Interacting electrons in two-dimensional Landau levels: Results for small clusters

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We study the two-dimensional electron gas in a quantizing magnetic field for the cases of Coulomb and harmonic interactions among the electrons. Numerical solutions for the quantum states of clusters of up to five electrons show that the strength of the excitation gap is a strongly oscillating function of the density not unlike what is observed in the anomalous quantum Hall effect. We present analytic results for the case of harmonic interactions and show that the variational wave function recently proposed by Laughlin for the Coulomb problem is in fact an exact eigenstate of the harmonic problem.

I. INTRODUCTION

The recent discovery of the quantum Hall effect[1] and the anomalous quantum Hall effect[2] has attracted considerable attention to the physics of Landau levels in two dimensions.[3] The central observation is a quantization of the Hall resistivity in the form

$$\rho_{xy}=\frac{h}{e^2 i}\ , \tag{1}$$

where i is a quantum number. There is also a concomitant strong decrease in the dissipation at the Hall plateaus:

$$\rho_{xx}\rightarrow 0\ . \tag{2}$$

It is generally believed that both these effects have their origin in the existence of a gap in the excitation spectrum of the system. For integral values of the quantum number i this gap is associated with the density of states gap (or mobility gap) between the Landau levels. In the anomalous quantum Hall effect values of the Hall resistivity are observed which require i to be a rational fraction of the form $i=p/q$ with q odd. This occurs when the chemical potential lies inside the lowest Landau level and the fractional filling of the level is i. In the absence of interactions the Landau level is highly degenerate and one does not expect an excitation gap. The existence of such a gap must presumably be due to some collective many-body effect arising from the interaction of the particles. The nature of this state is presently under active investigation.[4–9]

In order to understand many-body correlations it is necessary to understand the severe effect of the magnetic field on the particle dynamics—both classically and quantum mechanically. The magnetic field so severely restricts the dynamics that it is possible to exactly diagonalize the quantum Hamiltonian for small numbers of particles (provided that Landau-level mixing is neglected). This fact was first pointed out by Bychov, Iordanskii, and Éliashberg[10] and subsequently discussed by Laughlin.[11]

It is the purpose of this paper to report on some additional investigations into the dynamics of small groups of electrons in Landau levels with intentions of providing further clues to the origin of the anomalous quantum Hall effect.

II. COULOMB INTERACTIONS

Before delving into the quantum dynamics it is useful to consider the classical limit. If one assembles a cluster of electrons, their mutual Coulomb repulsion would normally cause them to fly apart, converting the electrostatic potential energy into kinetic energy. In the presence of a strong magnetic field, however, the particles simply orbit around one another, moving perpendicularly to the local electric field. If one considers the semiclassical motion of a quantum wave packet one finds the quantum drift velocity

$$\vec{v}_d=\frac{l^2}{\hbar}\vec{\nabla}V\times\hat{B}\ , \tag{3}$$

where V is the potential energy, $\hat{B}=\vec{B}/|\vec{B}|$, and l is the magnetic length, $l^2=\hbar c/eB$. This is precisely the same as the classical $\vec{E}\times\vec{B}$ drift law [the powers of $\hbar$ in (3) cancel out].

The existence of periodic or quasiperiodic classical motion suggests discrete bound states for the quantum case. The reason one can obtain bound states with a repulsive potential is related to the degeneracy of the Landau level. The degeneracy arises because the kinetic energy is a constant independent of momentum. Hence it is impossible for the stored potential energy to be released into kinetic energy and the particles remain bound together despite their repulsion.

Let us now examine the quantum mechanics in more detail. We will deal exclusively with the lowest spin state of the lowest Landau level. In the symmetric gauge with vector potential

$$\vec{A}=\tfrac{1}{2}(xB\hat{y}-yB\hat{x})\ , \tag{4}$$

the one-body eigenfunctions are[10,11] (in units where $l=1$)

$$\psi_m(x,y)=(2\pi 2^m m!)^{-1/2}(x-iy)^m e^{-(x^2+y^2)/4}\ , \tag{5}$$

where m is a non-negative integer. The (z component of the) angular momentum of this state is $-m$. Positive angular momenta are not allowed because the $\vec{B}\cdot\vec{L}$ term in the Hamiltonian makes them energetically unfavorable (i.e., they correspond to higher Landau levels). The neglect of Landau-level mixing allows a profound simpli-

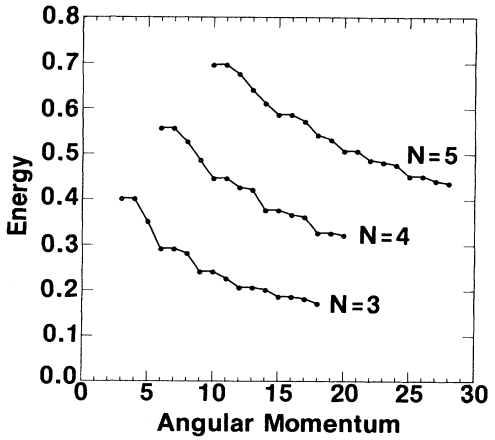

FIG. 1. Coulomb ground-state energy vs total angular momentum for clusters of $N=3$, 4, and 5 particles (in units of $e^2/\epsilon l$).

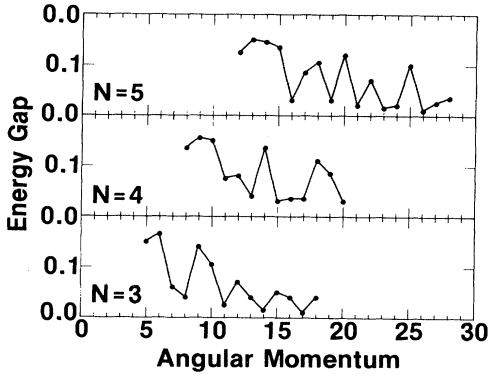

FIG. 2. Excitation gap vs total angular momentum. Note that the spacing between peaks satisfies $\Delta L=N$.

fication which makes it possible to exactly diagonalize the Hamiltonian for small numbers of particles because the number of basis states becomes finite. We take as basis states Slater determinants of the ψ_m. Because the Coulomb interaction conserves total angular momentum and because the individual angular momenta all have the same sign, the total number of allowed configurations is severely restricted. For example, with three particles ($N=3$) with total angular momentum three ($L=3$) there is only one state. It is made up of the one-particle states with $m=0,1,2$. No other combinations are allowed. This is a striking example of the severe restriction of the dynamics by the magnetic field.

The discrete Hamiltonian matrix may be evaluated knowing the matrix element of the Coulomb potential

$$M(m,n,k)\equiv\int d^2r_1\int d^2r_2\psi^*_{m+k}(r_1)\psi^*_{n-k}(r_2)\times\frac{e^2}{\epsilon r_{12}}\psi_m(r_1)\psi_n(r_2)\,, \tag{6}$$

where ϵ is the dielectric constant. In units where $e^2/\epsilon l=1$ we have the result

$$M(m,n,k)=\sum_{\alpha=0}^{m}\sum_{\beta=0}^{m+k}\sum_{\gamma=0}^{n}\binom{m}{\alpha}\binom{m+k}{\beta}\binom{n}{\gamma}\binom{n-k}{\alpha-\beta+\gamma}(-1)^{k+\alpha-\beta}\frac{\sqrt{\pi}}{2}\times\frac{2^{-2(m+n-\alpha-\gamma)}2^{-(m+n)}[2(m+n-\alpha-\gamma)]!(\alpha+\gamma)!}{[m!n!(m+k)!(n-k)!](m+n-\alpha+\gamma)!}\,, \tag{7}$$

where $\binom{m}{\alpha}$ is the binomial coefficient. The corresponding exchange matrix element may be found from

$$M_x(m,n,k)=M(m,n,n-m-k)\,. \tag{8}$$

It is now straightforward to automatically generate the basis states for a given N and L, evaluate the Hamiltonian matrix, and then find the energy eigenfunctions and eigenvalues. For the small clusters which we consider here the rank of H is under two hundred and it is therefore unnecessary to resort to any special diagonalization schemes.

In Fig. 1 we plot the ground-state energy E_0 (in units of $e^2/\epsilon l$) versus angular momentum for $N=3$, 4, and 5 (compare this with the results in Ref. 5). As the angular momentum increases the particles move further apart and the ground-state energy decreases. This decrease is not smooth however. One can see breaks in the curve at $L=3,6,9,12,\ldots$ for $N=3$, for example. This structure is made much clearer if we examine the first excitation energy $\Delta\equiv E_1-E_0$, which is shown in Fig. 2. Here we see strong periodic modulation of Δ as a function of angular momentum. For certain "magic numbers" in L the excitation gap rises dramatically.

We now consider the question of extrapolating these results for small numbers of particles to the thermodynamic limit. The relationship between filling factor ν and total angular momentum in a large homogeneous system is

$$\nu=\frac{N(N-1)}{2L}\,. \tag{9}$$

Laughlin[11] has noted that applying this for a small system with $N=3$, $L=9$ gives $\nu=\frac{1}{3}$. We can see from Fig. 2 that $L=9$ is indeed one of the magic numbers for $N=3$. On the other hand, so is $L=6$, which gives $\nu=\frac{1}{2}$, a value which is not observed experimentally.

In this same context we note that the spacing between peaks in the excitation gap obeys the rule

$$\Delta L=N\,. \tag{10}$$

Laughlin[4] has recently made an interesting suggestion for the form of the ground-state wave function and the elementary excitations. It is interesting to note that the angular momentum carried by these excitations obeys Eq. (10).

III. HARMONIC INTERACTIONS

The periodic structure in the excitation gap displayed in Fig. 2 and the appearance of magic numbers is reminis-

cent of nuclear shell structure ideas. One of the standard shell-model potentials[12] involves harmonic interactions among the particles. This has the virtue of being exactly soluble. Despite the fact that an attractive harmonic potential is completely unlike the repulsive Coulomb interaction, some interesting results may be obtained from consideration of this model.

We begin with the Hamiltonian

$$H=\sum_{i=1}^{N}\frac{1}{2m}\left[\vec{\mathrm{P}}_i+\frac{e\vec{\mathrm{A}}(\vec{\mathrm{r}}_i)}{c}\right]^2+\frac{\Lambda}{2}\sum_{\substack{i,j=1\\ i<j}}^{N}|\vec{\mathrm{r}}_i-\vec{\mathrm{r}}_j|^2 , \tag{11}$$

where i is a particle label and we again make use of the symmetric gauge. One can either view this as N two-dimensional oscillators coupled together or as one $2N$-dimensional oscillator. In any case it is exactly soluble. Again restricting ourselves to the lowest Landau level, the kinetic energy term is a constant for all the states and so may be ignored. We are left with only the potential energy which may be written (in units where $\hbar\omega_c=1$)

$$V=\lambda(V_0-v) , \tag{12}$$

where $\lambda\equiv l^2\Lambda/\hbar\omega_c$ and

$$V_0=N\left[L+\frac{N-1}{2}\right] , \tag{13}$$

$$L=\sum_i a_i^\dagger a_i , \tag{14}$$

$$v=\sum_{i,j} a_i^\dagger a_j . \tag{15}$$

Here $a_i^\dagger$ is the raising operator for the ith particle defined by (dropping the particle label):

$$a^\dagger=\tfrac{1}{2}(a_x^\dagger-ia_y^\dagger) , \tag{16}$$

$$a_x^\dagger=p_x+\tfrac{1}{2}ix , \tag{17}$$

$$a_y^\dagger=p_y+\tfrac{1}{2}iy . \tag{18}$$

There is a second type of ladder operator which mixes Landau levels. In arriving at the above results we have replaced terms containing these operators by their expectation value in the lowest Landau level.

Since V_0 and v commute we now need only diagonalize v. This is easily done by defining the vectors

$$W_1=\frac{1}{\sqrt{N}}(1,1,1,\ldots,1) ,$$

$$W_2=\frac{1}{\sqrt{2}}(1,-1,0,\ldots,0) ,$$

$$W_3=\frac{1}{\sqrt{6}}(1,1,-2,0,\ldots,0) ,$$

$$W_4=\frac{1}{\sqrt{12}}(1,1,1,-3,0,\ldots,0) ,$$

etc. Now define new operators

$$b_i^\dagger=\sum_j W_i(j)a_j^\dagger . \tag{19}$$

These operators diagonalize v

$$v=Nb_1^\dagger b_1 \tag{20}$$

and the Hamiltonian (12) becomes

$$V=\lambda N(L-b_1^\dagger b_1+N-1) . \tag{21}$$

It is straightforward to verify that $b_1^\dagger b_1$ is simply the angular momentum of the center of mass so that $L-b_1^\dagger b_1$ is the internal angular momentum. One sees that the energy levels are evenly spaced as would be expected for a harmonic system.

Having found the energy eigenvalues, how do we find the eigenvectors? The problem of antisymmetrization immediately arises. We have found the normal coordinates for the problem but we cannot simply make up a Slater determinant from functions of the normal coordinates. This is because the wave function must be antisymmetric under interchange of the particle coordinates not the normal coordinates. This is a standard difficulty which comes up in nuclear physics[12] and is well understood. For the present problem the appropriate states may be readily generated. Define the operators

$$g_i^\dagger=a_i^\dagger-\frac{1}{\sqrt{N}}b_1^\dagger . \tag{22}$$

Since $b_1^\dagger$ is symmetric under particle exchange, the $g^\dagger$ operators have the same behavior as the original $a^\dagger$ operators under particle exchange. Furthermore, $g_i^\dagger$ is nothing more than $a_i^\dagger$ with its center of mass part projected out. Thus with the use of (20)

$$v(g_i^\dagger)^m|0\rangle=0 . \tag{23}$$

This results from the fact that the $g^\dagger$ operators do not affect the center of mass motion. This means that we can construct any eigenstate from determinants of these operators

$$|\phi\rangle=(b_1^\dagger)^{m_0}\begin{vmatrix}(g_1^\dagger)^{m_1} & (g_2^\dagger)^{m_1} & \cdots\\ (g_1^\dagger)^{m_2} & \cdots & \cdots\\ \vdots & \vdots & \vdots\end{vmatrix}|0\rangle . \tag{24}$$

These states are properly antisymmetric and have eigenvalue

$$E=\lambda N\left[\sum_{i=1}^{N}m_i+N-1\right] . \tag{25}$$

It should be noted that not all these states are linearly independent because of the constraint

$$\sum_{i=1}^{N}g_i^\dagger=0 . \tag{26}$$

In typical nuclear physics problems ignoring this constraint leads to "spurious states" in which the center of mass is oscillating.[12] Because of the simplicity of the present problem we are able to explicitly project out the center of mass motion. The analog of the spurious states is that some of the correct states appear more than once in the enumeration (causing the lack of linear independence).

We are now in a position to compare the exact Coulomb

eigenfunctions and the exact harmonic eigenfunctions. This is most easily done by generating the unitary matrix

$$S_{ij}=\langle \phi_i \mid \psi_j \rangle \ , \tag{27}$$

where ϕ_i is an harmonic eigenfunction and ψ_j is a Coulomb eigenfunction. Consider now as an example the case $N=3$, $L=9$. There are a total of seven states (when the center of mass degree of freedom is included). It turns out that except for a 2×2 submatrix one obtains

$$S_{ij}=\delta_{ij} \ , \tag{28}$$

so the states are identical. Given conservation of angular momentum and center of mass motion, this result implies that there are very few allowed states of a given internal angular momentum and in many cases for small L the states are unique.[10] This again illustrates the severe restrictions placed on the dynamics by the magnetic field. One can understand this result from the following semiclassical picture: Despite the dissimilarity of the Coulomb and harmonic forces, the semiclassical trajectories are very similar. The particles rotate around each other in closed orbits with the only difference being in the direction of the motion and the speed as a function of the radius. For a harmonic potential the particles travel at a constant angular velocity independent of radius (hence the equispaced quantum levels). For the Coulomb potential the speed decreases with radius.

These considerations suggest that the overlap of the harmonic and Coulomb eigenstates will fall off rapidly with increasing N. This is indeed the case. For the simple example of the $N=3$, $L=9$ case the 2×2 nondiagonal submatrix of S arises from the fact that the two highest harmonic states are degenerate (because there are two different states with the same internal angular momentum). Thus the submatrix of S is arbitrary (and could be chosen to be diagonal). These degeneracies proliferate with increasing N and L and so the harmonic states probably do not form a useful basis for the Coulomb problem. Of course, this does not mean that some particular linear combination of degenerate harmonic states cannot be a close approximation to the Coulomb ground state.

With this in mind we now discuss the very interesting suggestion put forth by Laughlin[4] for an approximate Coulomb ground-state wave function

$$\psi=\prod_{\substack{j,k\\ j<k}}(z_j-z_k)^m \exp\left[-\tfrac{1}{4}\sum_i \mid z_i \mid^2\right] , \tag{29}$$

where $z_i=x_i-iy_i$. If m is an odd integer, then ψ is properly antisymmetric and it turns out that the filling factor is $\nu=1/m$. Taking note of the raising operator relation

$$a_i^\dagger \mid m_i \rangle=(m_i+1)^{1/2} \mid m_i+1 \rangle \tag{30}$$

and noting the single-particle wave function given by (5), we see that we have the representation in this basis

$$a_i^\dagger=\frac{1}{\sqrt{2}} z_i \ . \tag{31}$$

Hence Laughlin's state may be represented by (ignoring normalization)

$$\mid \psi \rangle=\prod_{\substack{j,k\\ j<k}}(a_j^\dagger-a_k^\dagger)^m \mid 0 \rangle \ . \tag{32}$$

But using (22), we have

$$\mid \psi \rangle=\prod_{\substack{j,k\\ j<k}}(g_j^\dagger-g_k^\dagger)^m \mid 0 \rangle \ . \tag{33}$$

Hence Laughlin's state, which is an approximate ground state for the Coulomb problem, is an exact eigenstate of the harmonic problem.[13]

IV. CONCLUSION

Following Bychov *et al.*[10] and Laughlin,[11] we have investigated by various analytical and numerical means the properties of small clusters of electrons in two-dimensional Landau levels. We have tried to present a physical picture, based on semiclassical considerations, of the great restriction placed on the dynamics of these systems by the magnetic field. Our main numerical result is that the lowest excitation energy gap is found to be a strongly modulated function of the total angular momentum (approximately inverse density). These considerations may shed some light on the reason for the existence of an excitation gap in the thermodynamic limit at certain special densities as is apparently necessary to explain the anomalous quantum Hall effect.

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