL_0^d < 1$.[5] The domain results (1) are not valid in the latter regime.

In the "weak-pinning" regime one expects that the zero-temperature correlation length will be identified with the domain size L_0. Indeed low-temperature renormalization-group arguments for the Ising and vector systems,[2] and a one-dimensional calculation[6] valid in the weak-pinning regime, yield correlation lengths which scale with the randomness in the manner predicted by (1). Grinstein and Mukamel[7] recently considered an exact solution of the strong-pinning regime of the Ising model in one dimension and found that the zero-temperature correlation length scales as predicted by the domain argument, i.e.,

$$\xi_p \sim 1/p \quad . \tag{2}$$

(Their model assumes that h is "infinite" or equivalently greater than $2J$.) This result is not surprising since in the strong-pinning regime each field site forms a domain around it, and hence the domain size or correlation length scales as $1/p$. In d dimensions we would expect the correlation length to scale as $p^{-1/d}$ in the strong-pinning regime. It is clear from (1a) that the correlation length will scale in the same fashion in the weak- and strong-pinning regimes only in one dimension. On the other hand, for the vector models we see that the strong-pinning result $\xi_p \sim p^{-1/d}$ will agree with the domain result (1b) only in two dimensions. In this paper we present an exact solution of the strong-pinning regime of the one-dimensional XY model and show, in particular, that the correlation length scales as in (2) rather than (1b). We also find that, as in the Ising case, the structure factor exhibits a Lorentzian-squared term only at $T \neq 0$. Additionally, we calculate the susceptibility and the Edwards-Anderson order parameter. We also study the dynamical behavior of the system using a simple relaxational model. We find that for large times t the magnetization decays anomalously with time as $t^{7/6}e^{-ct^{1/3}}$.

The model we consider is defined by the Hamiltonian

$$H = -J\sum_{\langle ij\rangle}\vec{S}_i\cdot\vec{S}_j - \sum_i \vec{h}_i\cdot\vec{S}_i \quad , \tag{3}$$

where $\vec{S}_i$ is an XY spin, given in polar coordinates by $(\cos\theta_i, \sin\theta_i)$. The fields $\vec{h}_i$ are distributed spatially with probability p. The field strength is infinite and the field directions are distributed uniformly over the unit circle. We will specify the field direction by a polar angle ϕ_i. From (1b) we see that the density of impurities per domain is given by

$$L_0 p \sim \left(\frac{J}{h}p\right)^{2/3} \quad . \tag{4}$$

Therefore for $h > 0(J)$, $L_0 p \lesssim 1$ and we are restricted to the strong-pinning regime. The infinite pinning fields allow an exact solution of the model since we need only consider finite chains bounded by impurity sites. We first consider the static correlation functions

$$S(l) = [\langle\vec{S}_n\cdot\vec{S}_{n+l}\rangle]_{\text{av}} \tag{5}$$

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and

$$\chi(l) = [\langle \vec{S}_n \cdot \vec{S}_{n+l} \rangle - \langle \vec{S}_n \rangle \cdot \langle \vec{S}_{n+l} \rangle]_{\text{av}} \ , \tag{6}$$

where the angle brackets denote a thermodynamic average and the square brackets denote the average over h_i and ϕ_i. The correlation function $S(l)$ is given by

$$S(l) = \sum_{N=0}^{\infty} (N+1) p^2 (1-p)^{N+l} \int_0^{\pi} \frac{d(\phi_{N+l+1} - \phi_0)}{\pi} \langle \vec{S}_n \cdot \vec{S}_{n+l} \rangle_{N+l+1}^{\phi_0, \phi_{N+l+1}}$$

$$+ \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} p^3 (1-p)^{M+N+l} \sum_{n=1}^{l-1} \int_{-\pi}^{\pi} \frac{d\phi_{N+n+1}}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_{M+l-n+1}}{2\pi} \langle \vec{S}_n \rangle_{N+n+1}^{\phi_{N+n+1}, \phi_0} \cdot \langle \vec{S}_{l-n} \rangle_{M+l-n+1}^{\phi_0, \phi_{M+l-n+1}} \ . \tag{7}$$

The first sum takes into account configurations with no magnetic field between the spins, while the second term averages over the configurations with a magnetic field at one intermediate site. The thermodynamic average $\langle \vec{S}_n \cdot \vec{S}_{n+l} \rangle_{N+l+1}^{\phi_0, \phi_{N+l+1}}$ is the spin-spin correlation function on a chain of length $N+l+2$ with boundary conditions specified by the angles ϕ_0 and ϕ_{N+l+1}. In the second term of (7) we have made use of the fact that for configurations which involve a field at an intermediate site one may replace $\langle \vec{S}_n \cdot \vec{S}_{n+l} \rangle$ by $\langle \vec{S}_n \rangle \cdot \langle \vec{S}_{n+l} \rangle$. Then, $\langle \vec{S}_n \rangle_{N+n+1}^{\phi_0, \phi_{N+n+1}}$ represents the average magnetization at site n in a chain of length $N+n+2$ with boundary conditions specified by the angles ϕ_0 and ϕ_{N+n+1}.

The evaluation of the thermdynamic averages appearing in (7) is carried out for $T=0$ and $T \neq 0$ separately. At $T=0$, the spin configuration between two field sites is given by a spin-wave solution with maximum possible wavelength subject to the boundary conditions. Thus for a chain of length $N+1$ with boundary angles ϕ_0 and ϕ_N we have

$$\vec{S}_n = \cos\left[\phi_0 + \frac{n(\phi_N + \phi_0)}{N}\right] \hat{i} + \sin\left[\phi_0 + \frac{n(\phi_N - \phi_0)}{N}\right] \hat{j} \ . \tag{8}$$

Using (8) in (7) we find, after performing all of the sums and integrals, that $S(l)$ does not exhibit a term of the form $l(1-p)^l$, and thus the Fourier transform of $S(l)$ which is the structure factor does not show a Lorentzian-squared behavior. Similar behavior is seen in the Ising case. The correlation length is given by

$$\xi_p = \frac{-1}{\ln(1-p)} \underset{p \to 0}{\sim} p^{-1} \ , \tag{9}$$

in agreement with the argument presented after (2).

For $T \neq 0$, we expand the Boltzmann factors appearing in the thermodynamic functions in a series of modified Bessel functions,[8] i.e.,

$$\exp[\beta J \cos(\theta_i - \theta_j)] = \sum_{n=-\infty}^{\infty} I_n(\beta J) \exp[in(\theta_i - \theta_j)] \ . \tag{10}$$

With this representation the partition function, the average magnetization, and the correlation function for a chain of length $N+1$ with boundary conditions specified by ϕ_0 and ϕ_N are as follows:

$$Z = I_0^N(\beta J) \left[1 + 2 \sum_{m=1}^{\infty} \left(\frac{I_m(\beta J)}{I_0(\beta J)}\right) \cos m(\phi_N - \phi_0)\right] \ , \tag{11a}$$

$$\langle \exp^{(i\phi_n)} \rangle_N^{\phi_0, \phi_N} = Z^{-1} \sum_{m=-\infty}^{\infty} I_m^n(\beta J) I_{m-1}^{N-n}(\beta J) \exp[im\phi_0 - i(m-1)\phi_N] \ , \tag{11b}$$

$$\langle \exp[i(\phi_n - \phi_{n+l})] \rangle_{N+m+l}^{\phi_0, \phi_{N+m+l}} = Z^{-1} \sum_{m=-\infty}^{\infty} I_m^{n+N}(\beta J) I_{m-1}^{l}(\beta J) \exp[im(\phi_0 - \phi_{N+n+l})] \ . \tag{11c}$$

Then we find from (7) that for large T satisfying $z \equiv [I_1(\beta J)/I_0(\beta J)] << 1$ the leading terms in $S(l)$ are

$$S(l) = Al(1-p)^l z^l + B(1-p)^l z^l \ , \tag{12a}$$

where A and B are given by

$$A = p + O(z^2) \ , \tag{12b}$$

$$B = 1 - p(1-p)^2 + O(z^2) \ . \tag{12c}$$

Equation (12a) implies that the inverse correlation length is given by

$$\xi^{-1} = \xi_T^{-1} + \xi_p^{-1} \ , \tag{13}$$

where $\xi_T^{-1} = -\ln z$ and ξ_p^{-1} is given by (9). These quantities are the thermal and random inverse correlation lengths, respectively. Fourier transforming (12a) we find that the structure factor has a Lorentzian form proportional to B and a Lorentzian-squared term proportional to A.

In calculating $\chi(l)$ we need only consider configurations with no intermediate fields, and thus it is given by

$$\chi(l)=\sum_{N=0}^{\infty}p^2(1-p)^{N+l}\sum_{n=0}^{N}\int_0^{\pi}\frac{d(\phi_{N+l+1}-\phi_0)}{\pi}(\langle\vec{S}_n\cdot\vec{S}_{n+l}\rangle_{N+l+1}^{\phi_0,\phi_{N+l+1}}-\langle\vec{S}_n\rangle_{N+l+1}^{\phi_0,\phi_{N+l+1}}\cdot\langle\vec{S}_{n+l}\rangle_{N+l+1}^{\phi_0,\phi_{N+l+1}}) \quad . \tag{14}$$

For large l satisfying $z^l << 1$ we find to leading order that $\chi(l)\sim(1-p)^l$, i.e., it is purely exponential. This difference in behavior between $\chi(l)$ and $S(l)$ at $T\neq 0$ is to be contrasted with the behavior of the XY model in two dimensions in the presence of a random field in the Gaussian approximation.[9] In that case, the disconnected piece of $\chi(l)$ vanishes and $\chi(l)$ and $S(l)$ are identical.

For the Edwards-Anderson order parameter we have

$$Q\equiv[\langle\vec{S}_n\rangle\cdot\langle\vec{S}_n\rangle]_{\text{av}}=\sum_{N=0}^{\infty}p^2(1-p)^N\sum_{n=0}^{N}\int_0^{\pi}\frac{d(\phi_{N+1}-\phi_0)}{\pi}\langle\vec{S}_n\rangle_{N+1}^{\phi_0,\phi_{N+1}}\cdot\langle\vec{S}_n\rangle_{N+1}^{\phi_0,\phi_{N+1}} \quad , \tag{15}$$

and after calculations analogous to those above we find

$$Q=\begin{cases}1, \; T=0,\\ p+2z^2p(1-p)+O(z^4), \; T\neq 0 \quad .\end{cases} \tag{16}$$

Unlike the Ising case, Q is saturated at $T=0$ for all values of p since the ground-state solution (8) is unique, whereas in the Ising case the kink separating up and down spin regions is free to slide.

It is also straightforward to consider the dynamical behavior of this system. We consider here a simple relaxational model defined by the equations[10]

$$\frac{\partial\theta_n(t)}{\partial t}=-\Gamma\frac{\delta H}{\delta\theta_n(t)}+\xi_n(t) \quad , \tag{17a}$$

$$\langle\xi_n(t)\xi_m(t')\rangle=2k_BT\Gamma\delta_{mn}\delta(t-t') \quad , \tag{17b}$$

$$\langle\xi_n(t)\rangle=0 \quad . \tag{17c}$$

If we write $\theta_n(t)=\theta_n(t)+\epsilon_n(t)$, where $\theta_n^0(t)$ is the ground-state solution corresponding to (8), then the linear (in ϵ) approximation to (17a) is

$$\frac{\partial\epsilon_n}{\partial t}=\Gamma J\cos\left(\frac{\phi_N-\phi_0}{N}\right)(\epsilon_{n+1}+\epsilon_{n-1}-2\epsilon_n)+\xi_n(t) \tag{18}$$

for a chain of length $N+1$, with boundaries specified by ϕ_N and ϕ_0. The infinite pinning fields require that $\epsilon_0=\epsilon_N=0$. Averaging over the noise $\xi_n(t)$, Eq. (18) can be solved easily, since $\langle\xi_n(t)\rangle=0$.

The solution with the longest relaxation time is given by

$$\langle\epsilon_n(t)\rangle=A_0(N,\phi_0,\phi_N)e^{-\alpha t}\sin\left(\frac{\pi n}{N}\right) \quad , \tag{19a}$$

where

$$\alpha=\alpha_0/N^2 \quad ,$$
$$\alpha_0=\pi^2J\Gamma\cos\left(\frac{\phi_N-\phi_0}{N}\right) \quad , \tag{19b}$$

and $A_0(N,\phi_0,\phi_N)$ is determined by the initial conditions. To study the relaxation of the total magnetization $M(t)$, we apply a small magnetic field H in the $\theta=0$ direction to the system for $t\leq 0$ and calculate $M(t)$ for $t>0$ when $H=0$. Using Eq. (17) we find that

$$A_0(N,\phi_0,\phi_N)=-\frac{2N}{\pi^2}H\sum_{n=0}^{N-1}\sin\left(\frac{\phi_N-\phi_0}{N}n+\phi_0\right)\sin\frac{\pi n}{N} \quad . \tag{20}$$

The magnetization is given by

$$M(t)=\sum_{N=1}^{\infty}p^2(1-p)^{N-1}\sum_{n=0}^{N-1}\int_{-\pi}^{\pi}\frac{d\phi_0}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi_N}{2\pi}\cos\left(\frac{\phi_N-\phi_0}{N}n+\phi_0+\epsilon_n(t)\right) \quad . \tag{21}$$

Linearizing Eq. (21) in $\epsilon_n(t)$ and performing the sum over n and the integrals over ϕ_0 and ϕ_N we find

$$M(t)\sim\sum_{N=1}^{\infty}N^3(1-p)^{N-1}e^{-\alpha t} \quad . \tag{22}$$

Replacing the sum by an integral we get

$$M(t)\sim t^{4/3}\int_0^{\infty}x^3\exp\left[-t^{1/3}\left(\frac{\alpha_0}{x^2}+|\ln(1-p)|x\right)\right]dx \quad , \tag{23}$$

where $x=N/t^{1/3}$. Using the saddle-point approximation we find that for large times $M(t)$ is given by

$$M(t)\sim t^{7/6}e^{-ct^{1/3}} \quad , \tag{24}$$

where

$$c=3\times 4^{-1/3}\alpha_0^{1/3}|\ln(1-p)|^{2/3} \quad . \tag{25}$$

A similar model for diffusion in a medium with randomly distributed traps has been studied.[11] It has been shown that the density of particles in such a medium decays as $e^{-t^{1/3}}$ for long times. The prefactor $t^{7/6}$ in (24) is specific to $M(t)$. Analysis of a different thermodynamic function is expected to yield a different prefactor.

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