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Static and dynamic properties of a two-dimensional Wigner crystal in a strong magnetic field

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Making use of an ansatz wave function for the ground state of two-dimensional electrons, which occupy only partially the lowest Landau level, we study theoretically the correlation energy, the shear modulus, and collective modes. We find that the electron lattice is locally stable for $0 \leq \nu < 0.45$, while the hole lattice is locally stable for $0.55 < \nu \leq 1$, where ν is the filling factor. The melting temperature T_M estimated from the shear modulus exhibits broad peaks around $\nu = 0.27$ and 0.73.

I. INTRODUCTION

Two-dimensional electron systems formed in metal-oxide—semiconductor field-effect transistors (MOSFET's) and superlattices have been the subject of intense theoretical and experimental studies in the last five years. Depending on carrier density and magnetic field these systems exhibit two-dimensional localization, quantization of the Hall effect, and possibly a Wigner crystal[1] or charge-density-wave state (CDW) at low temperatures. We shall study in this paper the stability of the CDW state in the extreme quantum limit, where the lowest Landau level with a single-spin component is only partially occupied. These electrons are interacting via Coulomb interaction, though we assume that the average Coulomb energy per electron is much smaller than the Landau energy spacing ($e^2/2l \ll \omega_c \equiv eB/m$). We shall make use of an ansatz wave function for the ground state, which describes the two-dimensional CDW state. This ansatz wave function gives rise to an electron charge distribution function consistent with that of the CDW state studied by Yoshioka and Fukuyama (YF).[2] Furthermore this ansatz wave function yields the Coulomb correlation energy in excellent agreement with the results obtained by Yoshioka and Fukuyama[2] and Yoshioka and Lee[3] (YL) within the Hartree-Fock approximation to the CDW state. The real advantage of the present method lies in the ease by which it enables us to study the lattice dynamics. We calculate the shear modulus of the electron lattice. The shear modulus is positive for the electron lattice for $0 \leq \nu < 0.45$ but becomes negative for $\nu > 0.45$, which implies that the CDW state is locally stable only for $0 < \nu < 0.45$. On the other hand, for $\nu > 1/2$, the hole lattice constructed on the filled Landau level is locally stable for $0.55 < \nu \leq 1$. We find that the shear modulus thus found is larger than the classical value[4] for $(0 < \nu < 0.35)$ and has a broad maximum around $\nu = 0.23$, implying that this value is favorable for the crystal formation, if we assume that the CDW melting is dictated by the dissociation of bound dislocation pairs[5,6] as in the classical lattice. We find in this way the maximum melting temperature $T_M = 0.0062e^2/l$ with $l = (eB)^{-1/2}$ for $\nu \cong 0.3$. Invoking the electron-hole symmetry proposed by YF,[2] we have a similar peak in the shear modulus and in T_M around $\nu = 0.7$. On the other hand, although our ansatz wave function yields commensuration energies at $\nu = \frac{1}{3}$, $\frac{1}{5}$, etc., they are too small to be physically relevant.

We also calculate the phonon spectrum in the long-wavelength limit within the present model; the lower phonon mode exhibits the $q^{3/2}$ dispersion as in the classical lattice in a magnetic field.[4]

II. ANSATZ WAVE FUNCTION

We shall consider a two-dimensional electron system described by the following Hamiltonian:

$$H = \sum_i \frac{1}{2m}[\vec{p}_i - e\vec{A}(r_i)]^2 + \tfrac{1}{2}\sum_{i \neq j} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} , \tag{1}$$

where $\vec{p}_i$ and $\vec{r}_i$ are the two-dimensional momentum and position vectors lying in the x-y plane. For later convenience we take the symmetric gauge for $\vec{A} = \frac{1}{2}B(-y,x,0)$, where B is the strength of a magnetic field applied in the z direction (normal to the two-dimensional plane).

The lowest-Landau-level wave function localized around $\vec{r} = \vec{R}$ is given by

$$\psi_{\vec{R}}(\vec{r}) = (2\pi l^2)^{-1/2} \times \exp\left[-\frac{1}{4l^2}[(\vec{r}-\vec{R})^2 - 2i(xY - yX)]\right] , \tag{2}$$

where

$$l^{-2} = eB \quad \text{and} \quad \vec{R} = (X, Y) . \tag{3}$$

Hereafter we shall use units of $\hbar = c = 1$. Our ansatz wave function for the ground state is a Slater determinant constructed by the wave functions (2) located at the regular two-dimensional lattice points $\vec{R}_j$. We have

$$\Psi(\{\vec{r}_i\}) = (N!)^{-1/2} \det |\psi_{\vec{R}_j}(\vec{r}_i)| , \tag{4}$$

where

$$\vec{R}_j \sim a_0 \left[n + \frac{m}{2}, \frac{\sqrt{3}}{2} m \right] . \tag{5}$$

a_0 is the lattice constant and n and m are integers. If we neglect the overlapping between two wave functions with different R_i's, the single-electron density is given by

$$\rho(\vec{r}) \cong \sum_i |\psi_{\vec{R}_i}(\vec{r})|^2 = (2\pi l^2)^{-1} \sum_i \exp\left[-\frac{1}{2l^2} (\vec{r} - \vec{R}_i)^2 \right] , \tag{6}$$

which is exactly found by YF (Ref. 2) and YL (Ref. 3) for the CDW ground state within the Hartree-Fock approximation. The Coulomb energy for the ansatz state (4) is evaluated as

$$E_c = \left\langle \Psi \left| \left[\tfrac{1}{2} \sum_{i \neq j} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} \right] \right| \Psi \right\rangle \Big/ \langle \Psi | \Psi \rangle , \tag{7}$$

as the wave functions $\psi_{\vec{R}_i}(\vec{r})$ do not form an orthonormal set. We expand Eq. (7) in terms of the n-electron correlations as

$$E_c = V(\{\vec{R}_i\}) = \tfrac{1}{2} \sum_{i,j} V(\vec{R}_i - \vec{R}_j) + \tfrac{1}{6} \sum_{i,j,k}{}' V_3(\vec{R}_i, \vec{R}_j, \vec{R}_k) + \cdots , \tag{8}$$

where the sums over i,j,k are carried out over different lattice points only.

Since the overlapping between two single-electron wave functions at different positions is small, we obtain in good accuracy

$$V(\vec{R}) = \frac{\sqrt{\pi} e^2}{4l} \operatorname{sech}(\tfrac{1}{8} R^2) I_0(\tfrac{1}{8} R^2) , \tag{9}$$

$$V_3(\vec{R}_1, \vec{R}_2, \vec{R}_3) = -\frac{\sqrt{\pi} e^2}{2l} \exp(-X_{123}) I_0(X_{123}) \{ 1 - \exp[-\tfrac{1}{4} (\vec{R}_{12}^{\,2} + \vec{R}_{23}^{\,2} + \vec{R}_{31}^{\,2} - 4iA_{123})] \} + \text{c.c.} , \tag{10}$$

where

$$X_{123} = \tfrac{1}{8} [\vec{R}_{12} \cdot \vec{R}_{13} - i(X_{12} Y_{13} - X_{13} Y_{12})], \quad \vec{R}_{ij} = \vec{R}_i - \vec{R}_j . \tag{11}$$

A_{123} is the area of the triangle with vertices at $\vec{R}_1$, $\vec{R}_2$, and $\vec{R}_3$, and $I_0(z)$ is the modified Bessel function. Hereafter we measure length of R_i in units of l. We give the derivation of Eqs. (9) and (10) in Appendix A. Since the electron density n is related to ν the filling factor of the lowest Landau level and the lattice constant a_0 by

$$n = \frac{\nu}{2\pi l^2} = \left[\frac{\sqrt{3}}{2} a_0^2 \right]^{-1} , \tag{12}$$

the overlapping integral between two wave functions at the nearest neighbor is given by

$$e^{-(a_0/l)^2/2} = e^{-2\pi/\sqrt{3}\nu} \cong e^{-3.6276\nu^{-1}} . \tag{13}$$

Therefore for $\nu < 0.5$, the overlapping integral gives rise to a contribution of the order of 10^{-3}. Then, except for the exchange contribution to the Coulomb energy, all other contributions due to the Fermi statistics of the electron are practically negligible for $\nu \leq 0.5$. Even for $0.5 < \nu \leq 1$, the inclusion of the lowest-order overlapping integrals will be adequate for evaluating the correlation energy. However, this is not so obvious in the case of the shear modulus. Indeed the shear modulus can be expressed as a difference of two large terms. Therefore the small modification in the potential of the order of 10^{-3} produces a significant effect. Therefore we believe that our results for the shear modulus around $\nu \cong 0.5$ are still qualitatively correct but may not be reliable quantitatively. In this circumstance it is of interest to note that V_3 contains a commensuration energy proportional to $\cos(A_{123}/l^2) = \cos(\pi/\nu)$ for the smallest triangle $\vec{R}_1, \vec{R}_2, \vec{R}_3$, which favors the triangular lattice when $\nu = \frac{1}{3}, \frac{1}{5}$, etc. However, the coefficient of the commensuration energy at $\nu = \frac{1}{3}$ is so small (10^{-7}) that we do not think this energy is physically relevant. Therefore for all values of ν (i.e., $0 < \nu < 1$), the Coulomb energy of the system is given by the first term of Eq. (8) within a few percent accuracy; the effect of the strong magnetic field is almost completely incorporated in $V(\vec{R})$ as modification of the Coulomb potential.

In order to consider the lattice dynamics we need the kinetic energy. This can be obtained from the following consideration. We shall first replace the lattice positions $\vec{R}_i$ to $\vec{R}_i + \vec{u}_i$, where the $\vec{u}_i$'s are small displacements. In general $\vec{u}_i$'s depend on time. Then the necessary kinetic energy is calculated as[7]

$$T = \left\langle \Psi \left| \left[iU^{-1} \frac{\partial U}{\partial t} \right] \right| \Psi \right\rangle \Big/ \langle \Psi | \Psi \rangle , \tag{14}$$

where U is the unitary operator changing $\vec{R}_i$ into $\vec{R}_i + \vec{u}_i$. We have

$$U = \exp\left[\sum_{i,\alpha} u_i^\alpha \frac{\partial}{\partial R_i^\alpha} \right] . \tag{15}$$

Substituting Eq. (15) in Eq. (14), we obtain

$$T = \tfrac{1}{2} \omega_c \sum_i \epsilon_{\alpha\beta} \dot{u}_i^\alpha u_i^\beta , \tag{16}$$

where $\omega_c = eB/m$ and $\epsilon_{\alpha\beta}$ is the antisymmetric tensor. Therefore the lattice dynamics of the electron lattice is described by the effective Hamiltonian

$$H_{\text{eff}} = T + \frac{1}{2}\sum_{i\neq j} V(\vec{R}_i - \vec{R}_j + \vec{u}_i - \vec{u}_j) , \tag{17}$$

where $V(R)$ is the first term in Eq. (8).

III. CORRELATION ENERGY

First let us calculate the correlation energy of the system, which is given by

$$NU_{\text{cor}} = \frac{1}{2}\sum_{i\neq j} V(\vec{R}_i - \vec{R}_j) = \frac{Ne^2}{2l}\sum_i V(\vec{R}_i) , \tag{18}$$

where N is the total electron number.

The potential $V(R)$ is shown in Fig. 1 together with the Coulomb potential. For $R < 2.7$, $V(R) < R^{-1}$; the divergence of the Coulomb potential at the origin is smoothed out due to the delocalization of the electron in the magnetic field. For $R > 2.7$, $V(R)$ is somewhat larger than the Coulomb potential. In the figure we have indicated the R value for the nearest neighbor for several ν's. For $R \gg 1$ (i.e., $R/l \gg 1$ in the ordinary unit), $V(R)$ is approximated as

$$V_1(R) = \frac{1}{R}(1 + R^{-2} + \tfrac{9}{2}R^{-4} + \tfrac{75}{2}R^{-6} + \cdots) . \tag{19}$$

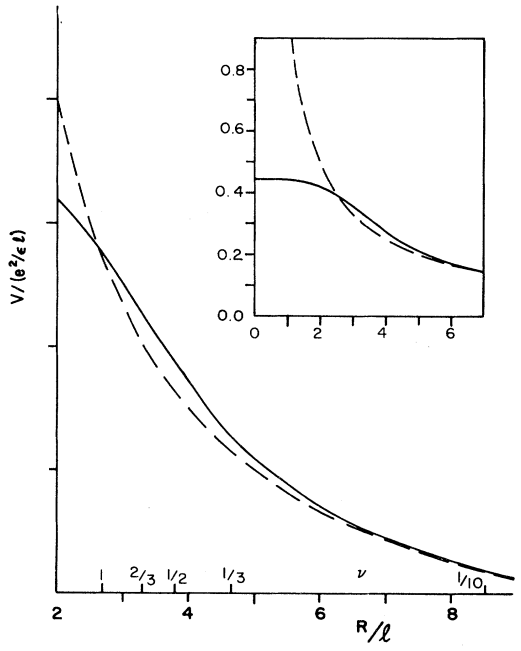

FIG. 1. Potential $V(R)$ is compared with the Coulomb potential R^{-1} (dashed curve). In the region of interest $V(R)$ is larger than R^{-1}. Fractional numbers on the horizontal axis indicate the nearest-neighbor distance of the triangular lattice for the corresponding filling factor.

The first term in Eq. (19) gives rise to a divergent contribution in Eq. (18). However, this divergence can be eliminated by the Madelung sum as done by Bonsall and Maradudin[4] for the classical electron system. The nearest lattice points lie always in the region $V(R) > R^{-1}$ except near $\nu = 1$; the correlation energy in a magnetic field is always larger than the classical value[4]

$$U_{\text{cor}}^{\text{class}} = -0.782\,133\frac{e^2}{l}\nu^{1/2} . \tag{20}$$

The sum (18) is carried out numerically, and shown in Fig. 2 together with the classical result. First of all we note that our correlation energy is in excellent agreement with a recent result by YL (Ref. 3) for $\nu \leq 0.5$. The discrepancy between our results and YL's is always less than one percent. We should point out that in order to get this agreement the inclusion of the Coulomb exchange term is essential near $\nu = 0.5$, although the exchange term becomes less than 10^{-2} for $\nu \leq 0.4$. Therefore our ansatz wave function reproduces the result of the mean-field calculation of the CDW state.[3] Second, when $\nu = 1$, the completely filled Landau state can be considered as a liquid state. In this limit, though $\rho(\vec{r})$ is uniform, the two-particle correlation function is given by[8]

$$\langle \rho(\vec{r})\rho(\vec{r}\,') \rangle = n^2(1 - e^{-(\vec{r}-\vec{r}\,')^2/2}) . \tag{21}$$

This liquid state gives

$$U_{\text{cor}}\,|_{\nu=1} = -\frac{e^2}{2l}(\pi/2)^{1/2} , \tag{22}$$

which is extremely close to our correlation energy at $\nu = 1$. Actually within our numerical error at $\nu = 1$, which is a few percent, the correlation energy of the electron lattice is identical to Eq. (22). However, this is not so surprising, if we consider the fact that the overlapping integrals between the nearest-neighbor wave functions are still quite small (only a few percent) even at $\nu = 1$. In the absence of the overlapping integral, the lattice should be the preferred state. Furthermore for $\nu > 0.5$, we can obtain a state with

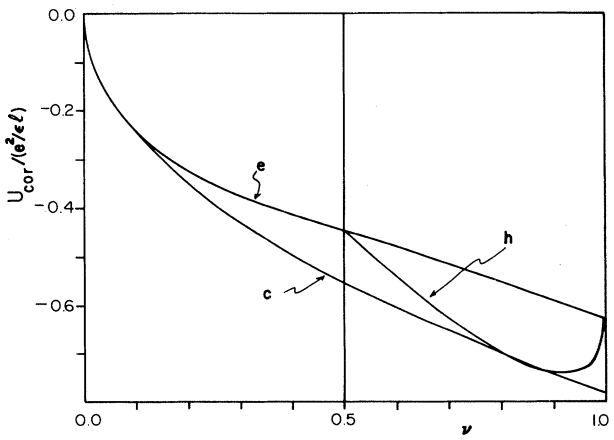

FIG. 2. Correlation energy per electron is shown as function of the filling factor. e, h, and c on the curves indicate the electron lattice, the hole lattice, and the classical result, respectively.

a lower correlation energy by assuming that the ground state is described by the completely filled Landau level plus a hole lattice with the hole density $(1-\nu)$ as proposed by YF.[2] Then the correlation energy of $\nu>0.5$ is given by

$$U^h_{\rm cor}(\nu)=U_{\rm cor}(1-\nu)-\frac{e^2}{2l}(\pi/2)^{1/2}(2\nu-1)\ , \tag{23}$$

where the second term comes from the Coulomb exchange energy between the holes and the underlying electron liquid. The derivation of Eq. (23) is given in Appendix B. We exhibit the correlation energy for the hole lattice thus constructed for $\frac{1}{2}<\nu<1$ in Fig. 2. Indeed it is clear that the hole lattice has the lower correlation energy than the electron lattice for $\nu>\frac{1}{2}$. Furthermore the correlation energy of the electron lattice has a different slope from that of the hole lattice at $\nu=\frac{1}{2}$ for increasing ν implying that the transition should be of the first order within the present model. Finally we have compared the liquid-state energy referred by Laughlin[8] at $\nu=\frac{1}{5}$, with the electron lattice energy. We find that the correlation energy of the electron lattice appears to be lower than that of the liquid state. Therefore we believe that, if there is any possibility for the liquid state rather than the electron lattice, it should be limited in the vicinity of $\nu=\frac{1}{2}$.

IV. SHEAR MODULUS

We shall now consider the shear modulus of the electron lattice (and the hole lattice for $\nu>\frac{1}{2}$). In this paper we limit our analysis to the harmonic approximation for simplicity, though in general the anharmonic terms would be rather important in the Coulomb system.[9,10] Expanding the second term in Eq. (16) in powers of $\vec{u}_i$, we obtain

$$V=NU_{\rm cor}+\tfrac{1}{4}\sum_{i\neq j}\left[\frac{\partial^2}{\partial R_i^\alpha\partial R_j^\beta}V(\vec{R}_i-\vec{R}_j)\right]\times(u_i^\alpha-u_j^\alpha)(u_i^\beta-u_j^\beta)+\cdots\ , \tag{24}$$

where we kept only the quadratic terms. We shall recast the quadratic terms in u_i as

$$\delta V=\frac{e^2}{2l}(2\pi)^{-2}\int d^2k\,\Phi^{\alpha\beta}(\vec{k})u^\alpha_{\vec{k}}u^\beta_{-\vec{k}}\ , \tag{25}$$

where the Fourier transform of u_i is defined by

$$\vec{u}_i=\frac{1}{\sqrt{N}}\int\frac{d^2k}{(2\pi)^2}\vec{u}_{\vec{k}}e^{i\vec{k}\cdot\vec{R}_i}\ , \tag{26}$$

and the k integral extends over the two-dimensional Brillouin zone. $\Phi^{\alpha\beta}(k)$ is given by

$$\Phi^{\alpha\beta}(\vec{k})=\sum_{\vec{R}}[1-\cos(\vec{k}\cdot\vec{R})]\partial_\alpha\partial_\beta V(R)\cong\left[\frac{2\pi n}{k}+C_0\right]k^\alpha k^\beta+C_1k^2\delta_{\alpha\beta}+O(k^3)\ , \tag{27}$$

where the R sum extends over the two-dimensional lattice. The constants C_0 and C_1 are evaluated numerically as function of ν. In Fig. 3 we show the shear modulus $C_t=(e^2/l)C_1$ and the coefficient in the longitudinal mode $C_l=(e^2/l)/(C_0+C_1)$ normalized by their classical values[4] ($C_{t0}=0.09775\nu^{1/2}e^2/l$ and $C_{l0}=-5C_{t0}$). We note that C_t has a broad peak around $\nu=0.23$, while C_l has a flatter peak around $\nu=0.28$. The origin of this broad peak becomes clear from Fig. 4, where we plot $\Phi_t=\frac{1}{8}(3RV'+R^2V'')$ as function of R. [C_1 is expressed as the lattice sum of $\Phi_t(R)$.] Compared with the case of the Coulomb potential, the second derivative of V has a broad structure near $R=5$. For $\nu=\frac{1}{3}$ the nearest lattice point lies in the vicinity of the peak of the above function and this nearest-neighbor contribution controls the shear modulus. Furthermore the shear modulus becomes negative for $\nu>0.45$, implying that the electron lattice is unstable for $\nu>0.45$. The point where the shear modulus becomes negative may change if we include V_3, etc., or the anharmonic corrections. However, we believe that the general tendency that the electron lattice becomes unstable near $\nu=\frac{1}{2}$ has general validity. Since the shear modulus of the hole lattice is symmetrical to that of the electron lattice, the hole lattice becomes locally stable for $0.55<\nu\leq 1$.

If we assume that in the extreme quantum limit the melting of the electron lattice is dictated by the Kosterlitz-Thouless mechanism,[5,6] we can calculate the melting temperature by

$$T_M=(4\pi)^{-1}na_0^2C_t=(2\pi\sqrt{3})^{-1}\frac{e^2}{l}C_1\ . \tag{28}$$

The predicted T_M is shown in Fig. 5, together with that of the classical theory[6] and the mean-field result of Fukuyama, Platzman, and Anderson.[11] We note that T_M has two broad maxima around $\nu=0.27$ and 0.73 and that T_M vanishes in the small region near $\nu=\frac{1}{2}$, where the electron

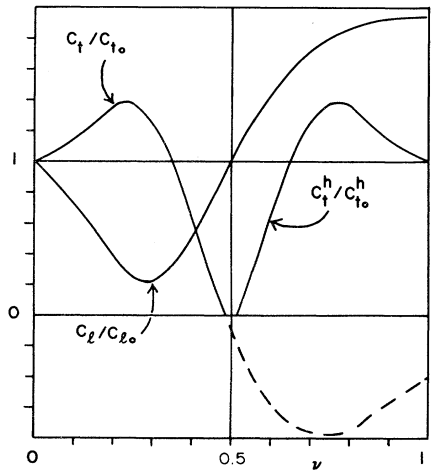

FIG. 3. Normalized shear modulus and a normalized term in the modulus of compression are shown as function of the filling factor. C_t for the electron lattice becomes negative for $\nu>0.48$, implying instability of the electron lattice.

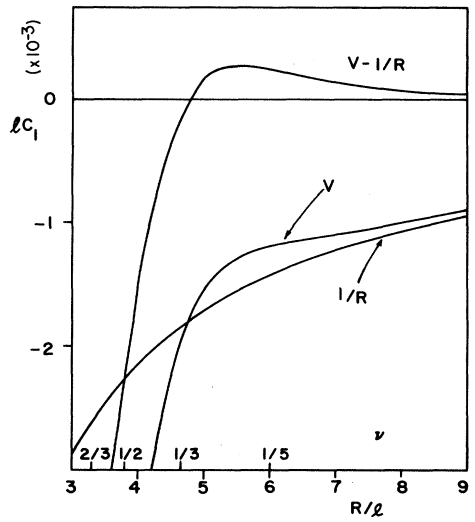

FIG. 4. Function $\Phi_t s$, which appears in the expression of the shear modulus are shown as function of the distance R. Φ_t corresponding to V is larger than that for R^{-1} when $R/l > 0.475$. However, for $R/l < 0.475$, Φ_t for V decreases sharply as R decreases.

lattice and the hole lattice are both unstable. We have also analyzed the square lattice with the same approximation and we find that the square lattice always has higher correlation energy than the triangular lattice does, although the difference in energies becomes less than 1% at $\nu=\frac{1}{2}$. It is noteworthy that the melting temperature of the quantum limit is substantially larger than that of the classical system especially near $\nu=0.27$, where T_M has a broad peak. It is of interest to note that the field-dependent correction to the classical C_1 in the limit of small ν is of the same order of magnitude and of the same sign as that due to the anharmonic corrections in the classical limit calculated by Fisher.[9] This indicates clearly the importance of the anharmonic corrections to C_1.

We shall conclude this section with the consideration of

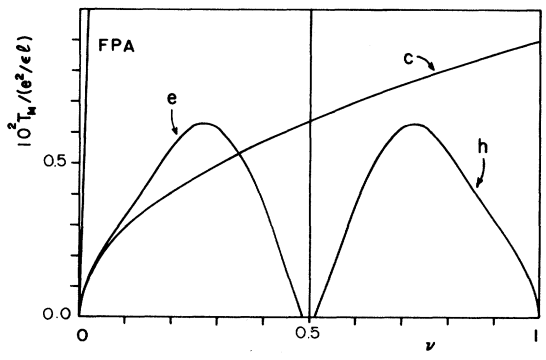

FIG. 5. Melting temperature obtained assuming the Kosterlitz-Thouless mechanism is shown for the electron lattice ($0<\nu<0.5$) and the hole lattice ($0.5<\nu<1$). C on the curve indicates the melting temperature for the classical system and FPA indicates that for mean-field theory.

the phonon modes in the electron lattice. The kinetic energy T in Eq. (16) is rewritten in terms of u_k as

$$T=\frac{\omega_c}{2(2\pi)^2}\int d^2k\,\epsilon_{\alpha\beta}\dot{u}^{\alpha}_{\vec{k}}u^{\beta}_{-\vec{k}}\ . \tag{29}$$

The equation of motion is then given by

$$i\omega\omega_c \vec{u}^{\,\alpha}_{\vec{k}}=\frac{e^2}{l}\epsilon^{\alpha\beta}\left[\left[\frac{2\pi n}{k}+C_0\right]k^{\beta}(\vec{k}\cdot\vec{u}_{\vec{k}})+C_1k^2u^{\beta}_{k}\right], \tag{30}$$

which yields

$$\omega=\pm\omega_c^{-1}e^2\left[C_1\left[\frac{2\pi n}{k}+C_0\right]\right]^{1/2}k^2\ . \tag{31}$$

Therefore, as in the classical theory,[4] the phonon has the $|k|^{3/2}$ dispersion. The other mode with $\omega\cong\omega_c$ can be constructed, if we introduce a single-electron wave function for the $n=1$ Landau level, as in the present limit this corresponds to the local excitation.

V. CONCLUDING REMARKS

Starting with an ansatz wave function describing the electron lattice (or the hole lattice) in a strong magnetic field we show that this wave function gives the correlation energy identical to that for the CDW state within the mean-field approximation. We show furthermore that the electron lattice and the hole lattice are locally stable for $0\le\nu<0.45$ and $0.55<\nu\le1$, respectively. Making use of the shear modulus of the electron lattice (and the hole lattice) thus determined, we estimate the melting temperature T_M, which has broad peaks at $\nu=0.27$ and 0.73. We believe that the detection of the phonon mode will provide a definitive evidence for the two-dimensional Wigner lattice in the extreme quantum limit.

When a uniform electric field is applied, for example, in the x direction, the electron lattice slides uniformly in the transverse direction giving rise to the Hall conductivity $\sigma_{xy}=-e^2\nu/\hbar$ or $e^2(1-\nu)/\hbar$ depending on whether the underlying lattice is the electron lattice or the hole; there is no quantization of the Hall effect within the present model. Therefore in order to account for the quantization of the Hall constant[12] at the fractional filling it is essential to introduce the relaxation process[13] possibly due to the impurity scattering.

Note added in proof. (1) After submission of this paper, Dr. Phil Platzman drew our attention to an unpublished work by H. Fukuyama, P. M. Platzman, and D. W. Anderson, who introduced the same ansatz function as ours and made a similar analysis of the commensurability energy. (2) In light of a recent paper by Yoshioka *et al.* [D. Yoshioka, B. I. Halperin, and P. A. Lee, Phys. Rev. Lett. 50, 1219 (1983)], we believe that for $\nu=\frac{1}{3}$ the correlated liquid proposed by Laughlin[8] has lower energy than the Wigner crystal. However, whether the correlated liquid has lower energy for $\nu=\frac{1}{5}$ appears to be undecided.

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APPENDIX A: POTENTIAL ENERGY OF THE ELECTRON LATTICE

We shall evaluate Eq. (7) as follows:

$$\left\langle \Psi \left| \sum_{\substack{i,j \\ i>j}} \frac{e^2}{|\vec{r}_i-\vec{r}_j|} \right| \Psi \right\rangle = \frac{e^2}{2} \sum_{i\neq j} \tilde{V}_2(\vec{R}_i,\vec{R}_j) + \frac{e^2}{6} \sum_{i,j,k}{}' \tilde{V}_3(\vec{R}_i,\vec{R}_j,\vec{R}_k) + \cdots \tag{A1}$$

and

$$\langle \Psi | \Psi \rangle = 1 - \tfrac{1}{2} \sum_{i\neq j} S_{ij} + \tfrac{1}{6} \sum_{i,j,k} S_{ijk} - \cdots , \tag{A2}$$

where

$$\begin{aligned}
\tilde{V}_2(\vec{R}_i,\vec{R}_j) &= \int \frac{d^2r_1 dr_2^2}{|\vec{r}_1-\vec{r}_2|} [\,|\psi_{\vec{R}_i}(\vec{r}_1)|^2 |\psi_{\vec{R}_j}(\vec{r}_2)|^2 - \psi_{\vec{R}_i}^*(\vec{r}_1)\psi_{\vec{R}_j}^*(\vec{r}_2)\psi_{\vec{R}_j}(\vec{r}_1)\psi_{\vec{R}_i}(\vec{r}_2)] \\
&= \frac{1}{(2\pi l^2)^2} \int \frac{d^2r_1 d^2r_2}{|\vec{r}_1-\vec{r}_2|} \left\{ \exp\left[-\frac{1}{2l^2}[(\vec{r}_1-\vec{R}_i)^2+(\vec{r}_2-\vec{R}_j)^2] \right] \right. \\
&\qquad - \exp\left[-\frac{1}{4l^2}\{(\vec{r}_1-\vec{R}_i)^2+(\vec{r}_2-\vec{R}_j)^2+(\vec{r}_1-\vec{R}_j)^2+(\vec{r}_2-\vec{R}_i)^2 \right. \\
&\qquad \left. \left. + 2i[(x_1-x_2)(Y_i-Y_j)-(y_1-y_2)(X_i-X_j)]\} \right] \right\} \\
&= \frac{\sqrt{\pi}}{2l} e^{-(R/l)^2/8} I_0[\tfrac{1}{8}(R/l)^2](1-e^{-(R/l)^2/4}) \quad \text{with } \vec{R}=\vec{R}_i-\vec{R}_j ,
\end{aligned} \tag{A3}$$

$$\begin{aligned}
\tilde{V}_3(\vec{R}_i,\vec{R}_j,\vec{R}_k) &= -\int \frac{d^2r_1 d^2r_2 d^2r_3}{|\vec{r}_1-\vec{r}_2|} [\,|\psi_{\vec{R}_i}(\vec{r}_1)|^2 \psi_{\vec{R}_j}^*(\vec{r}_2)\psi_{\vec{R}_k}^*(\vec{r}_3)\psi_{\vec{R}_k}(\vec{r}_2)\psi_{\vec{R}_j}(\vec{r}_3) \\
&\qquad - \psi_{\vec{R}_i}^*(\vec{r}_1)\psi_{\vec{R}_j}^*(\vec{r}_2)\psi_{\vec{R}_k}^*(\vec{r}_3)\psi_{\vec{R}_j}(\vec{r}_1)\psi_{\vec{R}_k}(\vec{r}_2)\psi_{\vec{R}_i}(\vec{r}_3)] + \text{c.c} \\
&= -\frac{1}{(2\pi l^2)^3} \int \frac{d^2r_1 d^2r_2 d^2r_3}{|\vec{r}_1-\vec{r}_2|} \left[\exp\left[-\frac{1}{4l^2}[2(\vec{r}_1-\vec{R}_i)^2+(\vec{r}_2-\vec{R}_j)^2+(\vec{r}_3-\vec{R}_k)^2] + (\vec{r}_2-\vec{R}_k)^2 \right. \right. \\
&\qquad \left. + (\vec{r}_3-\vec{R}_j)^2 + 2i[(x_2-x_3)(Y_j-Y_k)-(y_2-y_3)(X_j-X_k)] \right] \\
&\qquad - \exp\left[-\frac{1}{4l^2}\{(\vec{r}_1-\vec{R}_i)^2+(\vec{r}_2-\vec{R}_j)^2+(\vec{r}_3-\vec{R}_k)^2+(\vec{r}_1-\vec{R}_j)^2 \right. \\
&\qquad\quad + (\vec{r}_2-\vec{R}_k)^2+(\vec{r}_3-\vec{R}_i)^2 \\
&\qquad\quad + 2i[(x_1-x_3)Y_i+(x_2-x_1)Y_j+(x_3-x_2)Y_k \\
&\qquad\quad \left. \left. \left. - (y_1-y_3)X_i-(y_2-y_1)X_j-(y_3-y_2)X_k]\} \right] \right] + \text{c.c} \\
&= -\frac{\sqrt{\pi}}{2l} \exp(-X_{ijk}) I_0(X_{ijk}) \left[1 - \exp\left[-\frac{1}{4l^2}(\vec{R}_{ij}^{\,2}+\vec{R}_{jk}^{\,2}+\vec{R}_{ki}^{\,2}-4iA_{ijk}) \right] \right] ,
\end{aligned} \tag{A4}$$

where

$$X_{ijk}=\frac{1}{8l^2}[\vec{R}_{ij}\cdot\vec{R}_{ik}-i(X_{ij}Y_{ik}-X_{ik}Y_{ij})]\ , \tag{A5}$$

and A_{ijk} is the area of the triangle formed by $\vec{R}_i$, $\vec{R}_j$, and $\vec{R}_k$. Similarly,

$$S_{ij}=\int d^2r_1d^2r_2\psi^*_{\vec{R}_i}(\vec{r}_1)\psi^*_{\vec{R}_j}(\vec{r}_2)\psi_{\vec{R}_i}(\vec{r}_2)\psi_{\vec{R}_j}(\vec{r}_1)=\exp\left[-\frac{1}{2l^2}\vec{R}^{\,2}_{ij}\right]\ , \tag{A6}$$

$$S_{ijk}=\int dr_1^2dr_2^2dr_3^2\psi^*_{\vec{R}_i}(\vec{r}_1)\psi^*_{\vec{R}_j}(\vec{r}_2)\psi^*_{\vec{R}_k}(\vec{r}_3)\psi_{\vec{R}_j}(\vec{r}_1)\psi_{\vec{R}_k}(\vec{r}_2)\psi_{\vec{R}_i}(\vec{r}_3)=\exp\left[-\frac{1}{4l^2}(\vec{R}^{\,2}_{ij}+\vec{R}^{\,2}_{jk}+\vec{R}^{\,2}_{ki}-4iA_{ijk})\right]\ , \tag{A7}$$

and combining these we obtain

$$E_c=\left[\tfrac{1}{2}\sum_{i\neq j}\tilde{V}_2(\vec{R}_i,\vec{R}_j)+\tfrac{1}{6}\sum'_{i,j,k}\tilde{V}_3(\vec{R}_i,\vec{R}_j,\vec{R}_k)+\cdots\right]\left[1-\sum_{i\neq j}S_{ij}+\sum_{i,j,k}S_{ijk}\right]^{-1}$$

$$\cong\tfrac{1}{2}\sum_{i\neq j}V(\vec{R}_i-\vec{R}_j)+\tfrac{1}{6}\sum_{i,j,k}V_3(\vec{R}_i,\vec{R}_j,\vec{R}_k)\ , \tag{A8}$$

where

$$V(\vec{R}_i-\vec{R}_j)\cong\frac{e^2\tilde{V}_2(\vec{R}_i,\vec{R}_j)}{1-S_{ij}} \tag{A9}$$

and

$$V_3(\vec{R}_i,\vec{R}_j,\vec{R}_k)\cong e^2\tilde{V}_3(\vec{R}_i,\vec{R}_j,\vec{R}_k)\ . \tag{A10}$$

This gives Eqs. (9) and (10) of the text.

APPENDIX B: CORRELATION ENERGY OF THE HOLE LATTICE (for $\nu>\frac{1}{2}$)

First let us note that the electron density of the completely filled $n=0$ Landau level is uniform and $n=(2\pi l^2)^{-1}$, the density correlation is classical-plasma-like.[8] This is shown by evaluating

$$\langle\rho(\vec{r})\rho(\vec{r}')\rangle=g(\vec{r},\vec{r})g(\vec{r}',\vec{r}')-|g(\vec{r},\vec{r}')|^2\ , \tag{B1}$$

where

$$g(\vec{r},\vec{r})=\langle\psi^\dagger(\vec{r})\psi(\vec{r})\rangle=\frac{n}{\sqrt{\pi}\,l}\int dk\,e^{-(x-k)^2/l^2}=n \tag{B2}$$

and

$$g(\vec{r},\vec{r}')=\langle\psi^\dagger(\vec{r})\psi(\vec{r}')\rangle$$

$$=\frac{n}{\sqrt{\pi}l}\int dk\exp-[(x-k)^2+(x'-k)^2+i(x+x')(y-y')]/2l^2$$

$$=n\exp-[(\vec{r}-\vec{r}')^2+2i(x+x')(y-y')]/4l^2\ . \tag{B3}$$

Putting them together we have

$$\langle\rho(\vec{r})\rho(\vec{r}')\rangle=n^2(1-e^{-(\vec{r}-\vec{r}')^2/2l^2})\ . \tag{B4}$$

This is deduced recently by Laughlin,[8] who used a different argument. From this the correlation energy of the filled $n=0$ Landau level (i.e., the case $\nu=1$) is calculated as

$$U_{\rm cor}(\nu=1)=-\frac{e^2}{2(2\pi l^2)}\frac{1}{S}\int\frac{d^2r\,d^2r'}{|\vec{r}-\vec{r}'|}e^{-(\vec{r}-\vec{r}')^2/2l^2}=-\frac{e^2}{2l}(\pi/2)^{1/2}\ . \tag{B5}$$

Now in the presence of the hole lattice in addition to the filled $n=0$ Landau level we have

$$g(\vec{r},\vec{r})\cong n\left[1-\sum_i e^{-(\vec{r}-\vec{R}_i)^2/2l^2}\right] \tag{B6}$$

and

$$g(\vec{r},\vec{r}')\cong n\left[\exp-\{(\vec{r}-\vec{r}')^2+i[(x+x')(y-y')-(y+y')(x-x')]\}/2l^2 - \sum_i \exp-\{(\vec{r}-\vec{R}_i)^2+(\vec{r}'-\vec{R}_i)^2+2i[(y-y')X_i-(x-x')Y_i]\}/4l^2\right], \tag{B7}$$

where the second terms in (B6) and (B7) are due to the hole lattice. In writing the first term of (B7), we have changed the gauge to the symmetric one to be consistent with the localized Landau wave functions. This yields

$$\langle\rho(\vec{r})\rho(\vec{r}')\rangle = n^2\left[\left[1-\sum_i e^{-(\vec{r}-\vec{R}_i)^2/2l^2}\right]\left[1-\sum_i e^{-(\vec{r}'-\vec{R}_i)^2/2l^2}\right] - \left|\exp-\{(\vec{r}-\vec{r}')^2+i[(x+x')(y-y')-(y+y')(x-x')]\}/2l^2 - \sum_i \exp-\{(\vec{r}-\vec{R}_i)^2+(\vec{r}'-\vec{R}_i)^2+2i[(y-y')X_i-(x-x')Y_I]\}^2/4l^2\right|\right]. \tag{B8}$$

Then the correlation energy for the hole lattice is calculated easily as

$$U_{\rm cor}^{\rm hole}(\nu)=\frac{e^2}{2N}\int\frac{d^2r\,d^2r'}{|\vec{r}-\vec{r}'|}[\rho(\vec{r})\rho(\vec{r}')-(1-2\nu)n^2]=U_{\rm cor}(1-\nu)-\frac{e^2}{2l^2}(\pi/2)^{1/2}(2\nu-1)\,. \tag{B9}$$

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