Superfluid density, penetration depth, and integrated fluctuation conductivity of a model granular superconductor

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(Received 25 April 1983)

We study via Monte Carlo simulation the effective superfluid density n_s and the real part of the integrated fluctuation conductivity, γ_2, of a model granular superconductor in which the individual superconducting grains are coupled via Josephson tunneling. The phase-ordering transition temperature T_c is determined as the temperature at which n_s goes to zero. Above an intergrain normal-state resistance $R \sim R_0 = \hbar/e^2$, T_c falls significantly below the single-grain transition temperature T_{c0}, in agreement with our previous Monte Carlo results, and n_s deviates substantially from typical bulk behavior. At temperature $T=0$, we show analytically that n_s in site-diluted samples is proportional to the effective conductance of the sample in its normal state. It follows that the zero-temperature penetration depth $\lambda_p(0)$ of the granular superconductor varies as the square root of the normal-state resistivity. Near percolation, $\lambda_p(0) \propto (p-p_c)^{-t/2}$, where t is the percolation exponent describing effective conductivity in composites of normal metal and insulator. A sum rule is derived for γ_2, relating it to the Josephson coupling energy. γ_2 is found to have two characteristic contributions. One is due to thermodynamic fluctuations and appears near T_c in ordered and weakly diluted lattices of superconducting grains. The other arises from "impurity modes" associated with sites near vacancies in site-diluted lattices. This contribution persists at all temperatures near or below T_c, and dominates over the first contribution above a site dilution of about 10%. The possibility of observing these effects experimentally is discussed.

I. INTRODUCTION

Superconducting composites have recently been the subject of intense investigation, both experimental and theoretical.[1,2] Such materials are typically composed of one superconducting (S) constituent and one nonsuperconducting (N), the latter being a normal metal, a superconductor with a lower transition temperature, a semiconductor, or even an insulator such as air. They are of interest because their various properties (e.g., specific heat,[3,4] resistivity versus temperature,[5-10] upper critical field[11,12]) are quite different from those of ordinary, true superconductors. The reasons for this behavior are quite complex, having to do with percolation effects in disordered composites[13-15] and with thermodynamic fluctuations in both ordered and disordered superconductors.[16-21] The interplay between these influences is still being unraveled.

In this paper, we describe a study of a model superconducting composite which shows effects of both percolation and thermodynamic fluctuations. The model, first introduced by Deutscher *et al.*[16] and since then subjected to a number of studies,[17,18] is here investigated by Monte Carlo simulation supplemented by several exact results. The principal quantities to be calculated are the effective superfluid density and the fluctuation conductivity integrated over frequencies. Our calculations thus supplement our earlier Monte Carlo studies of the same model,[18] in which purely thermodynamic quantities, in particular, the specific-heat and amplitude and phase order parameters, were studied as functions of temperature and composition.

The model free-energy functional we study simulates a lattice of superconducting grains coupled together via Josephson tunneling. The lattice is described by a set of complex superconducting gap parameters $\{\psi_i\}$, one for each grain. The ψ_i's behave much like classical two-component spins of variable amplitude and phase, and the Josephson interaction is analogous to a ferromagnetic exchange integral. As has been shown previously,[22] this functional leads to two superconducting transitions, one at T_{c0} where the single grains become superconducting, the second at a temperature $T_c < T_{c0}$ where the superconducting grains phase lock to produce a state with long-range phase coherence, analogous to a ferromagnetic state in a spin system. Only this lower transition is a true phase transition, with a divergent correlation length.

It is straightforward to calculate the superfluid density n_s of this model, using Monte Carlo methods. According to the second London equation, n_s is proportional to the current induced when an external vector potential is applied to the sample. Applying a vector potential is equivalent to imposing a "twist" on the phases of the gap parameters. This connection then leads to the definition of n_s as an effective "spin-wave stiffness constant" for the sample. This stiffness constant can be expressed as an average, within the canonical ensemble, of a certain function of the ψ_i's; it can therefore be evaluated by the same methods used in our earlier papers. In like manner, we can calculate the equal-time current-current correlation function. Because of the Kubo relation, this function is proportional to the frequency integral of the real part of the conductivity.

In several cases, we are able to supplement our Monte

Carlo results with exact statements obtained analytically. Perhaps the most useful of these refers to diluted Josephson-coupled lattices at $T=0$. In this case, n_s is shown to be proportional to the effective conductivity of the same diluted lattice in its normal state. This leads to the prediction that the zero-temperature penetration depth $\lambda_p(0)$, which varies as $n_s^{-1/2}$, diverges near the percolation threshold p_c as $(p-p_c)^{-t/2}$ where t is the familiar exponent describing the variation of effective conductivity with p in a composite of normal metal and insulator. This prediction remains to be tested experimentally.

We turn now to the body of the paper. The model on which our calculations are based is described in Sec. II, and several exact results are obtained. Section III presents the results of the Monte Carlo simulations and a brief discussion follows in Sec. IV.

II. FORMALISM

A. Model and definitions

The basis of our calculations is the following model for the Helmholtz free energy of a granular superconductor (in units such that k_B, the Boltzmann constant, is unity):

$$F=-T\ln Z\ ,$$
$$Z=\int\left[\prod_{i=1}^{N_G}d^2\psi_i\right]\exp\left[-\frac{T_{c0}}{T}\left[\sum_i\frac{T-T_{c0}}{T_{c0}\delta}|\psi_i|^2+\sum_i\frac{0.053}{\delta}|\psi_i|^4+\sum_{\substack{i,j\\ i>j}}\frac{\pi}{16}\frac{R_0}{R_{ij}}|\psi_i-\psi_j|^2\right]\right] \tag{1}$$
$$\equiv\int\left[\prod_{i=1}^{N_G}d^2\psi_i\right]\exp\left[-\frac{\mathcal{F}}{T}\right]\ .$$

Here $\psi_i=|\psi_i|\exp(i\phi_i)$ is the dimensionless complex energy-gap parameter for the ith grain, related to the BCS energy gap by $\psi_i=T_{c0}\Delta_i$, T_{c0} is the single-grain transition temperature, δ is a dimensionless size parameter defined by $\delta=1/[N(0)vT_{c0}]$, where $N(0)$ is the electronic density of states per unit volume at the Fermi energy, and v is the grain volume; R_{ij} is the normal-state tunneling resistance between the ith and jth grains, and $R_0=\hbar/e^2\sim4000\ \Omega$ is a characteristic resistance. The integrals in Eq. (1) run over all complex values of the gap parameters.

The physics on which Eq. (1) is based is described in Refs. 16–18. In this paper, we assume that the grains lie on the sites of a simple-cubic lattice, introducing quenched disorder by site dilution. We also assume that there is Josephson coupling only between nearest-neighbor grains. The normal-state resistance between such grains, R_{ij}, will be denoted R. Our results will thus depend on δ, R, the absolute temperature T, and the volume fraction p of grains present. Although the phase-ordering transition in two dimensions is of particular interest,[19–21] it will not be considered in this paper.

One quantity of concern in this paper is the effective superfluid fraction γ, defined by

$$\gamma=\frac{1}{N_b}\left[\left\langle\sum_{\langle ij\rangle}{}'C_{ij}|\psi_i||\psi_j|\cos(\phi_i-\phi_j)\right\rangle-\frac{1}{T}\left\langle\left[\sum_{\langle ij\rangle}{}'C_{ij}|\psi_i||\psi_j|\sin(\phi_i-\phi_j)\right]^2\right\rangle\right]$$
$$\equiv\gamma_1-\gamma_2\ . \tag{2}$$

Here the primes denote that the sums are to be carried out over bonds parallel to the z axis, and C_{ij} is the coupling between grains i and j [$C_{ij}=\pi T_{c0}R_0/(16R_{ij})$]. The large angular brackets in Eq. (2) denote a thermodynamic average. Such averages are evaluated according to the prescription

$$\langle O\rangle=Z^{-1}\int\left[\prod_{i=1}^{N_G}d^2\psi_i\right]O(\psi_1,\ldots,\psi_{N_G})\exp\left[-\frac{\mathcal{F}}{T}\right]\ , \tag{3}$$

where O is any operator which depends on $\psi_1,\ldots,\psi_{N_G}$, and N_b is the number of bonds parallel to the z axis, whether present or absent.

The quantity (3) is simply a generalization of the "helicity modulus," introduced by Fisher, Barber, and Jasnow for liquid He,[23] to an order parameter with amplitude as well as phase variations. To see its connection with the superfluid density n_s, we note that in the presence of a vector potential $\vec{A}(\vec{x})$, the coupling term in $\mathcal{F}$ must be generalized as follows:

$$C_{ij}|\psi_i||\psi_j|\cos(\phi_i-\phi_j)\rightarrow C_{ij}|\psi_i||\psi_j|\times\cos(\phi_i-\phi_j+A_{ij})\ ,$$
$$A_{ij}=\frac{2e}{hc}\int_{(i)}^{(j)}\vec{A}\cdot\vec{dl}\ , \tag{4}$$

where the integral is to be taken between the centers of the ith and jth grains. Now imagine a cubic sample with periodic boundary conditions, and let the vector $\vec{A}$ be uniform and in the z direction (the periodic boundary conditions are necessary in order to prevent the sample from eliminating the vector potential via a redefinition of the phase variables). The helicity modulus γ is then easily shown to be

$$\gamma=\frac{h^2}{N_b a^2}\left[\frac{\partial^2 F}{\partial A^2}\right]_T\ , \tag{5}$$

where a is the separation between adjacent grain centers, and $\vec{A}=A_z\hat{z}$. Since the Josephson current density in the z

direction is

$$J_z=\frac{c}{2V}\left[\frac{\partial F}{\partial A}\right]_T , \tag{6}$$

where c is the speed of light, it follows that

$$\gamma=\frac{h^2}{N_b a^2}\frac{2V}{c}\left[\frac{\partial J_z}{\partial A}\right]_T=\frac{n_s}{n_0}\frac{h^2}{m^* a^2} , \tag{7}$$

where $n_0=V/N_b$ and we have used the second London equation, $\vec{J}=(n_s e^{*2}/m^* c)\vec{A}$, expressing the linear relation between current density and field in a superconductor. By a similar argument we can express the Josephson penetration depth into the superconducting array as

$$\frac{1}{\lambda_p^2}=\frac{4\pi}{(hc)^2}\frac{e^{*2}}{a}\gamma . \tag{8}$$

Note that λ_p is quite different from λ_g, the penetration depth of individual grains within the superconductor. The latter are assumed to be much larger than the grain size in the present model.

The other quantity studied in the present work is γ_2, the second term in Eq. (2). Since the Josephson current between links i and j is $I_{ij}=C_{ij}\sin(\phi_i-\phi_j)$, we can rewrite the second term in expression (2) as an equal-time current-current correlation function,

$$\gamma_2=\frac{1}{T}\frac{\hbar^2 a}{e^{*2}}\frac{1}{V}\int\int \vec{d}x\,\vec{d}x'\langle J_z(\vec{x},0)J_z(\vec{x}',0)\rangle , \tag{9}$$

V being the volume of the sample. This quantity may be connected to the Josephson fluctuation conductivity via the Kubo formula. In the classical limit[24] ($\hbar\omega \ll k_B T$), the Kubo formula gives (assuming an isotropic conductivity)

$$\mathrm{Re}\sigma(\vec{q}=0;\omega)=\frac{1}{T}\frac{1}{V}\int\int \vec{d}x\,\vec{d}x'\int_0^\infty dt\cos(\omega t)\langle J_z(\vec{x}t)J_z(\vec{x}'0)\rangle , \tag{10}$$

and hence

$$\gamma_2=\frac{\hbar^2 a}{\pi e^{*2}}\int_{-\infty}^{\infty}\mathrm{Re}\sigma(\omega)d\omega . \tag{11}$$

Evaluation of the fluctuation conductivity itself, rather than just its frequency integral, cannot be done by a Monte Carlo calculation such as in this paper, but would require a full dynamical calculation. Note also that the conductivity defined by (10) must be added to any other contributions to the conductivity, in particular, the normal-state conductivity.

It should be pointed out that our basic (1) does not include the path integral over the fluctuating vector potential A, i.e., thermodynamic fluctuations in the vacuum-electromagnetic-field energy. For a type-I superconductor, Halperin, Lubensky, and Ma[25] have shown that these fluctuations convert the superconducting transition to a weakly first-order one, although the effect is evident only over a few microdegrees. In the type-II limit, it appears (Dasgupta and Halperin[26]) that the superconductor has an inverted XY transition. Boyanovsky and Cardy[27] have treated the effect of impurities on this transition, within the ϵ expansion. In principle, such fluctuations should be included in our partition function, and it would probably be possible to do so within our Monte Carlo approach. It is quite difficult to predict their effect on measurable quantities in granular superconductors, without such calculations.

B. Some exact results

A number of exact statements can be made about both γ and the fluctuation conductivity based on the above expressions. Our previous Monte Carlo studies[18] have shown that the free-energy functional [Eq. (1)] has a phase transition in the T-p plane to a state with long-range phase coherence. Above the temperature $T_c(p)$ of this phase transition, γ vanishes. This implies that for $T>T_c(p)$,

$$\gamma_2=\gamma_1 . \tag{12}$$

The quantity on the right-hand side of Eq. (12) is simply (to within a constant factor) that portion of the energy which is sensitive to phase coherence. Assuming that amplitude fluctuations are irrelevant, this energy varies near $T_c(p)$ as

$$\gamma_1\sim(T-T_c)^{1-\alpha} , \tag{13}$$

where α is the specific-heat exponent for the three-dimensional x-y model (current estimates[28] give $\alpha\sim 0.01$, corresponding to a cusp in the specific heat). Equation (12) represents a sum rule for the fluctuation conductivity. It is amusing that Eq. (13) is identical to the behavior proposed by Fisher and Langer[29] for the dc conductivity in a ferromagnet near the magnetic ordering transition, although there appears to be no deep reason for the connection.

We turn next to the behavior of n_s near $T=0$ in diluted simple cubic lattices. The lattices will be assumed to be $N\times N\times N$ and we will obtain γ by considering the free energy (1) in the presence of a constant vector potential in the z direction, requiring periodic boundary conditions and taking $\vec{A}$ to be sufficiently small that the sum of the shifts A_{ij} in the z direction is much less than $\pi/2$. Near $T=0$, adjacent phases will certainly be nearly equal if they are connected by nonzero bonds, and so the cosine factors in (4) can be expanded in powers of their arguments. The resulting free energy takes the form

$$F=-T\ln\int\left[\prod_i |\psi_i|\,d\,|\psi_i|\right]\exp\left[-\frac{T_{c0}}{T}\left[\sum_i\frac{T-T_{c0}}{T_{c0}\delta}|\psi_i|^2+\sum_i\frac{0.053}{\delta}|\psi_i|^4+\sum_{i>j}\frac{\pi}{16}\frac{R_0}{R_{ij}}(|\psi_i|^2+|\psi_j|^2)^2\right]\right]$$
$$\times\int\left[\prod_i d\phi_i\right]\exp\left[-\frac{T_{c0}}{T}\left[\sum_{i>j}\frac{\pi}{16}\frac{R_0}{R_{ij}}|\psi_i||\psi_j|(\phi_i-\phi_j+A_{ij})^2\right]\right] . \tag{14}$$

TABLE I. Parameters describing superconducting lattices studied by Monte Carlo simulation.

Sample no.	Concentration p	Size parameter δ	Intergrain resistance R/R_0
I	1.0	0.01	0.1
II	1.0	0.01	1.0
III	1.0	0.01	10.0
IV	0.9	0.01	1.0
V	0.75	0.01	1.0
VI	0.5	0.01	1.0

We may estimate the second integral very accurately at low temperatures by replacing $|\psi_i|$ by its average. Then the phase and amplitude integrals in the free energy can be carried out separately, and the two terms in the free energy decouple. The first term is independent of A and does not affect γ. The second integral can be rewritten by a change of definition of the phase variables,

$$\phi_i' = \phi_i + \frac{e^*}{hc} z_i A \ , \tag{15}$$

where z_i is the z coordinate of the ith particle, with the new boundary condition

$$\phi(\vec{x}_i) = \begin{cases} 0, & z_i = 0 \\ Na\dfrac{e^*}{hc}A, & z_i = Na \ . \end{cases} \tag{16}$$

In the limit $T \to 0$, the second term of Eq. (14) contributes to the free energy an amount

$$\Delta F = \min \left[T_{c0} \left[\frac{\pi}{16} \sum_{\substack{i,j \\ i>j}} \frac{R_0}{R_{ij}} |\psi_i| \, |\psi_j| \, (\phi_i' - \phi_j') \right] \right] , \tag{17}$$

where min[ ] denotes the minimum value of the quantity in large parentheses subject to the constraints (16). If we interpret $Na(e^*/hc)A$ as a potential drop in the z direction, then (17) is simply the Joule heating in a random-resistor network in which the phases play the role of potentials, and the couplings C_{ij} correspond to conductances. We have

$$C_{ij} = T_{c0}(\pi/16)(R_0/R_{ij}) \, |\psi_i| \, |\psi_j| \ .$$

Since the Joule heating, in the presence of a given potential drop, is proportional to the effective conductance $\sigma(p)$ of the network, it follows that γ, which is proportional to the second derivative of (17), satisfies the simple relation

$$\frac{\gamma(p)}{\gamma(1)} = \frac{\sigma(p)}{\sigma(1)} \ . \tag{18}$$

This relationship is equivalent to the one obtained by Kirkpatrick[13] between conductances of diluted-resistor networks and the spin-wave stiffness constant of diluted Heisenberg ferromagnetics. It is valid for basically the same reason: γ is essentially the "spin-wave stiffness constant" of the granular superconductor.

III. RESULTS

We have calculated γ and γ_2 for a number of lattices, both ordered and site diluted, using the Monte Carlo technique described in our previous papers. Computations were carried out on lattices varying in size from $5\times5\times5$ to $12\times12\times12$, with periodic boundary conditions. Typically 2000 but sometimes as many as 10000 passes were made through the entire lattice. The parameters investigated in detail are listed in Table I.

Figure 1 shows the helicity modulus γ as a function of temperature for several ordered lattices of superconducting grains ($p=1$) with varying intergrain resistances. In all cases shown, the size parameter $\delta=0.01$; this corresponds to spherical particles of radii about 200 and 130 Å for Al and Pb, respectively. (Results for other values of δ are qualitatively similar.) All the curves show a progressive departure from bulklike behavior as R increases. For the smallest value shown ($R=0.1R_0$), γ is almost perfectly proportional to $\langle |\psi|^2 \rangle$, the superfluid density of a *bulk* superconductor. For $R=R_0$ and $10R_0$, the two differ considerably, and the phase-ordering transition temperature T_c (indicated by arrows in Fig. 1) decreases. Near T_c, γ varies as $(T_c-T)^{+\zeta}$, with $\zeta=\frac{1}{3}(2-\alpha)\sim\frac{2}{3}$; this exponent is not inconsistent with our results although our samples are far too small to test them properly. The corresponding behavior of the penetration depth λ_p is

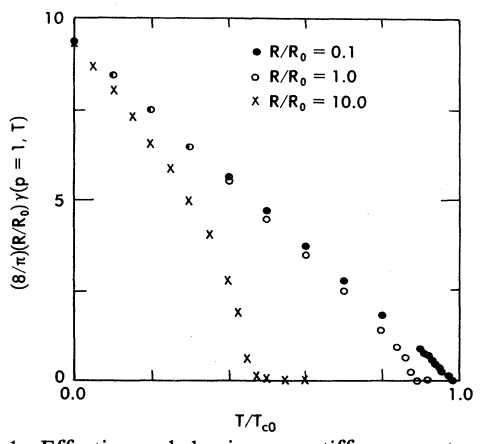

FIG. 1. Effective scaled spin-wave stiffness constant or helicity modulus $8R\gamma/\pi R_0$ as a function of temperature for an ordered lattice of superconducting grains ($p=1$) and three values of intergrain normal-state resistance.

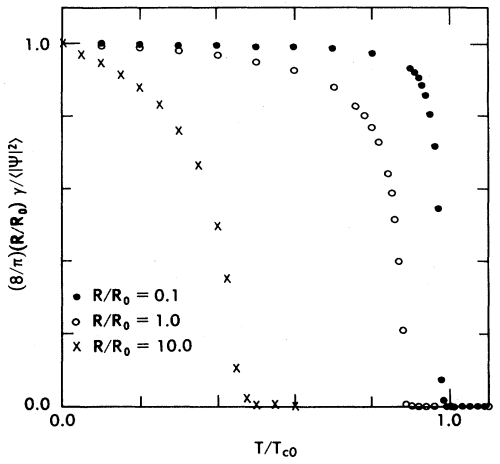

FIG. 2. Scaled helicity modulus $8R\gamma/\pi R_0$ divided by mean-squared energy gap parameter $\langle |\psi|^2 \rangle$ for the same cases as in Fig. 1.

$$\lambda_p \propto (T_c - T)^{-1/3} , \tag{19}$$

as compared to the mean-field BCS result $\lambda_{\rm BCS} \propto (T_c - T)^{-1/2}$.

The low-temperature behavior of γ is in part an artifact of the present model. The Ginzburg-Landau functional (1) tends to cause the squared average wave function to decrease linearly with temperature, even at low temperatures. The true BCS squared wave function, of course, varies at low temperature as

$$|\psi(T)|^2 = |\psi(0)|^2 - Ae^{-2\Delta(0)/T} , \tag{20}$$

where A is a constant independent of temperature. Nonetheless, even if this exponential behavior were properly included in our model, γ would vary at low temperatures as

$$\gamma(T) = \gamma(0) - A'T , \tag{21}$$

where the linear term arises from the "spin-wave excitations" of the phase variables. To show this linear term, we plot in Fig. 2 the helicity modulus normalized by the mean-square wave function,

$$\frac{\gamma(T)}{\langle |\psi_i(T)|^2 \rangle} . \tag{22}$$

Note that if a quantum-mechanical (finite-capacitance) term had been included in the free-energy functional, this linear dependence would have been replaced by a T^3 variation, as in Ref. 30.

Figure 3 shows the integrated fluctuation conductivity γ_2 for the same cases plotted in Figs. 1 and 2. The most striking feature of these results is the peak near T_c. This peak remains finite at T_c as we have confirmed by calculating the height of the peak for various sizes of lattice. Nevertheless, its position serves as an excellent means of fixing T_c; we find it is easier to locate T_c this way than from finding the point at which γ vanishes. Note also that the half-width of the peak progressively increases as R/R_0 increases, as does the height of the peak. This sug-

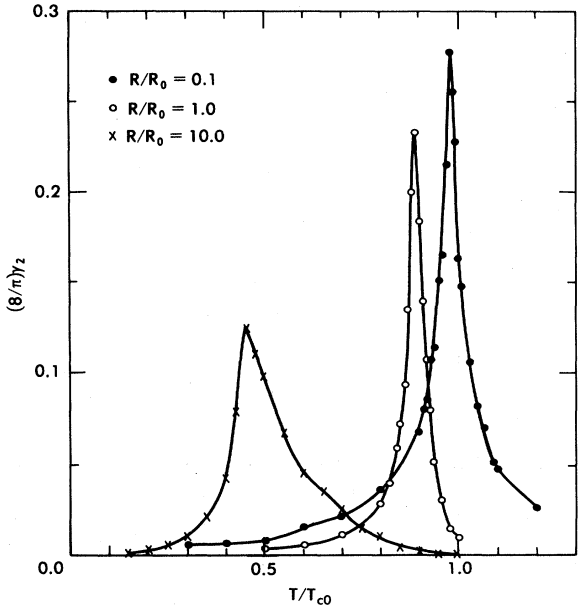

FIG. 3. Real part of the integrated fluctuation conductivity $8\gamma_2/\pi$ for the same cases as in Fig. 1. Solid lines are drawn to guide the eye.

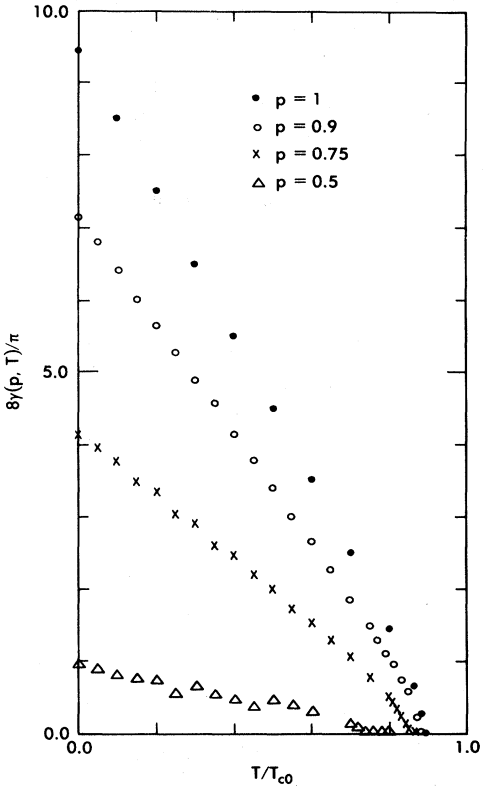

FIG. 4. Helicity modulus $8\gamma/\pi$ as a function of temperature, for several values of the site-occupation probability p. In all cases $R/R_0 = 1$ and the size parameter $\delta = 0.01$.

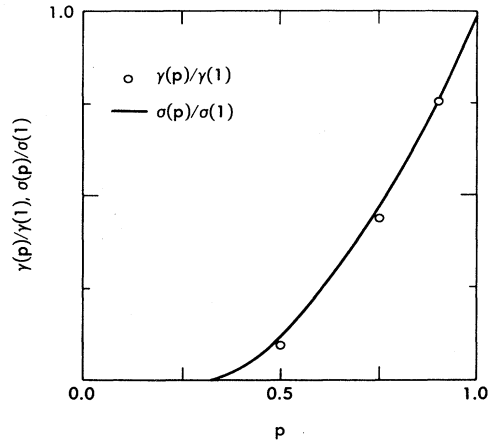

FIG. 5. Helicity modulus $\gamma(p)$ at zero temperature normalized to the value at $p=1$ for the case $R/R_0=1$ and $\delta=0.01$. Solid line is the effective conductance $\sigma(p)$ of the same lattice in its normal state, normalized to $p=1$, as calculated by Kirkpatrick (Ref. 13).

gests that fluctuation paraconductivity will be noticeable over a greater temperature range for samples with a higher normal-state resisitivity than for better-conducting lattices. Measurements on many superconducting composites tend to confirm this rough rule of thumb, although our result is strictly applicable only to the frequency-integrated conductivity.

We turn next to results for site-diluted lattices. Figure 4 shows the helicity modulus corresponding to $R/R_0=1$ and $\delta=0.01$, for several values of p. The percolation concentration for this geometry, i.e., the value of p above which the sites first form an infinite connected cluster (in the limit of an infinite sample) is $p_c=0.316$.[31] As is clear from the figure, T_c decreases monotonically with increasing p. The value of $\gamma(p)$ at $T=0$ very closely obeys Eq.

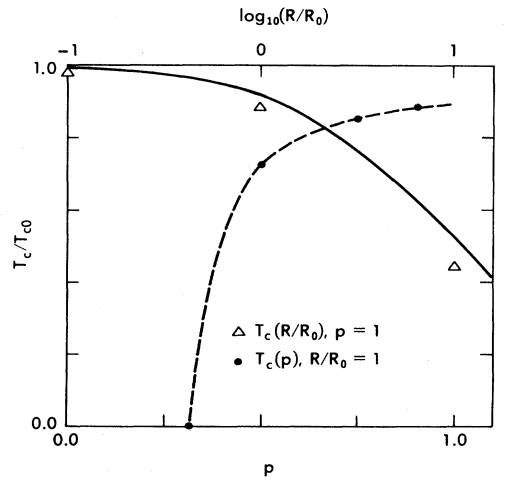

FIG. 6. Phase-ordering transition temperature T_c as a function of R/R_0 for $p=1$ and as a function of p for $R/R_0=1$. Solid curve represents the molecular-field prediction of the transition temperature [Eq. (23)]; dashed curve is merely to guide the eye.

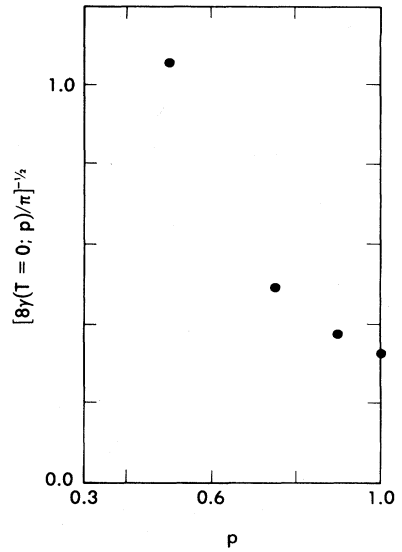

FIG. 7. Reduced zero-temperature penetration depth $\tilde{\lambda}_p(T=0;p)=[8\gamma(T=0;p)/\pi]^{1/2}$, as a function of site occupancy p.

(18) as is shown in Fig. 5, where we plot $\gamma(p)/\gamma(1)$ and $\sigma(p)/\sigma(1)$ for a simple cubic lattice.

We plot the results of Fig. 4 a different way in Fig. 6, where we show $T_c(p)$ as a function of normal-state lattice conductivity $1/(Ra)\equiv\sigma_p$ for ordered lattices, and as a function of p for the disordered lattices studied. In the case of ordered lattices, the transition temperature fairly well obeys a relation derived by treating the phase ordering within a molecular-field approximation,[32]

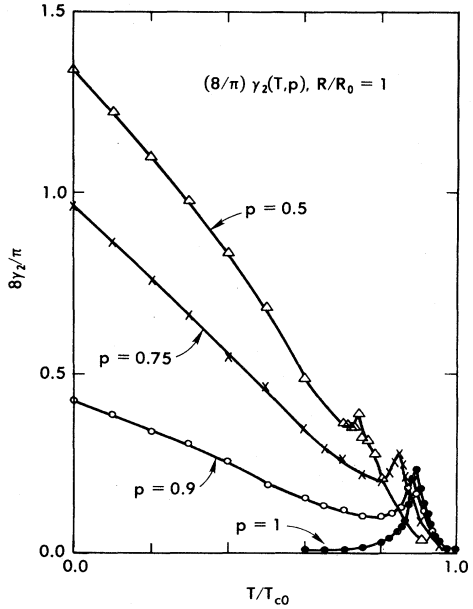

FIG. 8. Real part of the integrated fluctuation conductivity $(8/\pi)\gamma_2(T,p)$ as a function of temperature for various site occupancies p. In all cases $R/R_0=1$ and $\delta=0.01$.

$$\frac{T_c}{T_{c0}}=\frac{1}{1+(R'/R_0)},\quad R'=0.106\left[\frac{8}{3\pi}\right]R\ . \tag{23}$$

For disordered samples, we have not found a simple analytic relation expressing T_c as a function of p.

Figure 7 shows the reduced penetration depth $\tilde{\lambda}_p=\gamma^{-1/2}$ for the site-diluted samples at $T=0$, as a function of p. From Eqs. (8) and (18), the penetration depth obeys the relation $\lambda_p\propto\sqrt{\rho_n}$ at $T=0$. This is the same result long known for dirty bulk superconductors,[33] but it is here derived in a totally different way, for granular, Josephson-coupled superconductors. Also, since $\rho_n\propto(p-p_c)^{-t}$ at concentrations p close to p_c, with $t\sim 1.7$ in three dimensions, it follows that $\lambda_p(p,0)$ varies as $(p-p_c)^{-t/2}$. This result was previously proposed by Stroud and Bergman[34] on the basis of a scaling argument near percolation, but it is here derived as the $T=0$ limit of a Josephson-coupled disordered superconductor.

Figure 8 shows the integrated fluctuation conductivity γ_2 for various values of p and for $R/R_0=1$, $\delta=0.01$. In this case disorder has a striking qualitative effect: Whereas the peak in γ_2, for ordered samples, is narrowly confined to a temperature region around T_c, the analog for diluted lattices is superimposed on a broad background which persists to $T=0$. The background is due to site dilution, and presumably arises from ehanced phase fluctuations in the vicinity of the removed sites. Crudely speaking, these modes can be understood as the superconducting analog of impurity modes in the vicinity of vacancies in the lattice. The vanishing of the background for an ordered lattice can be understood by writing γ_2, in a fashion valid at low temperatures (assuming amplitude variations in the order parameter can be neglected),

$$\gamma_2=\frac{|\psi|^4}{T}\frac{\int\left[\prod_{i=1}^{N_G}d\phi_i\right]\left[\sum_{\langle ij\rangle}{}' C_{ij}\sin(\phi_i-\phi_j)\right]^2\exp\left[\beta\sum_{\substack{i,j\\ i>j}}C_{ij}|\psi|^2(\phi_i-\phi_j)^2\right]}{\int\left[\prod_{i=1}^{N_G}d\phi_i\right]\exp\left[\beta\sum_{\substack{i,j\\ i>j}}C_{ij}|\psi|^2(\phi_i-\phi_j)^2\right]}\ , \tag{24}$$

where $|\psi|^2$ is the expectation value of any of the $|\psi_i|^2$ at $T=0$ and $\beta=1/T$. For an ordered $N\times N\times N$ lattice with periodic boundary conditions, we expand the sine in a Taylor series in the phases. The first term in this expansion vanishes, and the first nonvanishing term is the cubic one, which can be shown to contribute to γ_2 an amount varying as T^2. In the diluted lattice, the leading term is temperature independent, leading to a contribution to γ_2 which remains finite at $T=0$.

IV. DISCUSSION

Of the quantities calculated in this work, the most readily comparable to experiment is the penetration depth λ_p. Recent detailed experiments have been carried out for λ_p as a function of temperature in a disordered granular superconductor, by Raboutou *et al.*[22] These workers obtain $\lambda_p\sim(T_c-T)^{-0.75}$ rather than the behavior $\lambda_p\sim(T_c-T)^{-0.3-0.4}$ which can be approximately inferred from our ordered and even our disordered samples. We can, however, bring our results into agreement with experiment if we make the additional assumption that varying T is somehow equivalent, in the samples of Raboutou *et al.*, to varying p. Then T_c is the point at which $p=p_c$, the percolation concentration. Our results then lead to $\lambda_p\sim(T_c-T)^{-t/2}$ where $t\sim 1.7$ in three dimensions. This result is in reasonable agreement with experiment.

How might p be effectively a function of temperature? Two possibilities may be suggested. One is an *explicitly* temperature-dependent p, as, for example, in a hypothetical granular superconductor in which different grains have different T_{c0}'s. There is some reason to imagine that this situation might hold in granular Al. If the grains were very strongly coupled together, then λ_p should vary as $(T_c-T)^{-t/2}$.

A second possibility is suggested by the Harris[35] criterion for the homogeneity of a phase transition in a disordered system, according to which the phase transition has the exponents of the pure system if the specific-heat exponent of the pure system $\alpha<0$. In the present case, α is that of the $d=3$ x-y model. It is believed to be negative, but only by a small margin.[28] If α were for some reason positive in granular superconducting composites, the exponents characterizing the transition might be percolative rather than x-y—like, hence producing a penetration depth varying in the manner observed by Raboutou *et al.* It would be interesting, however, to have measurements of $\lambda_p(0)$ vs p, in order to test the present theory directly.

Another possibility, not considered in this paper, is that the anomalous variation of penetration depth with temperature has something to do with fluctuations in the vector potential, as mentioned in Sec. II. If true, this would be a remarkable confirmation of the importance of these fluctuations in granular superconductors.

One of the more striking predictions of this work is the existence of two distinguishable contributions to the fluctuation conductivity $\sigma_1(\omega)$, one associated with *thermodynamic* fluctuations, and one with *composition* fluctuations (i.e., with disorder or impurities). The former is conspicuous only near $T=T_c(p)$ and, from our data, tends to be concealed by the second type of contribution except at small values of $1-p$. The disorder contribution is prominent at all temperatures and all values of p other than $p\sim 1$. Unfortunately, our calculations provide no information about the *frequency* distribution of $\sigma_1(\omega)$, but only about its integral so that an unambiguous comparison of

this result to experiment would be very difficult.[36]

To summarize, we have calculated in this paper both the effective superfluid density n_s (as well as the penetration depth implied by it) and the real part of the integrated fluctuation conductivity of a model granular superconductor, using Monte Carlo simulation in conjunction with some analytic results. We have considered both ordered and site-diluted samples. Our results suggest certain characteristic temperature and concentration dependences of the penetration depth in granular superconductors, as well as distinct effects in the fluctuation conductivity arising from both thermodynamic and composition fluctuations. It will be most interesting if a comparison between our results and experiment could be made in carefully controlled superconducting composites which approximately realize the assumption of our model.

ACKNOWLEDGMENTS

We would like to thank Professor C. Jayaprakash for a very informative discussion. This work was supported in part by the National Science Foundation (NSF) under Grant No. 81-14842 and by the NSF through the Materials Research Laboratory at The Ohio State University.

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