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Longitudinal dielectric behavior of a two-dimensional electron gas in a uniform magnetic field

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A new exact closed-form expression is given for the random-phase-approximation longitudinal polarizability function of a two-dimensional electron gas in a magnetic field. Detailed approximations are given for several limiting cases. Static screening and the nonretarded plasmon dispersion relation are examined in detail.

I. INTRODUCTION

The purpose of this paper is to present a tractable formulation of the longitudinal dielectric screening properties of a two- (or three-) dimensional (2D or 3D) electron gas in a static uniform magnetic field. A preliminary report of this analysis was presented in Ref. 1(a) and the development of these results will be fully explained here along with a discussion of some of their physical ramifications. The longitudinal dielectric screening properties of a 2D quantum plasma have been investigated before; specifically by Chiu and Quinn[1(b)] and Horing *et al.*[2]

Chiu and Quinn obtained the complete magnetopolarizability tensor for the system by solving the linearized equation of motion for the density matrix in the random-phase approximation (RPA). In their notation this paper is concerned only with the component χ_{yy} of this tensor. Their expression for this quantity (in the zero-temperature limit) is

$$\chi_{yy}(q,\omega)=-\frac{Ne^2}{m\omega_c^2\Omega^2}(1+I_{yy}) ,$$

$$I_{yy}=-1-2\frac{\Omega^2}{\chi^2}\left[1+\sum_{\alpha=-\infty}^{\infty}\frac{\Omega^2}{\alpha^2-\Omega^2}J_\alpha^2(X)\right]$$

$$-\delta\Omega^4\frac{1}{X}\frac{\partial}{\partial X}\sum_{\alpha=-\infty}^{\infty}\frac{J_\alpha^2(X)}{\alpha^2-\Omega^2} ,$$

where $\Omega=\omega/\omega_c$, $X=qv_F/\omega$, and

$$\delta=\frac{\pi c}{2\pi\zeta}\sum_{l=1}^{\infty}\frac{(-1)^l\cos(2\pi l\zeta/\omega_c)}{l}$$

(we have set $\hbar=1$ and ζ is the chemical potential). Chiu and Quinn in their treatment emphasize temperature high enough that the oscillatory term in δ is negligible and values of the field such that only one of the pole terms need be retained, so they can approximate the corresponding Bessel function appropriately to the long- and short-wavelength limits in discussing the properties of the magnetoplasmon spectrum.

Horing *et al.*[2] have studied the quantity χ_{yy}, which in their notation is $-(e^2/k^2)\mathrm{Im}I$, by solving the RPA integral equation for the longitudinal dynamic nonlocal inverse dielectric function. They then present a variety of series expansions appropriate to low wave numbers. In particular, they obtain various representations for the oscillatory terms neglected by Chiu and Quinn.[1b] Horing and Yildiz[2] go on to apply their results to the plasmon dispersion relation in the nonretarded limit, look at the de Haas—van Alphen (dHvA) structure of the oscillator strengths for the Bernstein resonances, and study static screening in the high-field quantum limit.

Our aim in this paper is to replace the infinite summations in the work of Chiu and Quinn and Horing and Yildiz by exact closed-form expressions in terms of a simple integral over a finite range. These expressions are valid for all values of the field, wave number, frequency, and temperature. They have the advantage of isolating the cyclotron resonance and dHvA oscillatory behavior, have simple expressions in the local and low-frequency limits, which reproduce the results of Chiu and Quinn and Horing and Yildiz, and are numerically tractable for other ranges of the parameters.

The starting point for the calculation is essentially Eq. (22) of Horing and Yildiz,[2] but to make the paper more self-contained we start off in Sec. II with a brief derivation of this result. In Sec. III we show how Horing and Yildiz's integrals can be performed exactly and show how this leads directly to Maldague's[3] finite-temperature generalization of Stern's[4] polarizability formula in the zero-field limit. In Sec. IV we perform the manipulation required to obtain the desired integral representation. In Sec. V we present various limits for the polarizability, along with some numerical results for the case where up to four Landau levels are occupied. We also examine the plasmon spectrum in the nonretarded limit and static screening in this case.

Although the concern of this paper is the 2D magnetoplasma, our method applies equally well for the 3D system, and for comparison, the two cases have been presented in parallel.

II. FORMALISM

The methodology adopted is that of Martin and Schwinger.[5] It was first noted by Matsubara[6] that the canonical density matrix is formally equivalent to the quantum-mechanical time-evolution operator if time is identified with imaginary temperature. Hence standard field-theoretic techniques can be used if the results can be analytically continued from the real- to the imaginary-time axis. The analysis of the longitudinal magnetoplasma dielectric properties in terms of thermodynamic Green's functions has been carried out by Horing[7] and we shall rely, when possible, on his results. The inverse dielectric function $K(1,2)$ [n denotes a space-time point $(\vec{r}_n,t_n)$] is defined in terms of the effective potential $V(1)$ at point 1 in a medium due to a weak local disturbance $U(2)$:

$$V(1)=\int K(1,2)U(2)\ . \tag{1}$$

For a medium described by the Hamiltonian $\mathscr{H}$, K is the thermodynamic average with respect to the canonical weight function

$$\rho(\mathscr{H})=e^{-\beta\mathscr{H}}=e^{-i\tau\mathscr{H}}\ , \tag{2}$$

which can be viewed as the translation operator through the imaginary-time interval $\tau=-i\beta$. The necessary averages with respect to $\rho(\mathscr{H})$ have period τ, so that the equation for K can be referred to the "time" interval $[0,\tau]$. For spatial and temporal homogeneity we have

$$K(1,2)=K(\vec{r}_1-\vec{r}_2;t_1-t_2)\ , \tag{3}$$

and as a consequence of this periodicity we can perform a Fourier analysis and write

$$K(\vec{p},t_1-t_2)=\frac{1}{\tau}\sum_{\nu\ \text{even}} e^{(i\pi\nu/\tau)(t_1-t_2)}k(\vec{p},\nu)\ . \tag{4}$$

This then leads to the spectral representation

$$K(\vec{p},t_1-t_2)=\delta(t_1-t_2)+\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t_1-t_2)}\left[\frac{k(\vec{p},\omega^+)}{1-e^{-i\omega\tau}}+\frac{k(\vec{p},\omega^-)}{1-e^{i\omega\tau}}\right]\ , \tag{5}$$

or, in terms of the spectral weight function

$$A(\vec{p},\omega)=i[k(\vec{p},\omega^+)-k(\vec{p},\omega^-)]\ , \tag{6}$$

to

$$K(\vec{p},t_1-t_2)=\delta(t_1-t_2)+\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega(t_1-t_2)}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\left[\mathscr{P}\frac{1}{\omega-\omega'}-\pi\cot\left[\frac{\omega\tau}{2}\right]\delta(\omega-\omega')\right]A(\vec{p},\omega')\ , \tag{7}$$

where

$$k(\vec{p},z)=1+\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{A(\vec{p},\omega)}{z-\omega}\ . \tag{8}$$

This provides the necessary analytic continuation since the integral in (8) is analytic off the real z axis and vanishes as $|z|\to\infty$ in the upper and lower half-planes. Martin and Schwinger have shown that the physical value of $K(\vec{r};t)$ is the Fourier transform of $k(\vec{p},\omega^+)$.

Now, from (1),

$$K(1,2)=\frac{\delta V(1)}{\delta U(2)}\ , \tag{9}$$

and for two-body interactions with potential $v(\vec{r})$,

$$V(1)=U(1)-i\int d\vec{r}_3 v(\vec{r}_1-\vec{r}_3)G_1(\vec{r}_3,t_3;\vec{r}_3,t_3^+)\ , \tag{10}$$

where G_n is the n-particle Green's function. Thus, from (9),

$$K(1,2)=\delta(1,2)-i\int v(1-3)\,\frac{\delta G_1(3,3^+)}{\delta U(2)}\ . \tag{11}$$

However, by the Schwinger variational principle,

$$\frac{\delta G_1(3,3^+)}{\delta U(2)}=G_1(2,2^+)G_1(3,3^+)-G_2(3,2;3^+,2^+)\ . \tag{12}$$

Therefore,

$$K(1,2)=\delta(1,2)+i\int v(1-3)[G_2(3,2;3^+,2^+)-G_1(2,2^+)G_1(3,3^+)]\ . \tag{13}$$

The RPA is obtained by introducing a Hartree decoupling of the two-particle Green's function

$$G_2(3,2;3^+,2^+)=G_1^0(3,3^+)G_1^0(2,2^+)\ ,$$
$$\frac{\delta G_1(1,1')}{\delta V(2)}=G_1^0(1,2)G_1^0(2,1')\ , \tag{14}$$

in which case (11) becomes

$$K(1,2)=\delta(1,2)-i\int\int v(1-3)\frac{\delta G_1(3,3^+)}{\delta V(4)}\frac{\delta V(4)}{\delta U(2)}\ , \tag{15}$$

or

$$K(1,2)=\delta(1,2)-i\int\int v(1-3)G_1^0(3,4)\times G_1^0(4,3^+)K(4,2)\ . \tag{16}$$

(The superscript 0 denotes the absence of interactions.) This is the RPA integral equation for the inverse dielectric response function. Because of spatial homogeneity, (16) can be solved by Fourier transforms and we obtain

$$k(\vec{p},\nu)=[1+iv(\vec{p})I(\vec{p},\nu)]^{-1}\ , \tag{17}$$

where

$$I(\vec{p},\nu)=\int_0^{\tau}dt\,e^{-i(\pi\nu/\tau)t}\times\int\frac{d\vec{q}}{(2\pi)^d}G_1^0(\vec{q};-t)G_1^0(\vec{q}-\vec{p};t) \tag{18}$$

(d is the dimensionality). Finally, after carrying through the analytic continuation described above, we obtain the RPA dielectric function

$$\epsilon(\vec{p},\omega)=[K(\vec{p},\omega^+)]^{-1}=1-v(\vec{p})\mathrm{Im}I\left[\vec{p},\frac{\tau\omega^+}{\pi}\right]+iv(\vec{p})\mathrm{Re}I\left[\vec{p},\frac{\tau\omega^+}{\pi}\right]\ . \tag{19}$$

The analytic properties of the retarded and advanced Green's functions for the noninteracting electron gas allow us to distort the t contour to run along the real-time axis, so (18) can be written

$$I\left[\vec{p},\frac{\tau\omega^+}{\pi}\right]=2i\int_0^{\infty}dt\,e^{i\omega^+t}\mathrm{Im}\int\frac{d\vec{q}}{(2\pi)^d}G_<^0(\vec{q};-t)G_>^0(\vec{q}-\vec{p};t)\ . \tag{20}$$

The Green's functions for two and three dimensions have been evaluated by Horing[7] and Horing and Yildiz.[2] From their results we have, for 2D,

$$\mathrm{Im}I\left[\vec{p},\frac{\tau\omega^+}{\pi}\right]=2i\int_{-\infty}^{\infty}d\omega' f_0(\omega')\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{\omega' s}\frac{m\omega_c}{4\pi}\coth\frac{\omega_c s}{2}\times\int_0^{\infty}dt\,e^{-i\omega t}\left[\exp\left[\frac{p^2}{2m\omega_c}\frac{\cos[\frac{1}{2}\omega_c(2t-is)]-\cosh(\omega_c s/2)}{\sinh(\omega_c s/2)}\right]-(i\to -i)\right]\quad(0<c<1) \tag{21}$$

and, for 3D,

$$\mathrm{Im}I\left[\vec{p},\frac{\tau\omega^+}{\pi}\right]=2i\int_{-\infty}^{\infty}d\omega' f_0(\omega')\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{s\omega'}(8\pi^{3/2})^{-1}\left[\frac{2m}{s}\right]^{1/2}\frac{m\omega_c}{\tanh(\frac{1}{2}\omega_c s)}\times\int_0^{\infty}dt\,e^{-i\omega t}\left[\exp\left[-\frac{p_z^2}{8ms}[(2t-is)^2+s^2]\right]\times\exp\left[\frac{p_\perp^2}{2m\omega_c}\frac{\cos\frac{1}{2}[\omega_c(2t-is)]-\cosh(\frac{1}{2}\omega_c s)}{\sinh(\omega_c s/2)}\right]-(i\to -i)\right] \tag{22}$$

[$\omega_c = |eH|/mc > 0$ is the cyclotron frequency, p_z ($p_\perp$) is the component of $\vec{p}$ parallel (perpendicular) to the magnetic field $\vec{H}$]. This is all that is required to get $\mathrm{Re}\epsilon = 1 + \chi(\vec{p},\omega)$, where χ is the polarizability. The imaginary part of ϵ can be found similarly, or by a Kronig-Kramers analysis. We shall render the complicated expressions (21) and (22) relatively simply by making several integral transformations.

III. SEPARATION OF THERMAL AND DYNAMIC EFFECTS: ZERO-FIELD LIMIT

First the ω' integration in (21) and (22) can be eliminated by noting that

$$\int_{-\infty}^{\infty} d\omega' f_0(\omega') e^{s\omega'} = \left[\frac{\pi e^{s\zeta}}{\beta}\sin\right]\left[\frac{\pi s}{\beta}\right], \tag{23}$$

where $f_0(\omega) = (1 + e^{\beta(\omega-\zeta)})^{-1}$ is the Fermi function, and ζ is the chemical potential. The next step is the elimination of the s integral. Consider

$$L_2 = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\frac{e^{ts}}{\sin\pi s}\coth(as)e^{-\lambda\coth(as)} = -\frac{\partial}{\partial\lambda}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{ts}\frac{s}{\sin\pi s}\left[\frac{e^{-\lambda\coth as}}{s}\right]. \tag{24}$$

This integral is an inverse two-sided Laplace transform and can be evaluated by the convolution theorem. We note[8]

$$\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{ts}\left[\frac{s}{\sin\pi s}\right] = \frac{1}{4\pi}\mathrm{sech}^2(\tfrac{1}{2}t) \tag{25}$$

and[9]

$$\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{ts}\left[\frac{e^{-\lambda\coth as}}{s}\right] = \begin{cases} e^{-\lambda}L_m(2\lambda), & t \neq 2an \\ \tfrac{1}{2}e^{-\lambda}[L_{n-1}(2\lambda)+L_n(2\lambda)], & t = 2an \end{cases} \tag{26}$$

where $m = [t/2a]$ (greatest integer less than or equal to $t/2a$) and L_n is the Laguerre polynomial of order n. Therefore, from the convolution theorem and the differential properties of the Laguerre polynomials,

$$\begin{aligned} L_2 &= \frac{e^{-\lambda}}{4\pi}\int_0^\infty \mathrm{sech}^2\tfrac{1}{2}(t-u)G_m(2\lambda)\,, \\ G_m(2\lambda) &= L_m(2\lambda) + 2L_{m-1}^{(1)}(2\lambda)\,, \end{aligned} \tag{27}$$

and $m = [u/2a]$, where $L_n^{(\alpha)}$ denotes an associated Laguerre polynomial. Therefore, since

$$\tfrac{1}{4}\beta\,\mathrm{sech}^2[\tfrac{1}{2}\beta(\zeta-u)] = -\frac{\partial f_0(u)}{\partial u}\,, \tag{28}$$

we have, for 2D,

$$\begin{aligned} \mathrm{Re}\epsilon(\vec{p},\omega) = 1 + \frac{2m\omega_c e^2}{p}\int_0^\infty du\left[\frac{\partial f_0(u)}{\partial u}\right]\int_0^\infty dt\, e^{-i\omega t}\sin\left[\frac{p^2}{2m\omega_c}\sin\omega_c t\right]\exp\left[-\frac{p^2}{2m\omega_c}(1-\cos\omega_c t)\right] \\ \times G_{[u/\omega_c]}\left[\frac{p^2}{m\omega_c}(1-\cos\omega_c t)\right]. \end{aligned} \tag{29}$$

We emphasize that (29) is exact within the RPA—no additional approximations have been made. To establish some degree of confidence in our procedures we shall take the zero-field limit in (29) and recover Stern's formula.[4]

As $\omega_c \to 0$ we can make the replacements $\sin\omega_c t \to \omega_c t$ and $1-\cos\omega_c t \to \tfrac{1}{2}\omega_c^2 t^2$, and take advantage of the uniform asymptotic estimate

$$n^{-\alpha}L_n^{(\alpha)}\left[\frac{x}{n}\right] \sim x^{-\alpha/2}J_\alpha(2x^{1/2}) \tag{30}$$

to obtain

$$G_{[u/\omega_c]}\left[\frac{p^2}{m\omega_c}(1-\cos\omega_c t)\right]\cong J_0(2pt\sqrt{u/2m})+\frac{2\sqrt{2m/u}}{pt}J_1(2pt\sqrt{u/2m})[u/\omega_c] . \tag{31}$$

Then, since

$$\lim_{x\to 0} x[a/x]=a \lim_{x\to\infty}[x]/x=a , \tag{32}$$

we have

$$\chi^{2D}(\vec{p},\omega)=\frac{2(2m)^{3/2}e^2}{p^2}\int_0^\infty dy\, y^{1/2}\left[\frac{\partial f_0}{\partial y}\right]\int_0^\infty \frac{dt}{t}\cos(u't)\sin(z't)J_1(t) , \tag{33}$$

where

$$u'=\frac{m\omega}{pk_F}\sqrt{\xi/y} , \quad z'=\frac{p}{2k_F}\sqrt{\xi/y} .$$

Now, by combining the trigonometric functions in the integrand of (33) and using Ref. 10, we have

$$\int_0^\infty \frac{dt}{t}\cos(u't)\sin(z't)J_1(t)=\tfrac{1}{2}[h(z'+u')+h(z'-u')] , \tag{34}$$

where

$$h(x)=\mathrm{sgn}x[\,|x|-(x^2-1)^{1/2}\Theta(|x|-1)] . \tag{35}$$

Hence

$$\chi^{2D}(\vec{p},\omega)=\frac{(2m)^{3/2}e^2}{p^2}\times\int_0^\infty dy\, y^{1/2}\left[-\frac{\partial f_0}{\partial y}\right]\{2z'-\mathrm{sgn}(z'+u')\Theta(|z'+u'|-1)[(z'+u')^2-1]^{1/2}-\mathrm{sgn}(z'-u')\Theta(|z'-u'|-1)[(z'-u')^2-1]^{1/2}\} . \tag{36}$$

This is precisely Maldague's[3] finite-temperature form of Stern's expression.[4]

IV. SEPARATION OF CYCLOTRON RESONANCE—BERNSTEIN POLE BEHAVIOR

Next consider the integral

$$T=\int_0^\infty dt\, e^{-ixt}\sin(a\sin t)\phi(\cos t) . \tag{37}$$

By decomposing the range of integration into segments of length 2π and taking advantage of periodicity we find on summing a simple geometric series,

$$T=\csc(\pi x)\int_0^\pi dt\,\sin(xt)\sin(a\sin t)\phi(-\cos t) . \tag{38}$$

Hence (29) can be expressed

$$\chi^{2D}(\vec{p},\omega)=\frac{2me^2}{p}\csc\left[\frac{\pi\omega}{\omega_c}\right]\int_0^\pi dt\,\sin\left[\frac{\omega t}{\omega_c}\right]\sin\left[\frac{p^2}{2m\omega_c}\sin t\right]\exp\left[-\frac{p^2}{2m\omega_c}(1+\cos t)\right]\times\int_0^\infty du\left[-\frac{\partial f_0}{\partial u}\right]G_{[u/\omega_c]}\left[\frac{p^2}{m\omega_c}(1+\cos t)\right] , \tag{39}$$

which is again exact within the RPA. This formula displays in a concise way all the features we expect of the polarizibility. The factor $\csc(\pi\omega/\omega_c)$, which has a pole whenever the frequency is a multiple of ω_c, is an aspect of cyclotron-resonance—Bernstein pole behavior. The integral contains all other magnetic field and thermal behavior. The latter is relatively uninteresting and can be treated within the Sommerfeld approximation. In the zero-temperature limit, which corresponds to $\partial f_0/\partial u = -\delta(u-\zeta)$, the expression becomes manifestly nonanalytic at $\omega_c = 0$, as was discussed in a preliminary report of this work.[11]

In the 3D case the integral corresponding to (24) is

$$L_3 = -\frac{\partial}{\partial\lambda}\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{s^{1/2}e^{ts}}{\sin\pi s}e^{-\gamma/s}\frac{e^{-\lambda\coth as}}{s}\ . \tag{40}$$

In addition to (25), we have[12]

$$\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{ts}s^{-1/2}e^{-\gamma/s} = \begin{cases}(\pi t)^{-1/2}\cos(2\sqrt{\gamma t}\,), & t>0\\ 0, & t<0\end{cases} \tag{41}$$

so by the convolution theorem

$$L_3 = \frac{e^{-\lambda}}{4\pi^{3/2}}\int_0^\infty du\,\mathrm{sech}^2[\tfrac{1}{2}(t-u)]\int_0^u dy\frac{\cos 2\sqrt{\gamma(u-y)}}{\sqrt{u-y}}G_{[y/2a]}(2\lambda)\ , \tag{42}$$

and consequently,

$$\begin{aligned}\mathrm{Re}\epsilon^{3\mathrm{D}}(\vec{p},\omega)-1 = {}& -\frac{4\omega_c m^{3/2}e^2}{\pi p^2}\int_0^\infty dt\,\cos(\omega t)\sin\left[\frac{p_z^2\omega_c t + p_\perp^2\sin\omega_c t}{2m\omega_c}\right]\exp\left[-\frac{p_\perp^2}{2m\omega_c}(1-\cos\omega_c t)\right]\\ & \times\int_0^\infty du\left[-\frac{\partial f_0}{\partial u}\right]\int_0^u dy\frac{\cos[2p_z t\sqrt{(u-y)/2m}\,]}{\sqrt{u-y}}\\ & \times G_{[y/\omega_c]}\left[\frac{p_\perp^2}{m\omega_c}(1-\cos\omega_c t)\right]\ .\end{aligned} \tag{43}$$

In this case thermal effects are of no consequence and can be adequately treated in the Sommerfeld approximation, which leads to corrections of order $(k_B T/\zeta)^2$. At $T=0$ K the corresponding polarizability is

$$\begin{aligned}\chi^{3\mathrm{D}}(\vec{p},\omega) = {}& -G'\omega_c\int_0^\infty\cos(\omega t)d\omega\int_0^\infty\frac{dy}{\sqrt{\zeta-y}}\sin\left[a_{11}t+\frac{a}{\omega_c}\sin\omega_c t\right]e^{-(a_\perp/\omega_c)(1-\cos\omega_c t)}\\ & \times\cos(2a_{11}^{1/2}t\sqrt{\zeta-y}\,)G_{n_y}\left[\frac{2a_\perp}{\omega_c}(1-\cos\omega_c t)\right]\ ,\end{aligned} \tag{44}$$

where

$$G' = \frac{4m^{3/2}e^2}{\pi p^2},\quad a_{11} = \frac{p_z^2}{2m},\quad a_\perp = \frac{p_\perp^2}{2m},\quad n_y = [y/\omega_c]\ .$$

Note that in this case, because of the additional t dependence associated with a_{11}, we cannot proceed as before to isolate a cyclotron resonance factor. Since $\chi^{3\mathrm{D}}$ has been completely analyzed for all field strengths,[13,14] we shall not dwell on (44), but go directly to consider the field dependence of $\chi^{2\mathrm{D}}$. Similar formulas for the complete magnetoconductivity tensor have been presented by Horing *et al.*[15] (without derivation).

V. DISCUSSION: LIMITS OF SPECIAL INTEREST

The quantity $\chi(\rho,\omega)$ calculated in this paper is $2\pi p\chi_{yy}(p,\omega)$ in the notation of Chiu and Quinn[1(b)] and $-(2\pi e^2/p)\mathrm{Im}I$ in the notation of Horing and Yildiz.[2] We shall first present approximate forms easily derived from (39) in various limiting cases.

A. dHvA regime

At field strengths for which $\zeta, p^2/2m \gg \omega_c$, and temperatures small compared to the Fermi temperature, we note that the integrand in (39) is negligible except near $t=0$, so that

$$\chi^{2\mathrm{D}}(\vec{p},\omega) \cong -\frac{2me^2}{p}\csc\left[\frac{\pi\omega}{\omega_c}\right]\int_0^\infty du\left[-\frac{\partial f_0}{\partial u}\right]e^{-x}G_{[u/\omega_c]}(2x)\int_0^\pi dt\,\sin\left[\frac{\omega t}{\omega_c}\right]\sin(x\sin t)\ , \tag{45}$$

where $x = p^2/2m\omega_c$. Now, by Féjer's formula,[16] which is uniformly valid for large n and all $x > 0$,

$$e^{-x}L_n^{(\alpha)}(2x) \sim \frac{1}{\sqrt{\pi}}(2nx)^{-1/4}\left[\frac{n}{2x}\right]^{\alpha/2}\cos\left[2\sqrt{2nx}-(2\alpha+1)\frac{\pi}{4}\right]. \tag{46}$$

At intermediate fields where $\zeta/\omega_c \gg 1$ we can make the approximation $[u/\omega_c] \cong u/\omega_c$. (This effectively prevents taking the low-field limit where the Landau level spacing becomes so small that phase averaging must be used, leading to $[u/\omega_c] \cong u/\omega_c + \frac{1}{2}$.) This merely smooths out the dHvA forms, which are sharp only at $T = 0$ K anyway, and gives

$$\chi^{2D}(\vec{p},\omega) \cong -me^2\left[\frac{\omega_c^3}{\pi p^3}\right]^{1/2}\csc\left[\frac{\pi\omega}{\omega_c}\right]\left[\frac{\sin(\pi/\omega_c)(\omega-p^2/2m)}{\omega-p^2/2m}-\frac{\sin(\pi/\omega_c)(\omega+p^2/2m)}{\omega+p^2/2m}\right]$$
$$\times\int_0^\infty du\, u^{-1/4}\left[-\frac{\partial f_0}{\partial u}\right]\left\{\cos\left[\left[\frac{2p}{\omega_c}\right]\sqrt{u/m}-\frac{\pi}{4}\right]\right.$$
$$\left.+(2\sqrt{mu}/p)\sin\left[\left[\frac{2p}{\omega_c}\right]\sqrt{u/m}-\frac{\pi}{4}\right]\right\}. \tag{47}$$

Finally, at $T = 0$ K, in terms of the standard dimensionless quantities $z = p/2k_F$, $u = m\omega/pk_F$, and $\eta = \zeta/\omega_c$, we have

$$\chi^{2D}(p,\omega) = -\left[\frac{me^2}{p}\right]\frac{\csc(\pi\omega/\omega_c)}{2^{5/4}\pi^{1/2}\eta^{3/2}}\left[\frac{\sin[4\pi z\eta(u-z)]}{u-z}-\frac{\sin[4\pi z\eta(u+z)]}{u+z}\right]$$
$$\times\left[z\cos\left[2^{5/2}z\eta-\frac{\pi}{4}\right]+2^{-1/2}\sin\left[2^{5/2}z\eta-\frac{\pi}{4}\right]\right], \tag{48}$$

which displays clearly the separation of cyclotron resonance and dHvA behavior.

B. High-field limit

This limit has been thoroughly treated by Horing *et al.*,[17] so we shall simply show how (39) leads to their results. First of all, we emphasize that for the 2D case, when other limits are to be taken, the zero-temperature limit should be taken last. Thus we first set $G_{[u/\omega_c]} = 1$, then the u integral gives the factor $f(0)$. Following Ref. 17 we also introduce the quantity $\lambda = p^2/2m\omega_c$. Then the t integral can be written

$$-\operatorname{Im}\int_0^\pi dt\,\sin\left[\frac{\omega t}{\omega_c}\right]e^{-\lambda e^{it}} = -\sum_{n=0}^\infty\frac{(-\lambda)^n}{n!}\int_0^\pi dt\,\sin(nt)\sin\left[\frac{\omega t}{\omega_c}\right]$$
$$= -\omega_c^2\sin\left[\frac{\omega\pi}{\omega_c}\right]\sum_{n=0}^\infty\frac{n\lambda^n}{n!}(\omega^2-n^2\omega_c^2)^{-1}. \tag{49}$$

Hence (39) becomes

$$\chi^{2D}(\vec{p},\omega) = -\frac{2m\omega_c e^2}{p}f(0)e^{-\lambda}\times\sum_{n=1}^\infty\left[\frac{\lambda^n}{n!}\right]\frac{n\omega_c}{\omega^2-n^2\omega_c^2}. \tag{50}$$

Finally, noting that from (23), $f(0) = (2\pi/m\omega_c)n_s$, where n_s is the areal density, we exactly reproduce Eq. (16) of Ref. 17.

C. Local limit

For low wave number (39) can be evaluated explicitly, whence we find

$$\chi^{2D}(p,\omega) = -\frac{e^2p\omega_c}{\omega^2-\omega_c^2}\times\int_0^\infty\left[-\frac{\partial f_0}{\partial u}\right](2[u/\omega_c]+1). \tag{51}$$

This exhibits a structure similar to that observed in the Hall conductivity[18] and suggests that screening effects in metal-oxide-semiconductor inversion layers may possess plateaus as a function of gate voltage (which produces a continuous variation in ζ).

D. Zero-frequency limit

From (39) we have

$$\chi^{2D}(p,0)=\frac{2me^2}{\pi p}F_N\left[\frac{p^2}{2m\omega_c}\right],$$
$$N=[\zeta/\omega_c],\quad T=0\text{ K}, \qquad (52)$$

where

$$F_N(a)=\int_0^\pi t\sin(a\sin t)e^{-a(1+\cos t)}\times G_N[2a(1+\cos t)]dt\ . \qquad (53)$$

The behavior of these expressions for high fields is indicated in Fig. 1.

The behavior of $\chi^{2D}(p,0)$ is shown in Fig. 2 for strong magnetic fields. With respect to the units $\hbar=2m=1,e^2=2$, we have arbitrarily set $\zeta=0.1$ with $\omega_c=0$, $\frac{1}{2}\zeta$, ζ, and 5ζ. It appears that the magnetic field suppresses the low wave-number singularity in the polarizability and the singularity at $p=2k_F$. However, the discrete nature of the Landau levels introduces a small amount of structure on the high momentum side of the maximum. It is clear that the polarizability converges to its zero-field value nonuniformly as the field tends to zero. The effect of the field on the potential induced by a point charge is to enhance the screening at large distances.

The effect of a magnetic field on the screening of a point charge has been thoroughly discussed by Horing and Yildiz.[2] In the high-field quantum limit they estimated the screened potential of a point charge by using a low wave-number approximation of the zero-frequency dielectric function. As an application of (52) we have calculated this potential numerically for the case $\omega_c=5\zeta,\zeta=0.1$; the results are shown in Table I and Fig. 3, where we also present the screened potential in the near quantum limit, where the two lowest Landau levels are occupied ($\omega_c=\zeta;\zeta=0.1$).

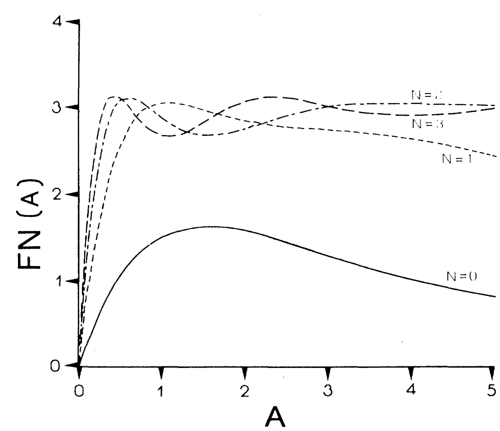

FIG. 1. Behavior of the function $F_n(a)$ introduced in Eq. (53).

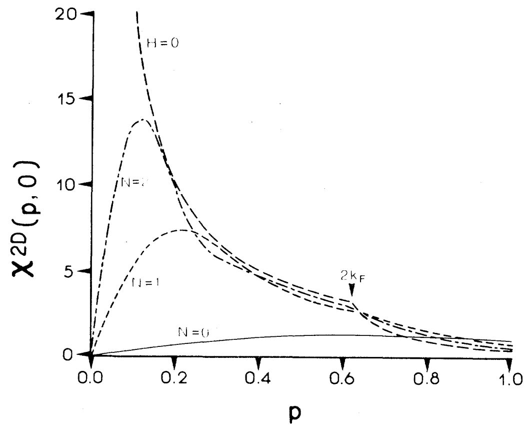

FIG. 2. Zero-frequency dielectric susceptibility $\chi^{2D}(p)$ for values of the magnetic field. Units where $\hbar=2m=1,e^2=2$ are used and the (zero-field) chemical potential has been arbitrarily set at $\zeta=0.1$.

TABLE I. Screened potential of a unit point charge. $V(1)$: Exact value for high-field quantum limit $\omega_c=5\zeta$. $V(2)$: Local screening approximation of Horing and Yildiz.

r	$V(1)$	$V(2)$
0.4	2.149 55	0.627 47
0.8	0.902 94	0.474 33
1.0	0.671 63	0.432 80
4.0	0.151 89	0.188 65
8.0	0.109 28	0.110 15
12.0	0.076 77	0.077 54
16.0	0.059 41	0.059 67
20.0	0.048 29	0.048 42
30.0	0.032 78	0.032 81
40.0	0.024 75	0.024 77
50.0	0.019 86	0.019 88
60.0	0.016 54	0.016 59
70.0	0.014 21	0.014 25
80.0	0.012 43	0.012 47
90.0	0.011 03	0.011 09
100.0	0.009 88	0.009 98

VI. PLASMON DISPERSION RELATION

We conclude with a discussion of the local nonretarded plasmon mode. In units where $\hbar=2m=1, e^2=2$, the 2D plasmon dispersion relation is

$$\epsilon_0+(2/\sqrt{\omega_c\alpha})\csc(\pi\Omega)\int_0^{\pi}dt\,\sin(\Omega t)\sin(\alpha\sin t)e^{-\alpha(1+\cos t)}G_n[2\alpha(1+\cos t)]=0\ , \tag{54}$$

where ϵ_0 is the ambient dielectric constant and

$$\Omega=\frac{\omega}{\omega_c},\quad \alpha=\frac{p^2}{\omega_c}\ . \tag{55}$$

In the local limit, if n is not too large, we can expand the integrand in powers of α. The integrals are then elementary and we obtain Chiu and Quinn's wave-vector expansion. The leading term gives

$$\epsilon_0=[2/(\omega_c\alpha)^{1/2}](\Omega^2-1)^{-1}G_n(0)\ , \tag{56}$$

and for $n=0$, the high-field quantum limit, we recover precisely Horing and Yildiz's result (with $\epsilon_0=1$),

$$\omega^2=\omega_c^2+2p\omega_c \tag{57}$$

(noting that the quantity ρ in their formula is $2\pi\omega_c/m$ in this limit). However, for high to intermediate fields, (54) is numerically very tractable. In Table II we present the plasmon mode for $n=0,1,2$, i.e., where up to three Landau levels are occupied. These values were obtained on a desk calculator (HP-67) in less than an hour. If a computer is used, $n \lesssim 50$ or even higher, which is a region that has hitherto been inaccessible, should present no problem. We note from Table II that the exact plasmon dispersion relation deviates from the local limit (57) beginning with relatively small values of the wave number. For lower values of the magnetic field, G_n in the integrand of (55) can be replaced by its uniform asymptotic value and the results go into Stern's zero-field expression as shown in (30)—(36).

In addition to the local plasmon, (54) has nonlocal roots, but since these are of little interest in the nonretarded limit we shall not consider them further. In summary, we have presented the development of a new exact relatively simple closed-form expression for the RPA polarizability of a 2D electron gas from which one cannot only recover the approximations presently in the literature, but also obtain with relative ease results for experimentally accessible field values which have not yet been investigated.

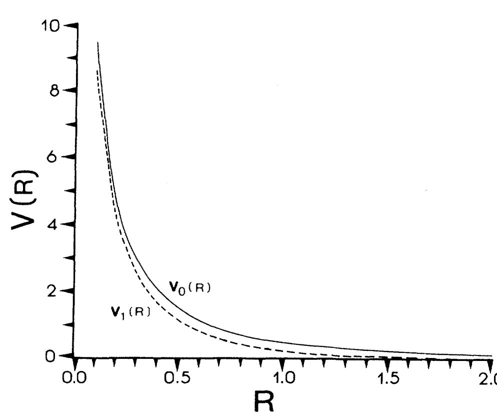

FIG. 3. Screened potential due to a unit point charge. $V_0(R)$: High-field quantum limit. $V_1(R)$: Lowest two Landau levels occupied.

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TABLE II. Plasmon dispersion relation for $\zeta=0.1$: (a) $\omega_c=0.5$, (b) $\omega_c=0.1$, (c) $\omega_c=0.05$. Reduced units are used where $\hbar=2m=1$, $e^2=2$, $\alpha=p^2/\omega_c$, $\Omega=\omega/\omega_c$. HY, which represents Horing and Yildiz, denotes the results of Ref. 2.

α	Ω (a)	Ω (HY)	Ω (b)	Ω (c)
0	1.00	1.00	1.00	1.00
0.01	1.13	1.13	1.68	1.64
0.02	1.18	1.18	1.82	1.65
0.03	1.21	1.22	1.85	1.64
0.04	1.24	1.25	1.86	1.64
0.05	1.26	1.28	1.86	1.63
0.06	1.28	1.30	1.85	1.62
0.07	1.30	1.32	1.84	1.62
0.08	1.31	1.34	1.83	1.61
0.09	1.32	1.36	1.82	1.60
0.10	1.33	1.38	1.81	1.59
0.15	1.37	1.45		
0.20	1.39	1.50		
0.30	1.40	1.60		
0.40	1.40	1.67		
0.50	1.39	1.73		
0.60	1.37	1.79		

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