Critical diffusion in a two-dimensional Ising system

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The relaxation of an Ising spin system which conserves the magnetization is analyzed using the time-dependent real-space renormalization group. The critical-dynamic exponent is found to be 3.74, close to the value suggested by the conventional theory.

I. INTRODUCTION AND DISCUSSION

The Ising spin system with a conserved magnetization is one of the simplest models in which the critical slowing down[1,2] is affected by a conservation law. An empirical master equation which describes the relaxation of the local magnetization was suggested by Kawasaki.[1] He generalized the Glauber model[3] by assuming that a spin flip takes place only between two spins which are nearest neighbors (NN) and with a transition rate which is determined by the detailed balance. Kawasaki calculated an upper limit to the dynamic exponent z, characterizing the slowing down of the diffusion process occurring in this model. A renormalization-group (RG) analysis in a 4-ϵ dimension of a similar model by Halperin *et al.*[4] revealed that this limit of z is the exact result. However, their expansion is not valid in low dimensions. Hence, although the relation $z=4-\eta$, where η as well as the other critical indices have their standard definitions, is believed to hold in all dimensions, a general proof to this assumption does not exist.

Recently we proposed, using the real-space RG technique, to analyze the time-dependent properties of systems near their critical point.[5–9] This technique which fits the low-dimensional system was used to calculate z of a one-dimensional (1D) magnetization-conserving Ising model.[9] In our RG technique it is preferable to deal with nonlocal perturbations. Thus we modified the Kawasaki model. We allow any two spins in the system to exchange their magnitude and not only the NN, as was postulated by Kawasaki. We found by an exact RG transformation that $z=3$ as expected.

In this paper we analyze the relaxation of a similar model but in two dimensions. Unfortunately, we are not able to perform an exact RG transformation. We chose the Kadanoff potential moving approximation,[10] based on the 1D spin-block transformation, and obtained $z=3.746$. This value is very close to the one at 4-η, and thus strengthens its validity. Since we have first published briefly the main results of this study,[11] the original Kawasaki model was studied by using the different real-space RG technique[12] (the cumulant approximation). Although the method in that case was limited to the study of the behavior of the correlation function, and the model was different from the present one, there was found a similar value of z, and that the operator $0_i (i=2,3,4)$ [see Eq. (19)] should be included in order to have an invariant dynamical operator. Thus both the Kawasaki model with the local spin conservation and the present model with the nonlocal spin conservation belong to the same universality class, and have the conventional behavior.

II. MODEL AND RG TRANSFORMATION

The model describes the time-dependent behavior of a large interacting spin system whose equilibrium is determined by an Ising Hamiltonian:

$$H=(K/g)\sum_i \sigma_i \sum_{j\,(\mathrm{NN})} \sigma_j \ , \tag{1}$$

where $\{\sigma_i=\pm 1\}$, g is the coordination number of the lattice, and σ_j are the NN of σ_i. The system is brought into constrained equilibrium. The constraint is removed at $t=0$ and the system relaxes toward the final equilibrium, described by (1). We assume the relaxation to be an instantaneous random spin-exchange transformation between two spins. This procedure is described by the empirical master equation for the spin's probability distribution $P(\{\sigma\},t)$:

$$\frac{\tau\, dP(\{\sigma\},t)}{dt}=-\sum_{i,j}(1-p_{ij})\,W_{ij}(\sigma)P(\{\sigma\},t) \ , \tag{2}$$

where τ is the bare time scale which characterizes the interaction with the heat bath. p_{ij} is a spin-exchange operator:

$$p_{ij}f(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_j,\ldots,\sigma_n)=f(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_i,\ldots,\sigma_n) \ , \tag{3}$$

and the transition probability distribution W_{ij} satisfies the detailed balance which ensures the ergodicity of the

system:

$$(1-p_{ij})W_{ij}P_e(\sigma)=0 \ . \tag{4}$$

The normalized equilibrium probability distribution $P_e(\sigma)$ is given by (1):

$$P_e(\sigma)=\exp(-H/k_BT)/Z\equiv\exp(\bar{H})/Z \ . \tag{5}$$

By using the perturbation from equilibrium, $\phi(\sigma,t)\equiv P(\sigma,t)/P_e$, Eq. (2) reads

$$\frac{\tau\, dP(\sigma,t)}{dt}=-\sum_{i,j}P_eW_{ij}(\sigma)(1-p_{ij})\phi \ . \tag{6}$$

The right-hand side (rhs) of (6) is composed of two parts. The first one, $P^{ij}\equiv P_eW_{ij}$, is by (4) symmetric under the $\sigma_i\leftrightarrow\sigma_j$ exchange. It can be taken as independent of these spins, due to the fact that the dependent part gives a vanishing contribution to (6). This property, that P^{ij} can be taken as independent of i and j, is responsible for the detailed balance. The second term, $(1-p_{ij})\phi$, is antisymmetric under the i-j spin's exchange, and takes care of the dynamic equilibrium restoring force. We will use

$$W_{ij}(\sigma_i,\sigma_j)=-\frac{1}{N}\left(\frac{P_e(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_j,\ldots,\sigma_n)}{P_e(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_i,\ldots,\sigma_n)}\right)^{-1/2}\exp\left[-\frac{K}{2}(\sigma_i+\sigma_j)\left(\sum_{k\,(\text{NN of }i)}\sigma_k+\sum_{l\,(\text{NN of }j)}\sigma_l\right)\right] \ . \tag{7}$$

The last factor in W_{ij} is entered to remove the i,j dependence in P^{ij}. It is not equal to unity only when all the rhs of (6) vanishes. There is no limitation on the distance between the spins. Hence, i and j are not correlated and the sum over them is of order N, where N is the number of spins in the system. To obtain the correct dependence, an extra factor of $(1/N)$ has been introduced in (7). In the previous study of the 1D case[9] we found that the relevant perturbation to the equilibrium is an energylike perturbation,

$$\phi=h\sum_{i,j}\sigma_i\sigma_j \ , \tag{8}$$

where i and j are NN. Our study will be restricted to this perturbation.

The time-dependent real-space RG (TRG) transformation is composed of two stages. First, the RG transformation is applied to the two sides of the master equation (6). Thus the transformation of the left-hand side (lhs) of (6) is nothing other than the standard RG static transformation.[13] This transformation maps the set of the spin variables $\{\sigma\}$ to a new set of spin variables $\{\mu\}$, defined on a lattice whose dimensions are scaled by a factor b, while preserving the free energy of the system. The transformation is performed by using the operator $T(\mu,\sigma)$ operating on $P(\sigma,t)$:

$$P(\mu)=\sum_{\sigma}T(\mu,\sigma)P(\sigma) \ . \tag{9}$$

The probability distribution can be represented as a point in the parameter space, $\vec{K}=\{K_\alpha\}$, of the interactions appearing in the Hamiltonian (and ϕ). The RG of the lhs of (6) is represented in the parameter space:

$$\vec{K}'=\underline{R}\vec{K} \ , \tag{10}$$

where $\{K'_\alpha\}$ are the interactions in $P'(\mu)$. The fixed point of this transformation, $\vec{K}^*=\underline{R}\vec{K}^*$, is associated with a critical point (or zero correlation).[14]

The RG transformation that we use in the present calculation is the spin-block transformation.[15] In each direction it is

$$T(\sigma_{2i},\sigma_{2i+1};\mu_\alpha)=\tfrac{1}{2}\cosh(K)(1+\mu_\alpha\sigma_{2i}\tanh q)(1+\mu_\alpha\sigma_{2i+1}\tanh q)\exp(-K\sigma_{2i}\sigma_{2i+1}) \ , \tag{11}$$

where $\tanh q=(\tanh K)^{1/2}$.

The transformation of the rhs of the master equation (6) is performed in a similar manner. Usually new terms are generated during the transformation. These terms have to be included *a priori* in the kinetic equation. This procedure is terminated when an invariant expression is obtained.[5] The rhs becomes a combination of terms associated with slow modes and transients. The rhs is written in a general form:

$$h\sum_{i,j}P^{ij}[\vec{K}\cdot\vec{O}(\sigma)] \ , \tag{12}$$

where h is the perturbation field, contributed by ϕ, $\vec{O}(\sigma)$ are the set of invariant spins operators which are created by the RG transformation of ϕ, and K are their amplitudes.

Under the RG transformation of ϕ, expression (12) becomes

$$h\sum_{\alpha,\beta}P'^{\alpha\beta}(\mu)[\underline{\Omega}\vec{K}\cdot\vec{O}(\mu)] \ , \tag{13}$$

where $P'^{\alpha\beta}$ is determined by the static RG of P_e and W. Thus the transformation of (12) is represented by

$$\vec{K}'=\underline{\Omega}\vec{K} \ . \tag{14}$$

The slow mode is the eigenoperator of $\underline{\Omega}$ with the largest eigenvalue[5] ω. The other eigenoperators describe transients.

The second stage of the TRG is the time scaling.

This is performed using a scale factor:

$$\tau' = b^z \tau; \quad b^z \equiv \lambda/\omega \ , \tag{15}$$

where λ is the scale factor of h. The transformation leaves the equation of motion for the slow mode invariant, while the weight of the transients becomes smaller. By using the standard RG arguments, z is identified as the dynamic critical exponent.[5]

The approximation in which the RG transformation is carried out is the potential moving approximation.[10] In this approximation part of the interactions in one direction are shifted to other locations on the lattice. That enables us to perform an effective 1D RG transformation in the other direction. This approximation gives a lower bond of the free energy, and has already been applied to the study of the magnetization[8] and energy relaxation. The effective 1D RG transformation can be either decimation or spin-block transformation. In 2D the recursion relations for the parameters K in the Hamiltonian (1) are

$$K_x' = bR^b K_x; \quad K_y' = R^b(bK_y) \ , \tag{16}$$

where R^b represents a 1D RG transformation which scales the space by a factor b. For more detail the reader is referred to Kadanoff[10] for the static RG, and to Achiam[8] for the TRG.

III. POTENTIAL MOVING APPROXIMATION

The RG transformation is carried out with the use of the potential moving approximation. This approximation effectively reduces the dimensionality of the problem to 1D. The relaxation of the 1D conserved magnetization Ising system has been studied before[9] by using the decimation transformation, based on the value $\beta/\nu = 0$ which is correct only in 1D. Thus we perform the calculation using the spin-block transformation.[15]

Starting with (7) and (8), the master equation (6) reads

$$\frac{\tau\, dP(\sigma,t)}{dt} = -\frac{1}{N}\sum_{i,j} P^{ij}(\sigma_i - \sigma_j) \times (\sigma_{i-1} + \sigma_{i+1} - \sigma_{j-1} - \sigma_{j+1}) \ . \tag{17}$$

$P(\sigma,t)$ is characterized in the parameter space by two parameters: K (1) and h (8). The RG transformation (9) transforms them as

$$K' = \operatorname{arctanh}[\tanh^2(K)]; \quad h' = \lambda h; \quad \lambda = 2[\tanh^{1/2}(K)][1+\tanh(K)]/[1+\tanh^2(K)] \ . \tag{18}$$

In the transformation of the rhs of (17) the following four spin operators have been generated:

$$\begin{aligned} O_1 &= (\sigma_i - \sigma_j)(\sigma_{i-1} + \sigma_{i+1} - \sigma_{j-1} - \sigma_{j+1}) \ , \\ O_2 &= \sigma_i \sigma_j \ , \\ O_3 &= \sigma_i \sigma_j (\sigma_{i-1} + \sigma_{i+1})(\sigma_{j-1} + \sigma_{j+1}) \ , \\ O_4 &= \sigma_i \sigma_j (\sigma_{i-1}\sigma_{i+1} + \sigma_{j-1}\sigma_{j+1}) \ . \end{aligned} \tag{19}$$

It is easy to calculate the new parameters of these operators by using the graphic representation appearing in Fig. 1. These terms conserve the magnetization: They all depend on $\sigma_i\sigma_j$, and $\mathrm{tr}_{\{\sigma\}} \sum_k \sigma_i\sigma_j\sigma_k = 0$. The new terms do not change the dynamic behavior. However, their existence means that the invariant parameter subspace of the problem must at least consist of O_1, O_2, O_3, and O_4. It is easy to see that this parameter subspace is invariant under the RG transformation, and the new parameters are transformed as follows:

$$k_2' = 2^6 \cosh^4(K) a_x^2 A^{-4} k_2 \ , \tag{20a}$$

$$k_3' = 2^4 a_x'^2 \cosh^2(K) \exp(2K) A^{-4} k_3 \ , \tag{20b}$$

$$k_4' = 2^6 a_x' \sinh(K) \cosh^3(K) A^{-4} k_4 \ , \tag{20c}$$

where

$$a_x' = \tanh^2 q/[1+\tanh(K)] \ ,$$

$$a_x = a_x'/\tanh q \ .$$

k_i, $i = 1, \ldots, 4$ are parameters of O_i, $i = 1, \ldots, 4$, respectively, and $A = 2[1+\tanh^2(K)]^{1/2}\cosh(K)$. The contribution from the transformation of O_1 and

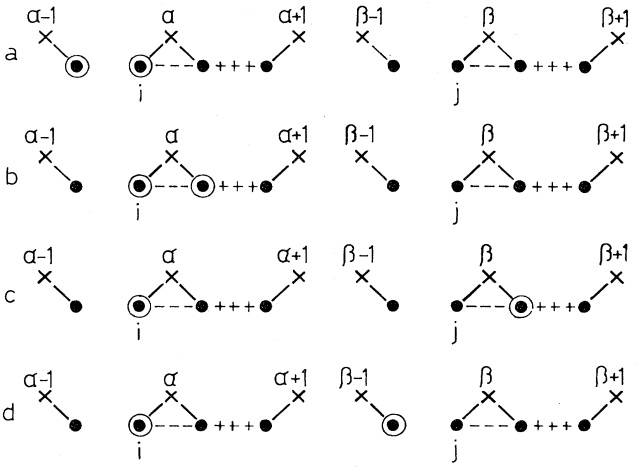

FIG. 1. Graphs which describe the RG transformation of O_1 [Eq. (8)]. The spins $\{\sigma\}$ are marked by dots. The new spins $\{\mu\}$ are marked by crosses. A bond of + or − symbols stands for a K or $-K$ interaction, respectively. A full bond between σ and μ contributes $\cosh(K) \times [(1+\sigma\mu\tanh(K)]/\,2^{1/2}$. A spin which appears linearly is marked by a circle around the dot.

from Eq. (20) are collected together to give

$$\underline{\Omega}=\begin{pmatrix} 2a_x'\cosh(K)\exp(K) & 0 & 0 & 0 \\ -2^2a^2\cosh^2(K) & a_x^2\cosh^2(K) & 0 & 0 \\ 2a_x'^2\sinh(K)\exp(K) & 0 & 2^2a_x'^2\exp(2K) & 0 \\ -2^2a_x'^2\sinh(K)\cosh(K) & 0 & 0 & 2^2a'^2\sinh(K)\cosh(K) \end{pmatrix} \times[1+\tanh^2(K)]^{-2}\cosh^{-2}(K)2^{-d} \ . \tag{21}$$

An extra factor of 2^{-d} has been introduced into (21). This factor results from the transformation $N=2^dN'$.

We can now check the 1D limit. In this case $K^*=\infty$, all the eigenvalues of $\underline{\Omega}$ are the same, $\omega_{1D}=2^{-2}$, and $\lambda_{1D}=2$. A substitution in (15) yields $z=3$, in agreement with our previous decimation calculation.[9]

Now we can use the potential moving approximation and scale the whole 2D space. The application of this approximation to the TRG has been discussed in length by the author in Ref. 8, and the reader is referred to this paper for more details. The result of the potential moving approximation is that we deal with perturbations in one direction only. In each direction we get the Ω which is found in (21), but with the following parameters:

$$\underline{\Omega}_x=\underline{\Omega}(\vec{K}_x); \quad \underline{\Omega}_y=\underline{\Omega}(K_y) \ . \tag{22}$$

Similar expressions are found for a system of higher dimensionality.

At the fixed point, $K_x^*=bK_y^*$. Thus the common contribution $\underline{\Omega}(K_x^*)$ from the TRG of the perturbation in different directions can be collected together. At the fixed point,[10] $K_x^*=0.609$; $\lambda=2.174$, where λ describes the scaling of the energy perturbation. The largest eigenvalue of $\underline{\Omega}$ is $\omega=0.162$ and the dynamic exponent is $z=3.747$.

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[1]K. Kawasaki, Phys. Rev. 145, 224 (1966).
[2]P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 46, 435 (1977).
[3]R. G. Glauber, J. Math. Phys. 4, 297 (1963).
[4]B. I. Halperin, P. C. Hohenberg, and S.-k. Ma, Phys. Rev. B 10, 139 (1974).
[5]Y. Achiam and M. J. Kosterlitz, Phys. Rev. Lett. 41, 128 (1978); Y. Achiam, Physica (Utrecht) (in press).
[6]Y. Achiam, J. Phys. A 11, 975 (1978).
[7]Y. Achiam, Phys. Rev. B 19, 376 (1979); Phys. Lett. 74A, 247 (1979).
[8]Y. Achiam, J. Phys. A 13, 1355 (1980).
[9]Y. Achiam, J. Phys. A 13, 1825 (1980).
[10]L. P. Kadanoff, Ann. Phys. (N.Y.) 100, 359 (1976).
[11]Y. Achiam, in the Fourteenth International Conference on Thermodynamics and Statistical Mechanics, Edmonton, Canada, 1980 (unpublished).
[12]S. J. Heilig, J. Luscombe, G. F. Mazenko, E. Oguz, and O. T. Valls, Phys. Rev. B 25, 7003 (1982).
[13]Y. Achiam, Phys. Lett. 72A, 35 (1979).
[14]K. J. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).
[15]D. R. Nelson and M. Fisher, Ann. Phys. (N.Y.) 91, 226 (1976).