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Eigenstates and properties of random systems in one dimension at zero temperature

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Eigenenergies and eigenfunctions are evaluated. Different localized eigenstates are proven to have exponentially different mobilities. This leads to exponentially high and exponentially narrow resonances of the conductance at eigenenergies. The probability distribution of resonance resistances is determined. The height and the width of a resonance allow one to evaluate the localization position and the localization length of an eigenstate. The phase correlation length is proven to be $\frac{1}{2}L_0$, where L_0 is the wave-function localization length. The latter strongly depends on the analytical nature of disorder. The dependence of L_0 on energy $\mathscr{E}$ may vary from algebraic to exponential. In the latter case a weak localization is achieved at reasonable energies $\mathscr{E} \propto [\ln(L/L_0)]^2$, where L is the length of the system. Different one-dimensional problems are reduced to the Schrödinger equation. In particular, at low frequencies ω, acoustic phonons and electromagnetic waves in a random media are localized. Their localization length is $L_0 \propto \omega^{-2}$.

I. INTRODUCTION: LOCALIZATION IN PHYSICS

Anderson[1] was the first to prove that a quantum particle in a random potential may be localized, so that its probability density may exponentially decay outside a certain "region of localization." Following Anderson, one usually considers localization in a macroscopic system. It is instructive, however, to explore a more general idea of localization: a finite motion in classical mechanics, an exponential maximum of the probability density in quantum mechanics, or an exponential maximum of the intensity of any wave. The maximum may be reached in a real space, in the momentum space, or with respect to any other variable. Such a localization is one of the most common phenomena in physics. As we shall see later, it exposes a deep similarity between phenomena very different in other aspects, and allows one to understand important features of localization by studying very simple models.

The simplest case of such a localization is the trivial localization of a free particle in a momentum space. (The localization is related to the conservation of momentum.) In a general case, in virtue of the conjugation of coordinate and momentum, localization in real space always implies an extended momentum state, and vice versa. Further, I consider the localization in real space. Some examples: (a) Particles in nucleus, electrons in atoms, atoms in molecules are localized in "microworld." The violation of the localization is correspondingly manifested in radioactivity of nuclei, ionization of atoms, and dissociation of molecules. (b) In "macroworld" ions, atoms, or molecules, or domain walls, are localized in solids and may be "delocalized" while melting. A strong localization of charges in amorphous and disordered semiconductors, inversion layers, and field-effect transistors (FET's) leads to the activated conductivity σ. Its temperature T dependence may be related to the thermal distribution rather than to inelastic collisions. [This is obvious from the independence of $\sigma(T)$ of the inelastic mean-free-path time.] In Mott three-dimensional (3D) case[2] $\ln\sigma(T) \propto -T^{-1/4}$. In the case of the nearest-neighbor hopping $\sigma(T)=\sigma_0 \exp(-\Delta E/k_B T)$, where ΔE is the activation energy and k_B is the Boltzmann constant. According to the Meyer-Neldel rule,[3] $\sigma_0=\sigma_{00}\exp(\Delta E/k_B T_c)$, where σ_{00} and T_c are constants. Brodsky[4] suggested that this dependence may be related to localization. He also suggested that the tunneling and breakdown in insulators may be related to the localized states, which are related to impurities. This seems to be confirmed by B. Ricco *et al.*[5]

In metals the localization at very low temperatures leads[6] to the insulator-type behavior of $\sigma(T)$, which increases with T. In inversion layers and FET's the localization leads to giant oscillations of σ with the gate voltage[7]: σ may change at least by 2 orders of magnitude when the gate voltage changes by 0.01%. A quantum particle in a quasiperiodic potential,[8] a Bloch electron in a magnetic field,[9] sound and electromagnetic waves[10] (in the absence of energy dissipation) may also be localized. (c) In "cosmoworld" planets in star systems, stars in double stars, matter in black holes, in a sense the whole universe in the closed model is also localized. The phenomena, which may be reduced[10,11] to the Schrödinger-type equation, and thus related to the localized solutions, are heat transfer, classical diffusion, 1D Ising model, and polystrand polymer thermodynamics.

In the absence of inelastic collisions, where an external potential does not change, the problem reduced to a *one-particle* problem. In the case of a charged quantum particle, this is clearly demonstrated by the Landauer formula.[12] It reduces the dimensionless resistance R (i.e., a *kinetic* characteristic of a random system) to a transmission coefficient t (i.e., a one-particle quantum-mechanical characteristic),

$$R = t^{-1} - 1 \,. \tag{1}$$

One-particle quantum mechanics yields the eigenstates. In 1D all eigenstates are localized.[6,13–15] Thus, their probability density at the ends is exponentially small with

L/L_0 when $L \to \infty$ (here L is the length of the system and L_0 is the localization length). Therefore, localized particles have exponentially small chances to run away from the random system. Vice versa, a particle with any energy different from an eigenenergy stays out of the random system, since its probability density exponentially decays inside the system, and has exponentially slim chances to get well inside it. Only an incident particle with the resonant energy equal to an eigenenergy has a better chance to get through the system.[16–18] This leads to exponentially (with respect to L/L_0) high and exponentially narrow resonances[16,17] of t [and therefore, by Eq. (1), of the conductance G] with the energy $\mathscr{E}$ and to the long-range correlation at the eigenenergy of a finite system.[19] The resonances are clearly seen on different scales in the numerical experiment in Figs. 1–5. The correlation function is presented in Fig. 6. (The details of the numerical experiment are described later.) The decrease in the resistance at least by 2 orders of magnitude, when the relative change in the Fermi energy is $\sim 0.01\%$, was observed in the experiments on accumulation layers.[7]

The knowledge of the position, height and width of a resonance allows one to determine the eigenenergy, its localization in the random system, and its localization length. Figure 7 demonstrates the (numerically calculated) probability density for the resonance eigenenergy in Fig. 5. The spectroscopy according to the "experiment" in Fig. 5 leads to the localization at the distance 29 from the end and to the localization length $L_0 = 2.2$. The "exact" values in Fig. 7 are correspondingly 31 and 2.2. Such a spectacular agreement is an accident [the relative accuracy should be $\sim (L_0/L)^{1/2}$] but it does demonstrate the efficiency of the method. Resonances in the case of a weak localization are presented in Fig. 8. The dependence

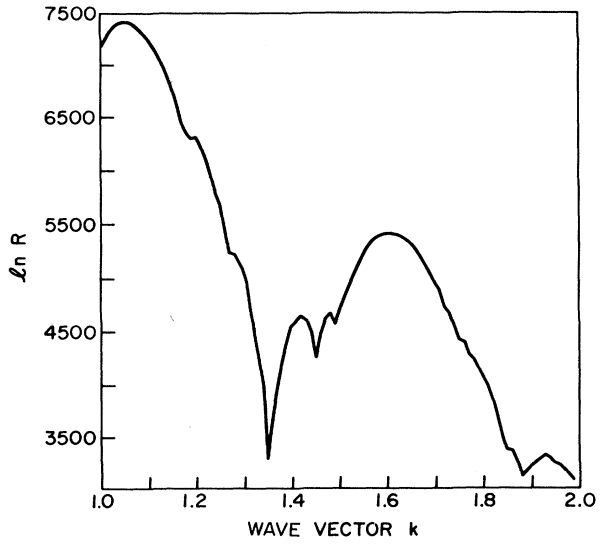

FIG. 1. The dependence of $\ln R$ (R is the dimensionless resistance on the dimensionless wave vector k—see Eq. (48). The strength of the dimensionless impurity potential is $v = 6$; its concentration is $c = 0.5$. The number of sites in the system is $L = 5000$. The "step" in k is 0.01.

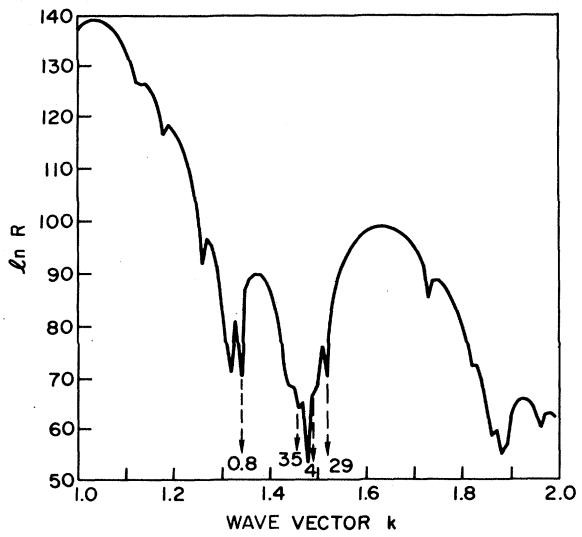

FIG. 2. Same as in Fig. 1 but for $L = 100$. Dashed arrows indicate the true value of $\ln R$ at the corresponding resonance values of k.

of the localization length on the Fermi energy is related to the nature of disorder and varies from linear to exponential.

If a nonmonochromatic set of particles hits a random system, then those outside the eigenenergy halfwidth (which is exponentially small with L/L_0 but still finite in a finite random system) are practically completely reflected. This may be denoted as a "junction reflection," since it is little dependent on the properties of a random system. The particles with the energies within the eigenenergy halfwidth are transmitted according to the resonance properties of this particular eigenenergy. The total number of transmitted particles is the product of the number of the "resonance" particles and their resonance transmis-

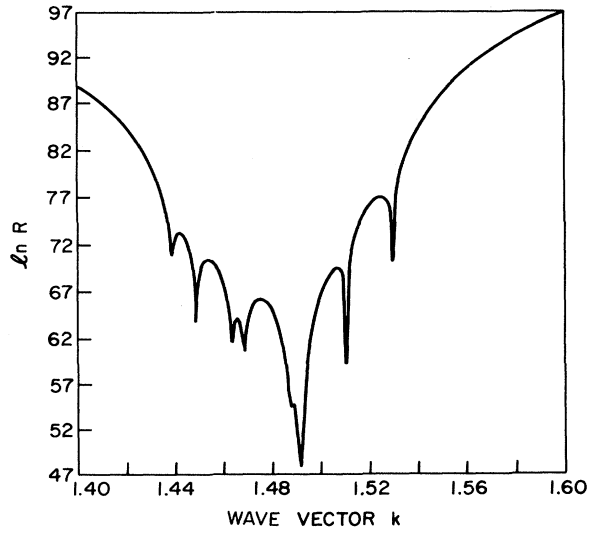

FIG. 3. Same as in Fig. 2 but for $\Delta k = 0.001$.

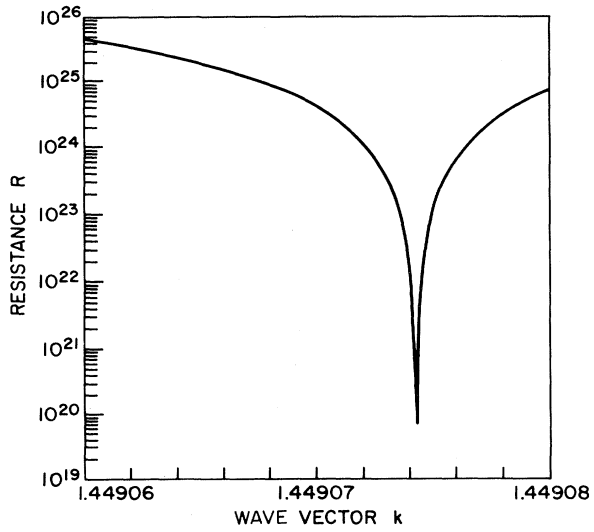

FIG. 4. Same as in Fig. 3 but for $\Delta k = 10^{-7}$

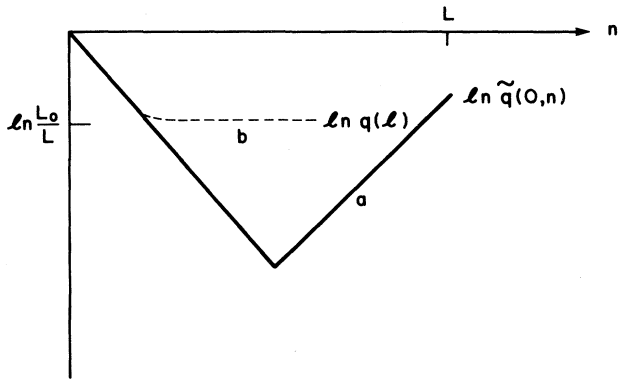

FIG. 6. Phase correlation at the eigenenergy (schematically). The correlation function between the initial phase and the phase at the nth site is $\tilde{q}(0,n)$. It is minimal at the localization point. The correlation function at the distance l is $q(l)$. The localization length is L_0, the total length is L.

sion through the random system. Thus the total transmission coefficient is the product of the junction and resonance transmission coefficients. In the case of charged particles this implies that the total conductance is the product of the junction and resonance conductances. (The multiplicativity of conductances in series is characteristic of localization.) A two-point measurement (when the potential drop is measured outside the barrier resistor) yields the total conductance, while a four-point measurement (the potential drop is measured inside the barrier resistor) yields the resonance conductance.

The localization by a 1D disorder is characteristic of the elastic propagation of any wave: electromagnetic, acoustic, etc. The presented approach is applicable to practically any linear 1D problem: Anderson model (with or without off-diagonal disorder), phonon eigenstates, quasiperiodic potential, Ising thermodynamics, classical diffusion, percolation, or chaos in any of these systems. In this paper I consider the strong localization ($L \gg L_0$). However, the resonance tunneling will manifest itself under all circumstances (see, e.g., Fig. 8) if the resolution is high enough. This may explain why for almost two decades different authors reported irregular oscillations with the electron density of a resistance[20–28,7] and $1/f$ noise[29] in various FET's and Si inversion layers. Characteristically, these oscillations were reproducible only at a given sample.

The above reasoning is obviously applicable to any random system with no inelastic scattering and to any resonance tunneling. The resonance tunneling may be related, e.g., to the complete reflection in geometrical optics (Fig.

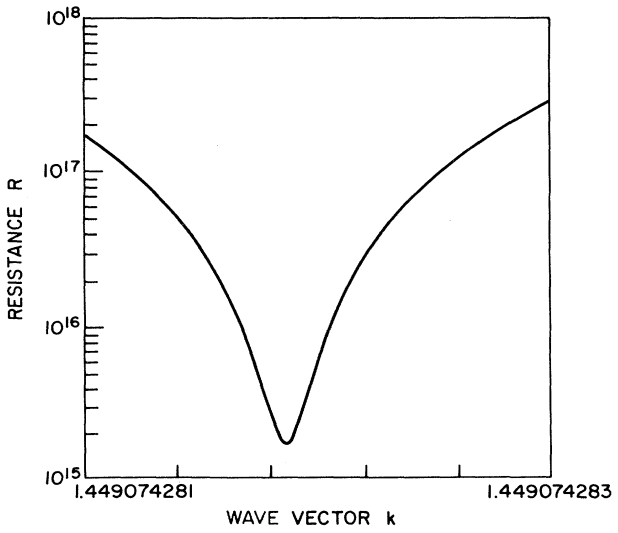

FIG. 5. Same as in Fig. 4 but for $\Delta k = 10^{-11}$. The resonance is at $k = 1.449\,074\,281\,87$ (all digits are meaningful).

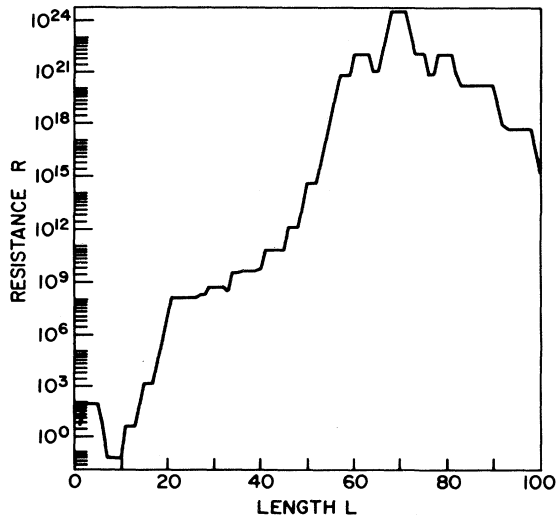

FIG. 7. Resistance, proportional to the probability density, as a function of site, for the random sequence and eigenenergy of Fig. 5. ($L = 100$ is the total length of the system.)

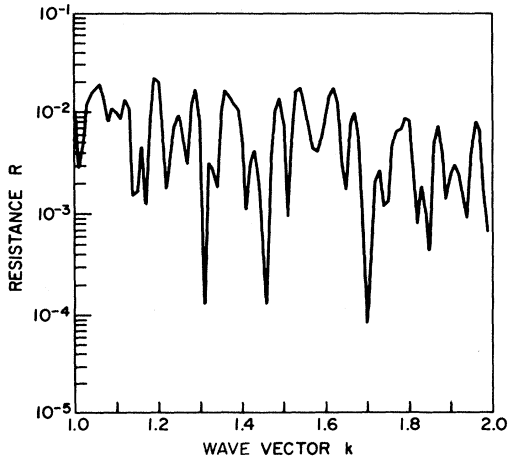

FIG. 8. Resistance R as a function of the dimensionless wave vector in the case of a weak localization. The impurity potential strength is $v=0.05$; its concentration is $c=0.5$; the length of the system $L=100$; "step" in k is $\Delta k=0.01$.

9, Brodsky's suggestion) and to the tunneling through and the breakdown in an insulator with impurities (Fig. 10, Brodsky's suggestion, verified by Ricco *et al.*[5]). Platzman[30] noted that the localization resonances of electromagnetic[10] and acoustic[11] waves may be easier to observe in a random media with artificially created large-scale imperfections.

II. PHYSICS OF LOCALIZED EIGENSTATES

A. A single barrier

In classical mechanics localization is absolute. Every particle with the energy above all potential barriers is never reflected, and every particle with the energy below the surrounding potential barriers is completely localized. In quantum mechanics only an infinitely high and/or broad barrier leads to a complete localization. Otherwise the quantum scattering is almost always finite and nonzero. Its characteristic feature is its remarkable dependence on the analytical nature of the potential. When the energy $\mathscr{E}$

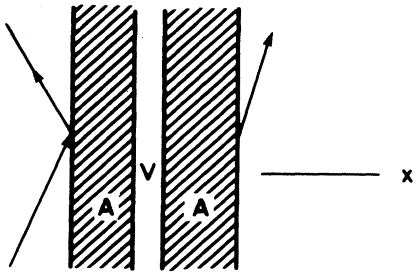

FIG. 9. Resonance transmission (via V) in the case of a complete reflection from A. Arrows denote incident, reflected, and transmitted waves.

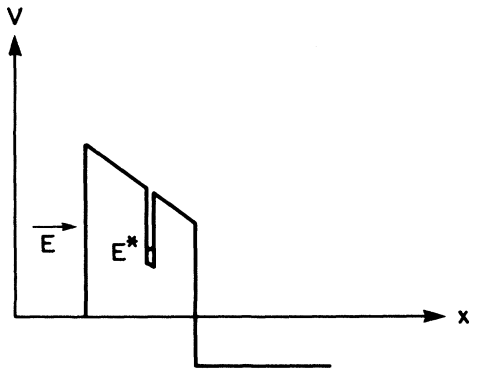

FIG. 10. Resonance tunneling via eigenenergy E^*; $E \geq E^*$ is the incident energy.

is low, then a "strong" δ-function potential leads to the transmission coefficient $t \propto \mathscr{E}$, while a quasiclassical potential (with a finite width and height) leads[31a(1)] to $\ln t \propto \mathscr{E}$. A high-energy particle has $(1-t) \propto 1/\mathscr{E}$ in the case of a δ function; $(1-t) \propto 1/\mathscr{E}^2$ when the potential barrier $U(x)$ is rectangular, $(1-t) \propto 1/\mathscr{E}^3$ when the potential energy has a discontinuity in the first derivative,[31(a2)] $(1-t) \propto 1/\mathscr{E}^{2+p}$ when $U^{(p-1)}(x)$ is continuous and $U^{(p)}(x)$ has a discontinuity, and $\ln(1-t) \propto -\mathscr{E}^{1/2}d$ when $U(x)$ is an analytical function[31(a3)] and d is a characteristic width of the potential. Note that in the latter case $d \to \infty$ (i.e., an infinitely wide potential) leads to *no* scattering: $t \to 1$.

B. Model of localization

The main specificity of quantum mechanics, which leads to the special properties of localized eigenstates, is related to a nonadditivity of potential barriers. Important features of localization are demonstrated already by the simplest case of a one-dimensional (1D) potential well between exponentially little transparent barriers. If the 1D barriers are infinitely high and/or broad, the well yields eigenstates; they are completely localized in a finite region, and their probability density exponentially decreases in the barriers. Other energies are forbidden in the well. Finite, but little-transparent barriers allow for the tunneling of a quantum particle to $\pm\infty$. The tunneling leads to a finite, but an exponentially long lifetime of a particle at a quasilevel, and thus to a finite width of the level $\delta\mathscr{E}$. The particles in the energy range $\delta\mathscr{E}$ stay in the well during the lifetime (although they do not "belong" in the well completely). The particles with other energies may penetrate into the well, but with exponentially little probability. They belong in the world outside the barriers and the well. (An example of this situation is a radioactive nuclei.) Such a well has "quasilevels" (E^* in Fig. 11). Their halfwidth $\delta\mathscr{E}$ is related to the tunneling through the more transparent barrier: $\delta\mathscr{E}/\mathscr{E} \sim t_1$; t_1 is the transmission coefficient through this barrier. It determines the long lifetime ($\propto 1/t_1$) of a particle in the well at the quasilevel. Long lifetime implies that the probability density ρ at the quasilevel is very high inside the well and de-

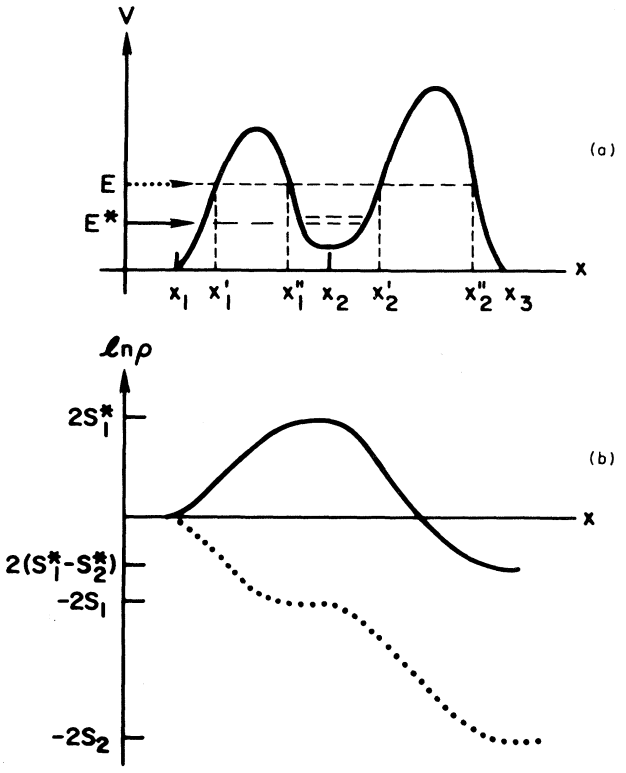

FIG. 11. Tunneling through potential barriers (a). Probability density ρ [(b), semilogarithmic scale] at "ordinary" energies (dotted line) and eigenenergies (solid line).

cays in the barriers [the solid line in Fig. 11(b)].

Consider the scattering of a particle with the energy $E \neq E^*$ by two potential barriers [Fig. 11(a)] with exponentially small transmission coefficients t_1 and t_2, with $t_1 > t_2$. The total transmission coefficient t_{12} is easily calculated (see Sec. IV) and equals $t_{12} \approx t_1 t_2$. The probability density ρ exponentially decreases in each of the barriers [the dotted line in Fig. 11(b)]. The ln$t$'s for the barriers add up: $\ln t_{12} \approx \ln t_1 + \ln t_2$, so the infinite set of barriers leads to $\ln t \to -\infty$ and $t \to 0$ (since $t_1, t_2, \ldots, \leq 1$).

The scattering of a particle with the energy E^* is completely different. Inside the well it finds itself at the quasilevel and stays there for an exponentially long time. Correspondingly, its probability density has an exponentially high maximum inside the well, exponentially decreasing in *both* barriers towards their ends [Fig. 11(b), solid line]. Its transmission coefficient equals $t_{12} \approx t_2 / t_1$. It may be even $t_{12} \simeq 1$, when $t_1 \simeq t_2$, i.e., when the barriers have the same *integral* characteristic-transmission coefficients. (In particular, periodic potential barriers, where all t's are equal, have completely transparent eigenstates, which form the allowed bands.) Such a resonance tunneling is characteristic of any system of little-transparent barriers and may be of importance in various physical systems.

C. Localization in a random system

An infinite set of barriers with different eigenenergies generally leads to a "true" localization of a particle with the given eigenenergy, since the ln$t$'s of all outside barriers add up. This leads to the idea of the localization in macroscopic random systems. At zero temperature ($T=0$) one may see that the localization of an eigenstate always leads to the resonance tunneling. By the very definition, the localization implies that the probability density ρ decreases from the localization region (LR) towards the ends of a conductor[31(b)] (see Fig. 12). So, at the ends $\rho_{1,2} \propto \exp(-2L_{1,2}/L_0)$, where L_0 is the localization length of the wave function and $L_1 = \tilde{\Lambda}$, $L_2 = L - \tilde{\Lambda}$ are the distances from LR to the ends. The transmission coefficients and therefore[12] the mobility and conductance

$$G \propto \rho_2 / \rho_1 \propto \exp(-2 \mid L_1 - L_2 \mid / L_0) .$$

Introducing the distance from LR to the middle point: $L_1 = \frac{1}{2}L - \Lambda$, $L_2 = \frac{1}{2}L + \Lambda$, where $0 \leq \Lambda \leq \frac{1}{2}L$, one obtains $t, G \propto \exp(-4\Lambda/L_0)$. The distance between the adjacent eigenenergies is $\Delta\mathscr{E} \simeq 1/L\,\rho_s(\mathscr{E})$, where $\rho_s(\mathscr{E})$ is the density of states per unit length. The νth eigenstate width $\delta\mathscr{E}_\nu$ is related to the tunneling distance $\tilde{\Lambda} = \frac{1}{2}L - \Lambda$ to the nearest end of the system; thus $\delta\mathscr{E}_\nu \propto \exp(-2\tilde{\Lambda}/L_0)$. In a random system, where there can be no long-range order, LR's and therefore Λ's are randomly distributed. So, typically among s LR's, which divide $\frac{1}{2}L$ into a $(s+1)$ segment, there will be a minimal $\Lambda \sim L/2(s+1)$. Its $G \propto \exp[-2L/(s+1)L_0]$. When $s \sim L/L_0$, then $G \sim 1$, despite the localization of this state. The distance between these "transparent" energies is $\Delta\mathscr{E}^* \sim \Delta\mathscr{E}(L/L_0) \sim 1/\rho_s L_0 \equiv k_B T_0$ (k_B is the Boltzmann constant). Outside the Lifshitz tail of "fluctuation states," $\rho_s(\mathscr{E}) \sim 1/hV_F$, where V_F is the Fermi velocity. Since $\Delta\mathscr{E} = V_F \hbar \Delta k$, k is the Fermi wave vector, so outside the Lifshitz tail $\Delta k \sim \pi/L$ and $\Delta k^* \sim \pi/L_0$. Thus the dependence of the resistance R on k should look as in Fig. 13, in

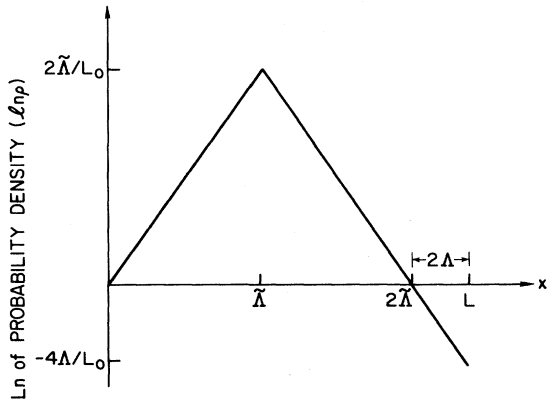

FIG. 12. Localization in a random system (schematically); ρ is the probability density, L is the length: $\tilde{\Lambda}$ and Λ are, correspondingly, the distances from the localization point to the nearest end and to the middle of the system.

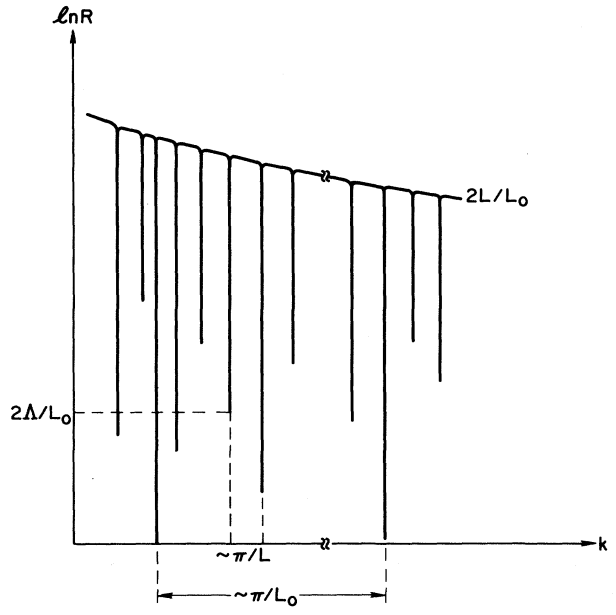

FIG. 13. Resistance R resonances; k is the dimensionless wave vector (schematically).

agreement with Figs. 1—5. The knowledge of $R(\mathscr{E}_\nu)$ and $\delta\mathscr{E}_\nu$ allows one to determine Λ and L_0, i.e., the localization region and the localization length. This is possible because of the exponentially narrow resonances, which are specific for each system. In fact, they present a kind of a "micropicture" of the localization regions, which (in agreement with experiments) is perfectly reproducible for any (microscopically) given sample, but is not reproducible from sample to sample. When $L\to\infty$, the picture is self-similar and is a perfect example of chaos.

According to the previous considerations, at the eigenenergy, $\ln R=4\Lambda/L_0$. So, $\Lambda=\frac{1}{4}L_0\ln R$. The probability dW of Λ (equally distributed at $L/2$) in the interval $d\Lambda$ is

$$dW\equiv P(\Lambda)d\Lambda=d\Lambda/(L/2)\ ,$$

i.e.,

$$P(\Lambda)=2/L\ . \tag{2a}$$

Therefore, the probability of $\ln R$ in the interval $d\ln R$ is

$$dW=(2/L)d\Lambda=(L_0/2L)d\ln R\ . \tag{2b}$$

Thus, the probability density of $\ln R$ at an eigenenergy is

$$P(\ln R)=L_0/2L\ , \tag{2c}$$

i.e., independent of R.

A numerical calculation[32] (by DiVincenzo) for 400 resonances (at 100 different samples) is in good agreement with this statement—see Figs. 14(a) and 14(b).

At the resonances, the particle probability density looks as in Fig. 7, or (in the case of a transparent eigenenergy presented in Fig. 15) as in Fig. 16. An exponentially small (with respect to L/L_0) shift from the resonance energy smears out the localization (Fig. 17) and then leads to a

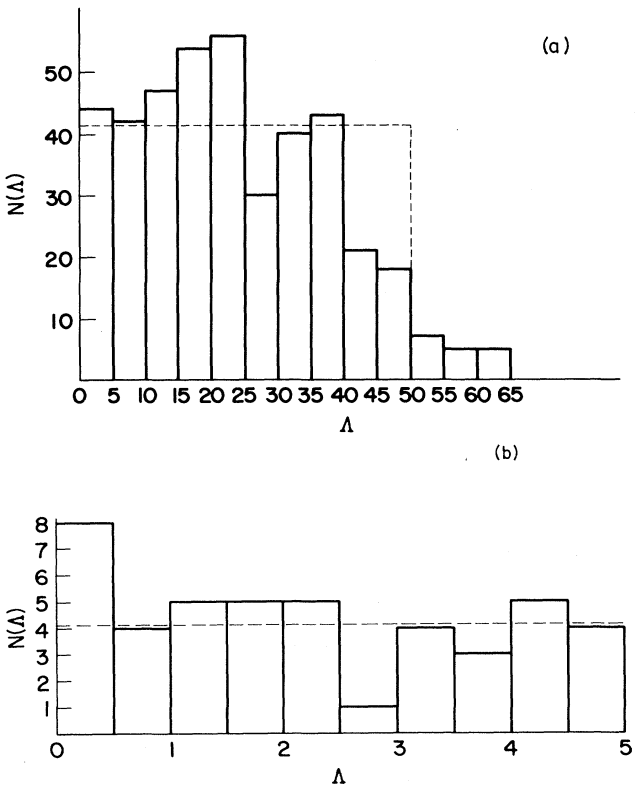

FIG. 14. (a) Histogram for resonant $\Lambda=\frac{1}{4}L_0\ln R$. Here L_0 is the localization length, R is a resonant resistance; $L=100$, $c=0.5$, $v=4$, $k^2\simeq 2.88$. The number of samples is 100; the number of resonances per sample is 4. (b) Same as in (a) but only for $0\le\Lambda\le 5$.

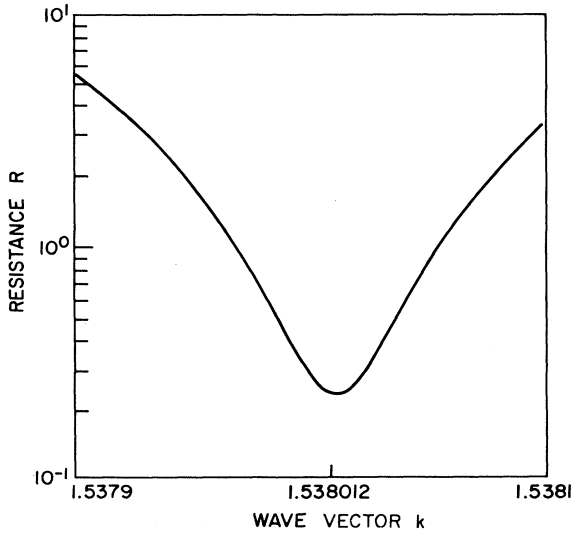

FIG. 15. One of the deepest resonances of the resistance R in a random system of the length $L=100$ with 50 impurities. Impurity potential strength is $v=1.7$; the "step" is $\Delta k=10^{-6}$.

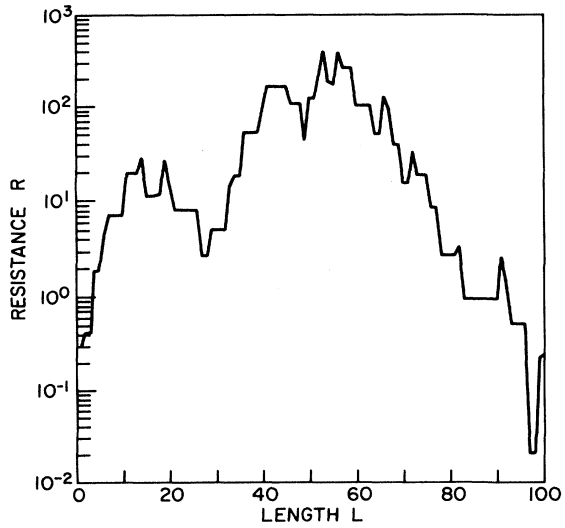

FIG. 16. Resistance R, proportional to the probability density, for the resonance of Fig. 15.

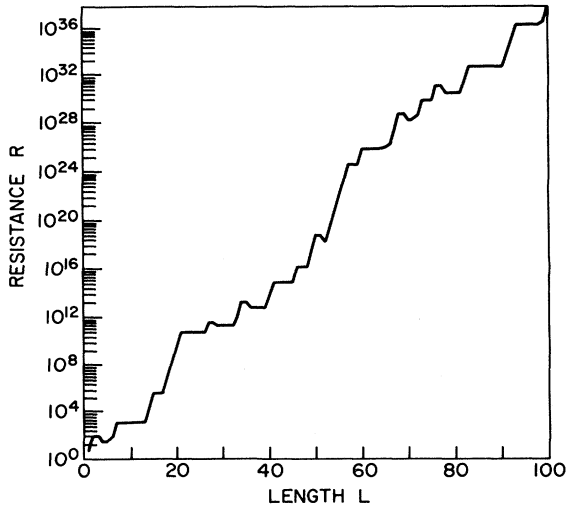

FIG. 18. Same as in Fig. 17 but for $k=1.4$, i.e., at the distance 5.10^{-2} from the resonant k.

"conventional" probability density, exponentially increasing from one end to another (Fig. 18). The resonances and localization could be computed only in virtue of a relatively small L/L_0. Ten times larger L/L_0 would be unresolvable at a computer.

The states of particles in a random system demonstrate the simplest example of an irreversability. Any "regular" wave function of the type presented in Fig. 18 has the conjugate wave function,[33] increasing in the opposite direction. (Physically, it corresponds to the exponentially decreasing probability density, generated by an incoming wave.) Whenever the exponentially increasing ρ is present in the wave function, it dominates. So, the *exponential* accuracy in the initial condition is needed to eliminate it and obtain the exponentially decreasing solution.

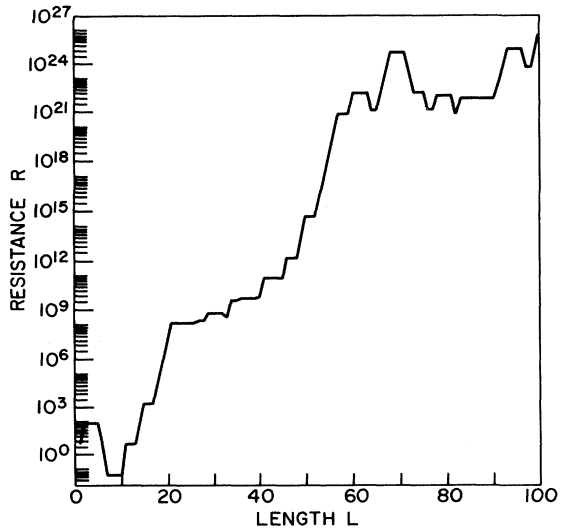

FIG. 17. Same as in Fig. 16 but for $k=1.4491$, i.e., at the distance 3.10^{-5} from the resonant k.

D. Special eigenstates

Various special cases of a 1D localization are related to special eigenstates. On a lattice with a unity spacing, $k=m\pi$, m is an integer, implies no phase shift between lattice sites and in certain cases (δ-function potentials, vacancies on an underlying periodic set with one type of the potentials) is always an eigenvalue.[34,15] This eigenstate is slowly decreasing: $\rho \propto 1/L^2$. Even more important, the localization length diverges at $k=m\pi$ as $L_0 \propto |k-m\pi|^{-1}$ or as $L_0 \propto |k-m\pi|^{-1/2}$ (for more details see Refs. 34 and 15), and the region of "anomalous" eigenstates near this "mobility point" (where $L_0=\infty$) is rather broad—see Fig. 19. Another special eigenenergy is at the middle of the band in the case of off-diagonal disorder. According to Stone and Joannopoulos,[35] there $\ln R \propto L^{1/2}$, $L_0=\lim(2L/\ln R)=\infty$. In the case of the Schrödinger equation, $k=\pi/2$ anomalously often is also the eigenvalue. By Mel'nikov,[36] there the ensemble average of $\langle R\rangle$ leads to $\ln\langle R\rangle \propto L$, while $\langle G\rangle \propto L^{-3/2}$. The latter relation demonstrates that anomalously often $G \gtrsim L^{-3/2}$ at $k=\pi/2$. This G dominates $\langle G\rangle$, but makes exponentially small contributions into $\langle R\rangle$. Finite L shifts this eigenvalue from $k=\pi/2$ (see Fig. 20).

E. Localization length

A 1D scattering obviously reduces to "individual" scatterings. The individual scattering by a single potential strongly depends on the analytical nature of the potential.[31(a)] At low energies $\mathscr{E}$, the dependence of an individual transmission coefficient t_1 on the energy changes from linear to exponential. At high energies, the dependence varies from $1-t_1 \propto 1/\mathscr{E}$ to $(1-t_1)$ exponentially decreasing with $\mathscr{E}$. An exponentially small $(1-t_1)$ leads to the exponentially large $L_0 \propto (1-t_1)^{-1}$ (see Sec. VII). When t_1 is exponentially small, $L_0^{-1} \propto \ln t_1$ (see Sec. VII). Then a particle is mainly localized in the region *between*

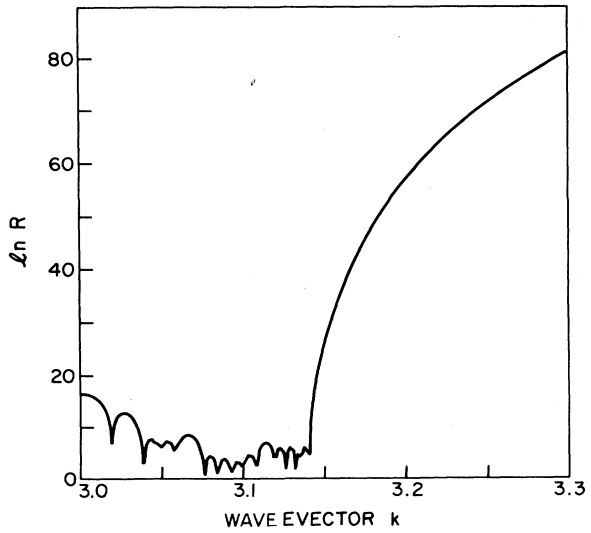

FIG. 19. Resonances and the resistance in the vicinity of the extended eigenstate $k=\pi$. Here $L=100$, $v=10$, $c=0.5$, $\Delta k=0.0015$.

the adjacent potential barriers, while the wave-function decay length L_0 is related to t_1, i.e., to the *barrier*. In the Lifshitz tail there is also the third length—the (exponentially decreasing with $\mathscr{E}$) distance between the localized (fluctuation) eigenstates. Thus, already in 1D there may be different characteristic lengths rather than one assumed by one-parameter scaling,[13] and there may be no universality.[11]

III. TRANSFER MATRIX

Consider the Schrödinger equation

$$\Psi''+[k^2-V(x)]\Psi=0\ , \tag{3}$$

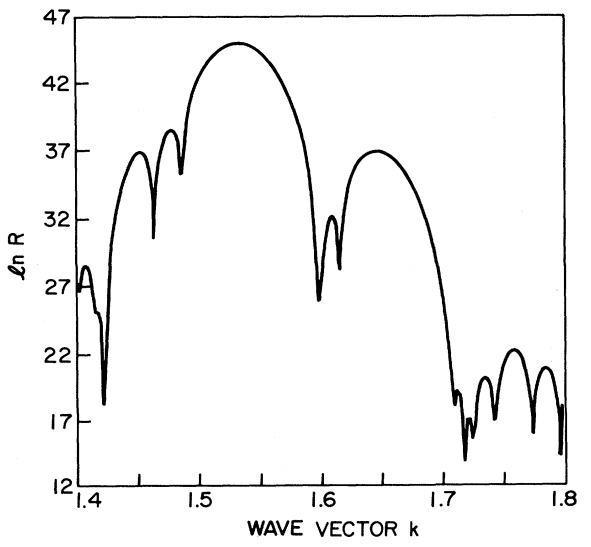

FIG. 20. Resonances in the vicinity of $k=\pi/2$. Here $L=100$; $v=3.0$; $c=0.5$; $\Delta k=0.002$. Finite length removed eigenvalues from $k=\pi/2$.

where $\hbar^2V/2m$ is the potential energy and $\hbar^2k^2/2m$ is the total particle energy. Suppose (see Fig. 11) x_1 and x_2 are the minima of $V(x)$, where $V(x_1),V(x_2)<k^2$. Suppose the function $\Psi_1(x)$ satisfies Eq. (3) and the boundary conditions

$$\Psi_1(x_1)=1,\ \ \Psi_1'(x_1)=ik_1,\ \ k_1=k(x_1)\ , \tag{4a}$$

$$k(x)=[k^2-V(x)]^{1/2}\ . \tag{4b}$$

These boundary conditions imply that $\Psi_1(x)$ has the same probability and current densities as $\exp[ik_1(x-x_1)]$. A simple calculation demonstrates that in the vicinity of x_1,

$$\Psi_1(x)-\exp[ik_1(x-x_1)]\propto(x-x_1)^4\ . \tag{5}$$

(For more details on any statement in this section see Appendix A.) If $V'(x)$ has a discontinuity at $x=x_1$, or x_1 is a regular point rather than the point of a minimum, then

$$\Psi_1(x)-\exp[ik_1(x-x_1)]\propto(x-x_1)^3\ . \tag{6}$$

Any solution to Eq. (3) can be presented as a linear combination of independent solutions Ψ_1 and $\bar{\Psi}_1$ (a bar denotes the complex conjugation):

$$\Psi(x)=A_1\Psi_1(x)+B_1\bar{\Psi}_1(x)\ . \tag{7}$$

Similarly,

$$\Psi(x)=A_2\Psi_2(x)+B_2\bar{\Psi}_2(x)\ , \tag{8}$$

where Ψ_2 yields Eq. (3) and the boundary conditions

$$\Psi_2(x_2)=1,\ \ \Psi_2'(x_2)=ik_2,\ \ k_2=k(x_2)\ . \tag{9}$$

Introduce the transfer matrix θ from x_2 to x_1:

$$\begin{bmatrix}A_1\\B_1\end{bmatrix}=\theta\begin{bmatrix}A_2\\B_2\end{bmatrix},\ \ \theta=\begin{bmatrix}\theta_{11}&\theta_{12}\\\theta_{21}&\theta_{22}\end{bmatrix}\ , \tag{10}$$

If Ψ is a solution to Eq. (3), then $\bar{\Psi}$ is a solution too. Since the current is conserved, it must be the same at $x=x_1$ and at $x=x_2$. These two conditions lead to the following form of θ in a general case [cf. Ref. 13(b)]:

$$\theta=(k_2/k_1)^{1/2}\begin{bmatrix}\cosh(S)\exp(i\alpha)&\sinh(S)\exp(i\tilde{\beta})\\\sinh(S)\exp(-i\tilde{\beta})&\cosh(S)\exp(-i\alpha)\end{bmatrix}\ . \tag{11}$$

The transmission t and reflection r coefficients to x_2 for the wave, incident from x_1, equal

$$t=(k_1/k_2)\,\mathrm{sech}^2(S),\ \ r=\tanh^2S\ . \tag{12}$$

Thus, S characterizes the reflection by the potential.

When the barrier is symmetric (thus, in particular, $k_1=k_2$), and its center is at $x=0$, then

$$\tilde{\beta}=\frac{\pi}{2}\ . \tag{13}$$

The matrix θ can be evaluated exactly for δ-function and rectangular potentials. The former case is special at low and high energies; the latter case is special at high energies. The approximate formula for θ can be derived in a

general quasiclassical case, which is very instructive physically.

The quantity $(\tilde{\beta}-\pi/2)$ characterizes the asymmetry of the barrier. In a quasiclassical case, when $k^2 < V_{\max}$, then

$$\beta \equiv \tilde{\beta}_1 - \frac{\pi}{2} \approx \int_{x_1''}^{x_2} k(x)dx - \int_{x_1}^{x_1'} k(x)dx \ . \tag{14}$$

The quantity α is related to the total phase shift at the potential. When $k^2 < V_{\max}$, then in a quasiclassical case

$$-\alpha_1 \approx \int_{x_1}^{x_1'} k(x)dx + \int_{x_1''}^{x_2} k(x)dx \ . \tag{15}$$

Naturally, when $k^2 \gg V_{\max}$, then $\alpha,\beta \to 0$ together with $k \to \infty$.

According to Eq. (12), transmission and reflection are related to S. Thus, in a general case any individual scattering is related, by Eq. (11), to three real parameters with a clear physical meaning. The description of the scattering by Eq. (11) is in fact much more convenient than by the initial Schrödinger equation with the *function* $V(x)$ (rather than with three numbers S,α,β). Of course, a similar three-parameter transfer matrix can be introduced when, say, $k_1^2 < 0$ (see Sec. VIII).

In the special case of the δ-function potential $v\delta(x)$,

$$t = (1+v^2/4k^2)^{-1} \ . \tag{16a}$$

Thus, a high barrier or a deep well with $v^2 \gg k^2$ lead to t linearly increasing with the particle energy $\mathscr{E} \propto k^2$:

$$t \propto k^2 \propto \mathscr{E} \ , \tag{16b}$$

while a particle with high energy has

$$r = 1 - t \propto 1/k^2 \propto 1/\mathscr{E} \ . \tag{16c}$$

A tunneling through a quasiclassical barrier [see Fig. 11(a)] leads in Eq. (11) to[31(a1)]

$$S_1 = \int_{x_1'}^{x_1''} |k(x)|\,dx = \int_{x_1'}^{x_1''} [V(x)-k^2]^{1/2}dx \ , \tag{17}$$

where $V(x_1') = V(x_1'') = k^2$. By Eqs. (12) and (17) the transmission coefficient *exponentially* increases with the particle energy—cf. Eq. (16b).

A quasiclassical scattering above the barrier or well, when $k^2 > V_{\max}$ and $r \ll 1$, crucially depends on the analytical nature of $V(x)$. By Eq. (16c), the δ-function leads to $r \propto 1/\mathscr{E}$. A symmetric rectangular barrier of the height V_0 and the width t leads to

$$t = \{1 + \tfrac{1}{4}[(\kappa/k)-(k/\kappa)]^2 \sin^2(\kappa d)\}^{-1} \ , \tag{18a}$$

$$\kappa = (k^2 - V_0)^{1/2} \ . \tag{18b}$$

When $k \to \infty$, then

$$r = 1 - t \propto 1/\mathscr{E}^2 \ . \tag{18c}$$

A quasiclassical potential with a discontinuity in the first derivative at $x = x_0$,

$$V(x) = \begin{cases} V_0[1-(x-x_0)/\lambda_1] & \text{if } x \geq x_0 \\ V_0[1-(x-x_0)/\lambda_2] & \text{if } x \leq x_0 \end{cases} \ , \tag{19}$$

leads[31(a2)] to

$$r = (V_0^2/64k_0^6)(\lambda_1^{-1}-\lambda_2^{-1})^2 \alpha 1/\mathscr{E}^3 \ . \tag{20a}$$

A quasiclassical potential with a discontinuous $V^{(p)}(x)$ at $x = x_0$ leads to

$$r \propto 1/\mathscr{E}^{2+p} \ . \tag{20b}$$

Finally, an analytical $V(x)$ leads to[31(a3)]

$$r = \exp\left[-4\,\mathrm{Im}\int_{x_m}^{\tilde{x}} k(x)dx\right] \ , \tag{21a}$$

where x_m is the point where $V(x)$ is maximal: $V(x_m) = V_{\max}$ and $\tilde{x}$ is the complex point where $V(\tilde{x}) = k^2$. Typically Eq. (21a) yields

$$\ln r \propto -k \propto -\mathscr{E}^{1/2} \ , \tag{21b}$$

i.e., leads to the exponential dependence of r on $\mathscr{E}$.

A good guide to the scattering by an analytical potential is provided by

$$V(x) = V_0/\cosh^2(x/\lambda) \ . \tag{22}$$

This potential leads to[31(a4)]

$$t = \sinh^2(\pi k\lambda)\{\sin^2(\pi k\lambda) + \cosh^2[\pi(\lambda^2 V_0 - \tfrac{1}{4})^{1/2}]\}^{-1} \ . \tag{23a}$$

When $\lambda^2 V_0 \gg k^2, \lambda^2, 1$, then

$$t \propto \sinh^2(\pi k\lambda) = \sinh^2(\pi\lambda\mathscr{E}^{1/2}) \ , \tag{23b}$$

when $\lambda^2 k^2 \gg V_0\lambda^2, 1$, then

$$1 - t \propto \exp(-2\pi k\lambda) = \exp(-2\pi\lambda\mathscr{E}^{1/2}) \ . \tag{23c}$$

In all cases when $1 - t \ll 1$ and $k_2 = k_1$, by Eq. (12),

$$S \approx \sqrt{2(1-t)} \ . \tag{24a}$$

Every symmetric potential is reduced to an "effective δ-function potential" $V_{\text{eff}}\delta(x)$, which leads to the same θ as Eq. (11). One should choose

$$V_{\text{eff}} = 2k \sinh[S(k)] \tag{24b}$$

and present the wave function in the form

$$\Psi = \begin{cases} A_1 \exp(ikx - \frac{1}{2}i\alpha) + B_1 \exp(-ikx + \frac{1}{2}i\alpha) & \text{for } x < 0 \\ A_2 \exp(ikx - \frac{1}{2}i\alpha) + B_2 \exp(-ikx + \frac{1}{2}i\alpha) & \text{for } x > 0 \ . \end{cases} \tag{24c}$$

Naturally, V_{eff} and α depend on the energy. The length α/k also depends on k. As long as the distance between the centers of actual potentials is larger than α/k, such a presentation may be useful.

IV. RESONANCE TUNNELING

Now consider a tunneling through two barriers of Fig. 11(a), denoting them by subscripts "1" and "2" correspondingly. The transfer matrix through both barriers is the product of their transfer matrices. The direct calculation, which accounts for Eqs. (11) and (12), leads to the transmission coefficient t_{12} (see Appendix B for all the details),

$$t_{12}=|\cosh S_1\cosh S_2+\sinh S_1\sinh S_2\exp(i\omega)|^{-2}, \quad (25)$$

where

$$\omega=\tilde{\beta}_1-\tilde{\beta}_2-\alpha_1-\alpha_2 . \quad (26)$$

The quasiclassical tunneling, by Eqs. (14) and (15), yields

$$\omega=2\int_{x_1''}^{x_2'} k(x)dx . \quad (27)$$

The Bohr quantization determines quasiclassical levels in the well between the barriers[31(a5)]

$$\int_{x_1''}^{x_2'} k(x)dx=(Q+\tfrac{1}{2})\pi , \quad (28)$$

where Q is an integer. In this approximation at eigenenergies $\omega=(2Q+1)\pi$, and Eq. (25) leads to

$$t_{12}=1/\cosh^2(S_1-S_2)\approx 4\exp(-2/|S_1-S_2|) . \quad (29)$$

In particular, $S_1=S_2$ leads to $t_{12}=1$. Here $S_{1,2}\gg 1$ are determined by Eq. (17). Outside the resonance at the levels, by Eq. (25),

$$t_{12}\propto\exp[-2(S_1+S_2)] . \quad (30)$$

The width of the levels, by Eq. (25), is

$$\delta\omega\propto[\exp(-2S_1)+\exp(-2S_2)] . \quad (31)$$

V. RECURRENCE RELATION

Now it is convenient to consider the wave, incident from $x=+\infty$, and the transfer matrix from x_2 to x_3 in Fig. 11(a). The same reasoning as in Sec. III leads to the same form of the transfer matrix as in Eq. (11) with the same S and α, but with the factor $(k_2/k_3)^{1/2}$ rather than $(k_2/k_1)^{1/2}$, and $\tilde{\beta}$ replaced by $(\pi-\tilde{\beta})\equiv\beta$ (see Appendix B). Also, it is convenient to introduce $\cos(h)=\operatorname{sech}(S)$ with $0\le h\le\pi/2$ and $\tan(h)=\sinh(S)$. Finally [cf. Refs. 11 and 13(b)]

$$\begin{bmatrix}A_3\\B_3\end{bmatrix}=(k_2/k_3)^{1/2}\begin{bmatrix}\exp(i\alpha_2)\sec(h_2) & \exp(i\beta_2)\tan(h_2)\\ \exp(-i\beta_2)\tan(h_2) & \exp(-i\alpha_2)\sec(h_2)\end{bmatrix}\begin{bmatrix}A_2\\B_2\end{bmatrix} . \quad (32)$$

The Schrödinger equation is linear. Therefore, it is sufficient to consider currentless (i.e., real) wave functions, with

$$\Psi=A_\kappa\Psi_\kappa+\bar{A}_\kappa\bar{\Psi}_\kappa,\quad \kappa=2,3 , \quad (33)$$

in Eqs. (7) and (8). Choose

$$A_\kappa=\exp(\tfrac{1}{2}g_\kappa-\tfrac{1}{2}i\phi_\kappa) . \quad (34)$$

Then the transfer matrix θ_κ from x_κ to $x_{\kappa+1}$ leads to the recurrence relation (for details see Appendix C):

$$\phi_\kappa=2F_\kappa-\alpha_{\kappa-1}-\beta_{\kappa-1},\quad F_\kappa=\arctan X_\kappa , \quad (35)$$

$$g_{\kappa+1}=g_\kappa+\ln(k_\kappa/k_{\kappa+1})+H_\kappa=\sum_{\kappa_1=1}^{\kappa}H_{\kappa_1}+\ln(k/k_{\kappa+1}) , \quad (36a)$$

$$H_\kappa=\ln[(1+X_\kappa^2)/(w_{\kappa-1}+w_{\kappa-1}^{-1}X_\kappa^2)] , \quad (36b)$$

$$X_{\kappa+1}=w_\kappa(X_\kappa-r_\kappa)/(1+r_\kappa X_\kappa) , \quad (37)$$

where

$$w_\kappa=\tan^2(\tfrac{1}{4}\pi-\tfrac{1}{4}h_\kappa) ,\qquad r_\kappa=\tan[(\alpha_\kappa-\beta_\kappa+\alpha_{\kappa-1}+\beta_{\kappa-1})/2] . \quad (38)$$

Obviously, this recurrence is valid for any κ. To the left of all barriers one may chose

$$X_0=\pm 1,\quad \alpha_0=h_0=0,\quad \beta_0=\pi/2 . \quad (39)$$

The knowledge of two independent solutions relates the incident and reflected waves to the transmitted waves and leads to the transmission coefficient[15]

$$t=4[\exp(g_{N+1}^{+})+\exp(g_{N+1}^{-})+2]^{-1} , \quad (40)$$

where superscripts $(\pm)$ correspond to $X_0=\pm 1$. The Landauer formula[12] for the dimensionless resistance $R=t^{-1}-1$ yields

$$R_N=[\exp(g_{N+1}^{+})+\exp(g_{N+1}^{-})-2]/4 , \quad (41a)$$

where N is the total number of scatterers. According to Eqs. (33), (34), and (41a) the resistance at a given length is approximately equal to the probability density ρ. By Eqs. (41a) and (36) when $L\to\infty$, then

$$g_{L+1}^{(\pm)}\sim LH_{\rm av},\quad H_{\rm av}=\lim_{L\to\infty}\left[L^{-1}\sum_{\kappa=1}^{L}H_\kappa\right] . \quad (41b)$$

Thus, typically the resistance and the probability density exponentially increase from one end to another. The average localization length of the wave function is determined by the ensemble average $\langle H_{\rm av}\rangle$ of $H_{\rm av}$, namely $L_0=2/\langle H_{\rm av}\rangle$. The localization length determines the decay of the wave functions, thus determines the overlapping of different eigenfunctions and the correlation between wave functions localized at the distance L from each other. At the distances $L\gg L_0$ wave functions and the corresponding contributions H_κ in Eq. (36) are independent. Thus, when $\kappa\gg L_0$, $g_{\kappa+1}$ in Eq. (36) consists of $\sim\kappa/L_0$ independent terms. Therefore g's yield to the Gaussian distribution. The corresponding exponentially rare fluctuations of g_κ are related to the eigenstates (cf. Appendix B). Their g's increasing from both ends [cf. Figs. 11(b) and 12] match each other and yield exponential localization of all states. The exponentially low probability of the eigenstates leads to their exponentially small "width" (cf. Appendix B).

The determination of R as a function of the particle energy (from the recurrence relations) allows one to determine the eigenenergies (i.e., the energies where R is minimal) and their eigenfunctions (again from the corresponding recurrence relations). This method, as well as the corresponding recurrence relations, are applicable to phonons,[11] and to any kind of the waves: acoustic, electromagnetic,[10,11] spin waves, hydrodynamic ones, as well as to the random Ising[11,37] and diffusion problems.

VI. PHASE RANDOMIZATION

It is natural to expect that the phase ϕ_κ in Eqs. (33) and (34) is randomized[37(b)] in a random sequence and becomes independent of the initial phase ϕ_0. Any wave function can be decomposed into the independent wave functions, introduced in Sec. V, with $X_0^\pm=\pm 1$ [and thus, by Eqs. (39) and (35), $\phi_0^+=0$, $\phi_0^-=-\pi$]. Therefore, it is sufficient to demonstrate that the corresponding $\phi_\kappa^\pm$ yield $\Delta\phi_\kappa\equiv\phi_\kappa^+-\phi_\kappa^-\to 0$ when $\kappa\to\infty$.

In this section I demonstrate, following Ref. 19, that $\Delta\phi_\kappa$ exponentially decreases with κ. More specifically (see Appendix D),

$$\Delta\phi_\kappa\propto 1/R_\kappa\ , \tag{42a}$$

where R_κ is the resistance of κ scatterers. Thus, the correlation length of $\Delta\phi_\kappa$ is $L_0/2$ [since $R(L)=\exp(2L/L_0)$]. So, if the "initial" phase at x_κ is ϕ_κ, then in a given random system the phase $\phi_{\kappa'}$ at $x_{\kappa'}$ will be independent of ϕ_κ with the exponential accuracy, and will depend only on the sequence between x_κ and $x_{\kappa'}$. That is, when ϕ_κ changes from 0 to 2π, then $\phi_{\kappa'}$ changes by $\propto 1/R(x_{\kappa'}-x_\kappa)$. Only this change contributes into the phase correlation between x_κ and $x_{\kappa'}$. Therefore,

$$\begin{aligned} q(|x_\kappa-x_{\kappa'}|)&\equiv\langle\phi_\kappa\phi_{\kappa'}\rangle-\langle\phi_\kappa\rangle\langle\phi_{\kappa'}\rangle \\ &\propto\langle 1/R(|x_\kappa-x_{\kappa'}|)\rangle \\ &\propto\exp(-2|x_{\kappa'}-x_\kappa|/L_0)\ . \end{aligned} \tag{42b}$$

The angular brackets denote the ensemble averaging. The ensemble averaging may be replaced by the averaging over different initial positions. In all cases, the most convenient quantity to study correlation function is (cf. the Ising model in Ref. 37)

$$\tilde{q}(x_\kappa,x_{\kappa'})=\phi_{\kappa'}(\phi_\kappa^+)-\phi_{\kappa'}(\phi_\kappa^-)\ ,$$

where $\phi_{\kappa'}(\phi_\kappa^\pm)$ is the phase at $x_{\kappa'}$, generated by the phase $\phi_\kappa^\pm$, where $\phi_\kappa^+=0$, $\phi_\kappa^-=-\pi$ are independent phases. The quantity $\tilde{q}(x_\kappa,x_{\kappa'})$ describes the phase correlation specifically between the points x_κ and $x_{\kappa'}$ in a given random sequence, and allows for the detailed analysis of the phase correlations. Since $\tilde{q}\propto 1/R$, the extensive quantity which may be scaled and whose average has regular ($\propto 1/\sqrt{L}$) fluctuations, is $\ln\tilde{q}$. However, the correlation at the distance l is described by the ensemble average $q_a(l)\equiv\langle\tilde{q}(x_\kappa,x_\kappa+l)\rangle$. Alternatively, it can be related to the average over all possible initial positions. For instance in the case of a disorder on a lattice with unity interlattice spacing (and $x_\kappa=\kappa$) the correlation at l is related to

$$q_i(l)=\frac{1}{L-l}\sum_{n=1}^{L-l}\tilde{q}(n,n+l)\ .$$

For almost all energies, $\ln\tilde{q}$, $\ln q_a$, and $\ln q_i$ are related to $\ln R$, $\ln\langle R\rangle$, $[L^{-1}\sum_n R(n,n+l)]$ and have the same order of magnitude[14,38,15]; all correlations exponentially vanish. But eigenenergies drastically change the situation. According to Sec. II, the resistance at the eigenstate is proportional to the probability density and has an exponentially high maximum inside the system, at the localization point. Therefore at the eigenenergy the phase correlation between $\kappa=0$ and a given κ first exponentially decreases with κ then reaches an exponentially deep minimum in the localization region and then exponentially *increases* with κ—see Fig. 6(a). The correlation between the point with approximately equal values of R (which are located at approximately equal distances from the localization region), is ~ 1, no matter how far off these points are from each other. The phase, which has already exponentially well forgotten its "origin," after the localization point starts "rediscovering" it. When the localization region is close to $L/2$, the correlation $\tilde{q}$ may be the highest between the ends.

A more meaningful characteristic of the system is the correlation $q_i(l)$ at the distance l. Most of initial n's and the corresponding $(n+l)$'s will be on the slopes of $R(L)$. Their

$$\tilde{q}(n,n+l)\propto\exp(-2l/L_0)$$

is exponentially small when $l\gg L_0$. However, a fraction $\sim L_0/L$ will have $n\sim\Lambda-l/2$ and $\tilde{q}(n,n+l)\sim 1$. Thus

$$q_i(l)\sim\exp(-2l/L_0)+(L_0/L)\ .$$

When $L\to\infty$, then $\ln q_i(l)\propto -l$, as should be expected. However, at *large* $l>\frac{1}{2}L_0\ln(L/L_0)$ the correlation saturates[39] to (L_0/L)—see Fig. 6(b).

VII. PHYSICAL IMPLICATIONS: CONDUCTANCE AND ITS RESONANCES

Equations (41a), (41b), and (36)–(38) relate the localization length to the characteristics of an individual scattering. Two cases are of special interest.

When the scattering is weak, the individual reflection coefficient is close to 1. By Eq. (32), this means $h_\kappa\simeq 0$. Thus, $w_\kappa\simeq 1$, by Eq. (38), and one can expand $H_{\rm av}$ in Eq. (41b) with respect to $|1-w_\kappa|\ll 1$. On the other hand, $H_{\rm av}$ does not change when $w_\kappa\to w_\kappa^{-1}$, because Eqs. (36) and (37) are invariant to $w_\kappa\to w_\kappa^{-1}$, $X_\kappa\to -X_\kappa^{-1}$. If $w_\kappa=1+w_\kappa'$, where $|w_\kappa'|\ll 1$, then $w_\kappa^{-1}\simeq 1-w_\kappa'$. Therefore, the $H_{\rm av}$ decomposition is even in w_κ':

$$H_{\rm av}\propto(w_\kappa')^2\sim 1-t_\kappa\ , \tag{42c}$$

where t_κ is, by Eqs. (12), (38), and (32), an individual transmission coefficient. The dependence of t_κ on the analytical nature of the potential when the energy is high was discussed in Sec. II. Thus, the localization length strongly depends on the analytical properties of an individual potential. At high energies the analytical potential leads to the exponential dependence of L_0 on the energy:

$L_0 = L_{00} \exp(a\mathscr{E}^{1/2})$, a is a constant. This means, that even in 1D the localization becomes weak at moderate energies,

$$\mathscr{E} \propto [\ln(L/L_{00})]^2 , \tag{42d}$$

and implies an incredible

$$R = \exp[(2L/L_{00})\exp(-a\mathscr{E}^{1/2})] ,$$

i.e., the exponent in the exponent, when $L \to \infty$.

A strong individual scattering means, by Eq. (32), $h \simeq \pm\pi/2$ and thus, by Eq. (38), $w \ll 1$ (when $h \simeq \pi/2$) or $w \gg 1$ (when $h \approx -\pi/2$). When $w_\kappa \ll 1$, and $r_\kappa \sim 1$, then, by Eq. (37), $X_\kappa \ll 1$. So,

$$X_{\kappa+1} \simeq -w_\kappa r_\kappa \tag{43}$$

and

$$H_{\rm av} \sim [\ln(1/w_\kappa)]_{\rm av} . \tag{44}$$

When $w_\kappa \gg 1$, then $X_\kappa \gg 1$,

$$X_{\kappa+1} \simeq w_\kappa / r_\kappa , \tag{45}$$

and

$$H_{\rm av} \sim (\ln w_\kappa)_{\rm av} . \tag{46}$$

Equations (44) and (46) can be combined into

$$H_{\rm av} \sim (\ln t_\kappa^{-1})_{\rm av} , \tag{47}$$

when $t \ll 1$. Thus, when $t \ll 1$ exponentially increases with $\mathscr{E}$, and $\ln t \propto \mathscr{E}^{1/2}$ (at low $\mathscr{E}$), then the localization length depends on $\mathscr{E}$ as $L_0 \propto \mathscr{E}^{1/2}$. In various limiting cases, L_0 can be evaluated analytically (see Ref. 15 and Appendix C). In a general case, Eqs. (41) and (36)—(39) allow for easy numerical experiments. In the case of δ-function potentials they were derived in Ref. 15; in the case of arbitrary nonoverlapping potentials in Ref. 11. Consider the simplest case of δ functions on a lattice. The corresponding Schrödinger equation is

$$\Psi'' + \left[\kappa^2 - \sum_{n=1}^{L} v_n \delta(x-n)\right]\Psi = 0 , \tag{48}$$

where v_n is v with the probability c and is zero with the probability $(1-c)$. Further on everywhere $c = 0.5$. Then Fig. 8 demonstrates a weak localization for a weak potential $v = 0.05$ at the length $L = 100$. Figure 21 presents the case of an intermediate potential $v = 1.7$ at a large length $L = 5000$. The same potential at a shorter length $L = 100$ is demonstrated by Fig. 22. The deepest resistance minima in this plot is resolved in Fig. 15. By Eq. (16a), a relative strength of the potential is determined by $v/2k$. A strong potential $v = 6$ (at the same sequence as $v = 1.7$) at a large length $L = 5000$ is presented in Fig. 1. [Note an "unusually" deep minima. Note also that $R \sim \exp(7500)$ is easily calculated.] For a shorter length $L = 100$, $R(k)$ is presented in Fig. 2 and (on the finer scales) in Figs. 3—5. The calculated $R = R(L)$, which according to Eqs. (41), (33), and (34) is approximately equal to the probability density ρ, is demonstrated for $v = 6$, $L = 100$ in Fig. 7 (at the eigenenergy) and for $v = 1.7$, $L = 100$ in Fig. 16 (at the eigenenergy), Fig. 17 (very close to the eigenenergy; note the "future" localization maximum), and Fig. 18 (an "ordinary" energy).

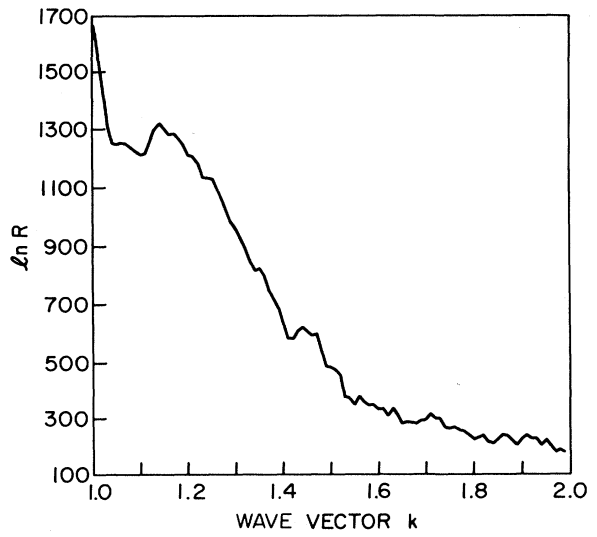

FIG. 21. Resonances for $L = 5000$; $v = 17$; $c = 0.5$; $\Delta k = 0.01$. At such a scale resonances are very little pronounced.

The knowledge of the height R_r and the width δk of the resistance resonances in Figs. 5 and 15 allows one to determine the localization position Λ for the corresponding eigenenergies. According to Sec. II C the distance from the localization position to the nearest end equals

$$\tilde{\Lambda} \equiv \tfrac{1}{2}L - \Lambda = \frac{L}{2 + |\ln R_r / \ln \delta k|} .$$

The directly determined probability densities for these eigenenergies in Figs. 7 and 16 yield the exact values of $\tilde{\Lambda}$. The results are for $v = 1.7$: "evaluated" $\tilde{\Lambda} = 50$, "exact" $\tilde{\Lambda} = 47$; for $v = 6.0$: evaluated $\tilde{\Lambda} = 31$, exact $\tilde{\Lambda} = 29$.

Figure 14(a) represents a histogram of the distribution

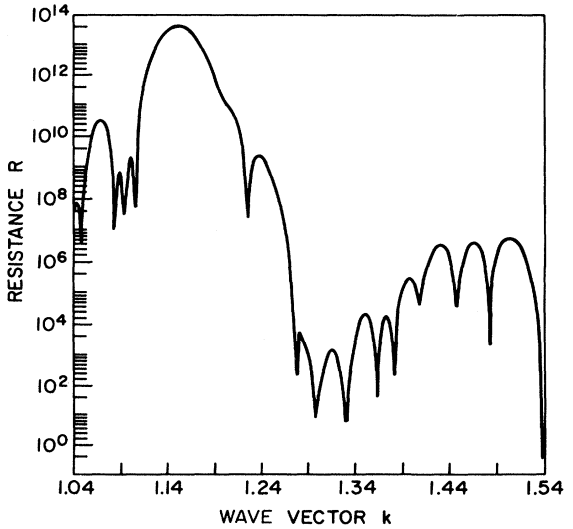

FIG. 22. Same as in Fig. 21 but $L = 100$ and $\Delta k = 0.002$.

of $\Lambda=\frac{1}{4}L_0 \ln R$ for 400 resonance minima of R at $L=100$, $v=4$. One hundred random sequences were chosen (with $c=0.5$). In each sequence four R minima closest to $k^2=2.88$ were chosen. Within the normal fluctuations, the distribution is in agreement with Eq. (2c), where $L_0 \approx 4.5$. The "tail" in Fig. 14(a) at $\Lambda > L/2$ is related to the fluctuations in L_0. Also, $\Lambda \simeq L/2$ is close to the ends which somewhat affect the distribution. Note that resonance $R \sim 1$ and $R \sim \exp(40)$ are equally probable.[40] In the most interesting region of small $\ln R < 5$ the histogram is presented in Fig. 14(b). Fewer peaks (~ 40 of them) lead to larger fluctuations, but again within normal limits.

The derived equations allow also for other applications. It was suggested[41,10(a),15] that a representative resistance has the same order of magnitude as the ensemble average resistance, if the *number* of impurities is kept *fixed*. The latter average can be analytically evaluated,[15] and numerical experiments[15] verify the suggestion of Refs. 41 and 10a. Numerical experiments similar to those of this section allow one to test the relations[14,38,42] between the representative and ensemble average resistances and the limitations on the universality.[11]

The paramagnetic spin splitting of the energy accounts for the effect of a static magnetic field in a 1D system. Finite electric field changes the potential, $U(x) \rightarrow U(x) - eEx$, and shifts its minima. (See the straightforward calculation in Ref. 43.) It may be easily accounted for in the suggested scheme. The generalizations to acoustic[11] and electromagnetic waves,[10(a),10(b)] to a quasiperiodic potential,[40] to a 1D Ising model[37(a),19] polymer thermodynamics, and classical diffusion in a random media are also straightforward (see below). But the scope of possible applications may be even wider [e.g., to charge-density wave (CDW)].

VIII. ACOUSTIC PHONONS, ELECTROMAGNETIC WAVES, ETC.

A harmonic chain with force constants λ_j, displacements $q_j \exp(-i\omega t)$ (t is time), and equal masses yields the dimensionless equation,

$$-\Omega^2 q_j = \lambda_j (q_{j+1} - q_j) + \lambda_{j-1}(q_{j-1} - q_j) \ . \tag{49}$$

The units are such that $\lambda_j = 1$ has the highest concentration. In the continuous case, when q_j, λ_j slowly change with j, Eq. (49) reduces to the equation for sound waves.

Introduce

$$\xi_j = \lambda_j (q_{j+1} - q_j) \ . \tag{50}$$

Then Eq. (49) leads to

$$-\Omega^2 \xi_j / \lambda_j = \xi_{j+1} - 2\xi_j + \xi_{j-1} \ . \tag{51}$$

Introduce

$$\lambda_j^{-1} = 1 + g_j, \quad \Omega^2 = 2(1 - \cos k), \quad -\pi \le k \le \pi \ . \tag{52}$$

Then Eq. (51) yields

$$\xi_{j+1} - 2\cos(k)\xi_j + \xi_{j-1} = -2(1 - \cos k) g_j \xi_j \ . \tag{53}$$

Two real quantities ξ_{j-1}, ξ_j can be expressed through one complex "amplitude" δ_j:

$$\xi_j = \delta_j \exp(-ikj) + \bar{\delta}_j \exp(+ikj) \equiv d_j + \bar{d}_j \ , \tag{54a}$$

$$\xi_{j+1} = d_j \kappa + \bar{d}_j \bar{\kappa}; \kappa = \exp(-ik) \ , \tag{54b}$$

where a bar denotes the complex conjugation. By Eqs. (54a) and (54b),

$$\xi_{j-1} = d_{j-1} + \bar{d}_{j-1} \ , \tag{54c}$$

$$\xi_j = d_{j-1}\kappa + d_{j-1}\bar{\kappa} \ . \tag{54d}$$

According to Eqs. (54a) and (54d)

$$d_j + \bar{d}_j = d_{j-1}\kappa + \bar{d}_{j-1}\bar{\kappa} \ . \tag{55}$$

Substituting ξ_{j+1}, ξ_j, and ξ_{j-1} from Eqs. (54b), (54a), and (54c) into Eq. (53), one obtains

$$d_j[\bar{\kappa} - 4g_j \sin^2(k/2)] + \bar{d}_j[\kappa - 4g_j \sin^2(k/2)] = d_{j-1} + \bar{d}_{j-1} \ . \tag{56}$$

Equations (55) and (56), after simple algebraic transformations, lead to the transfer matrix from $\delta_{j-1}, \bar{\delta}_{j-1}$ to $\delta_j, \bar{\delta}_j$:

$$\begin{bmatrix} \delta_j \\ \bar{\delta}_j \end{bmatrix} = \begin{bmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{bmatrix}^j \begin{bmatrix} 1 - ig_j \tan(k/2) & -ig_j \tan(k/2) \\ ig_j \tan(k/2) & 1 + ig_j \tan(k/2) \end{bmatrix} \begin{bmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{bmatrix}^{-j} \begin{bmatrix} \delta_{j-1} \\ \bar{\delta}_{j-1} \end{bmatrix} \ . \tag{57}$$

This transfer matrix is equivalent to the transfer matrix for the Schrödinger equation

$$\Psi'' + \left[k^2 - \sum v_j \delta(x - j)\right]\Psi = 0 \ , \tag{58}$$

where

$$v_j = 2kg_j \tan(k/2) \ . \tag{59}$$

The last equation rigorously maps[11] the phonon problem onto the problem of a quantum particle in Eq. (58). (Different masses of atoms may be considered similarly.) For instance, the localization length for a quantum particle was calculated in Ref. 15. A straightforward application of the corresponding formula for L_0 to the case of acoustic phonons with $k \ll 1$ leads to the localization length L_0,

$$L_0 \propto 1/k^2 \ . \tag{60}$$

As it should be, $L_0 \rightarrow \infty$ when $k \rightarrow 0$, i.e., when phonon frequency $\Omega \rightarrow 0$.

The harmonic chain equation describes a number of other systems.[45] For instance, in the case of diffusion, q_j in Eq. (49) can be interpreted as the Laplace transformation of the density $\rho_j(t)$,

$$\rho_j(t)=\int_0^\infty q_j(\omega)e^{-\omega^2 t}d\omega\ , \tag{61}$$

and λ_j as the hopping probability from the jth to the $(j+1)$th site. The same type of the equation describes the Bloch electron in a magnetic field.[9] An electromagnetic wave in 1D also reduces to the Schrödinger-type equation. (This is the case, e.g., in Fig. 9.) For instance, suppose a dielectric constant $\epsilon=$ const, a magnetic permeability $\mu=\mu(x)$, and a magnetic field $\vec{\mathrm{H}}_t$ is

$$\vec{\mathrm{H}}_t=\vec{\mathrm{H}}(x)\exp(i\omega t-i\,\vec{\mathrm{k}}_t\cdot\vec{\mathrm{r}})\ , \tag{62}$$

where t is time and $\vec{\mathrm{r}}$ denotes the transverse coordinates. Then the equation

$$\vec{\mathrm{H}}''(x)+(-k_t^2+\epsilon\mu\omega^2/c^2)\,\vec{\mathrm{H}}(x)=0 \tag{63}$$

is of the Schrödinger type for each of the $\vec{\mathrm{H}}$ components. As in the case of phonons, $L_0\propto/\omega^2$ for Eq. (63). A 1D Ising problem can also be mapped[11] on the Schrödinger equation. Consider, for instance, the Hamiltonian $\mathscr{H}$,

$$\mathscr{H}=-\sum(J_j s_j s_{j-1}+H_j s_j),\quad s_j=\pm1,\quad J_j>|H_j|\ . \tag{64}$$

The partition function $Z_s^{(j)}$ for j sites with the spin s at the jth site yields the equation

$$Z_s^{(j)}=\sum_{s'}\theta_{ss'}^{(j)}Z_{ss'}^{(j,-1)}\ . \tag{65}$$

Here

$$\theta_{ss'}^{(j)}=\exp(\tilde{J}_j ss'+\tilde{H}_j s)\equiv 2\sinh(2\tilde{J}_j)\tilde{\theta}_{ss'}^{(j)}\ , \tag{66}$$

$$\tilde{J}_j=J_j/k_BT,\quad \tilde{H}_j=H_j/k_BT\ . \tag{67}$$

k_B is the Boltzmann constant, T is temperature, and

$$\tilde{\theta}^{(j)}=\begin{bmatrix}\exp(\tilde{J}_j+\tilde{H}_j)/2\sinh(2\tilde{J}_j) & \exp(-\tilde{J}_j+\tilde{H}_j)/2\sinh(2\tilde{J}_j)\\ \exp(-\tilde{J}_j-\tilde{H}_j)/2\sinh(2\tilde{J}_j) & \exp(\tilde{J}_j-\tilde{H}_j)/2\sinh(2\tilde{J}_j)\end{bmatrix}\ . \tag{68}$$

Now consider potential energy, presented in Fig. 23. The corresponding dimensionless Schrödinger equation is

$$\Psi''+\left[k^2-\sum_n v_n s[x-(n-1)]s(n-x)\right]\Psi=0\ , \tag{69}$$

$$s(x)=\begin{cases}1, & x>0\\ 0, & x<0,\quad x_n=n\ .\end{cases} \tag{70}$$

Present Ψ in the region $x_{n-1}<x<x_n$ in the form

$$\Psi=A_n\exp[-i\kappa_n(x-x_n)]+B_n\exp[+i\kappa_n(x-x_n)]\ , \tag{71}$$

where

$$\kappa_n^2=v_n-k^2\ . \tag{72}$$

The sign of κ_n will be chosen later. A simple calculation leads to

$$\begin{bmatrix}A_n\\ B_n\end{bmatrix}=\left[\frac{\kappa_{n+1}}{\kappa_n}\right]^{1/2}\begin{bmatrix}\frac{1}{2}(\kappa_n/\kappa_{n+1})^{1/2}(1+\kappa_{n+1}/\kappa_n)\exp(Z_{n+1}) & \frac{1}{2}(\kappa_n/\kappa_{n+1})^{1/2}(1-\kappa_{n+1}/\kappa_n)\exp(-Z_{n+1})\\ \frac{1}{2}(\kappa_n/\kappa_{n+1})^{1/2}(1-\kappa_{n+1}/\kappa_n)\exp(Z_{n+1}) & \frac{1}{2}(\kappa_n/\kappa_{n+1})^{1/2}(1+\kappa_{n+1}/\kappa_n)\exp(-Z_{n+1})\end{bmatrix}\begin{bmatrix}A_{n+1}\\ B_{n+1}\end{bmatrix}\ . \tag{73}$$

The 2×2 matrix in Eq. (73) is identical to $\tilde{\theta}^{(n)}$, from Eq. (68), if

$$Z_{n+1}=-H_n,\quad \kappa_{n+1}/\kappa_n=\tanh(J_n+H_n)\ . \tag{74}$$

IX. SUMMARY

The problems of phonons, electromagnetic waves, classical diffusion, Ising thermodynamics, and Bloch electrons in a magnetic field reduce to the Schrödinger equation with the determined potential. The Schrödinger equation in a general case is reduced to a three-parameter transfer matrix. The recurrence relation for the exponents in the components of the transfer matrix is derived. The localization and correlation lengths and their dependence on the parameters of a random system are determined. Eigenenergies and eigenstates are determined and analyzed. Their localization is proven to be uniquely determined by the dependence of the resistance on the Fermi energy.

ACKNOWLEDGMENTS

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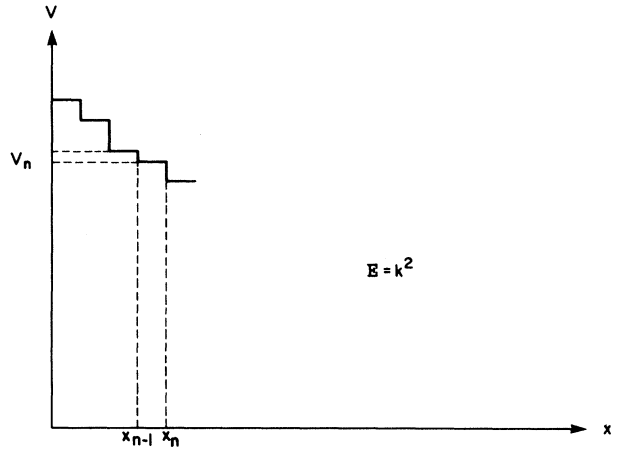

FIG. 23. The Ising thermodynamics maps on such a potential $V(x)$ in the Schrödinger equation.

APPENDIX A: TRANSFER MATRIX

Consider a general form of a transfer matrix through an arbitrary 1D potential barrier (or well)—see Fig. 11(a). Present the Schrödinger equation in the form

$$\Psi''+(k_1^2-V_1)\Psi=0\ , \tag{A1}$$

where

$$k_1^2=(2m/\hbar^2)E-V(x_1),\quad V_1(x)=V(x)-V(x_1)\ . \tag{A2}$$

$(\hbar^2/2m)V(x)$ is a potential energy, E is a total energy of a particle, $V_1(x_1)=0$, and x_1 is, e.g., the point of a local minimum where $E>U(x_1)$. If $V_1(x)\equiv 0$ in some region, any point in this region can be chosen for x_1. Choose the solution Ψ_1 to Eq. (A2), which yields

$$\Psi_1(x_1)=1,\quad \Psi_1'(x_1)=ik_1\ . \tag{A3}$$

Consider in the vicinity of x_1 the wave function

$$\Psi=A_1\Psi_1(x)+B_1\bar{\Psi}_1(x)\ . \tag{A4}$$

A bar denotes the complex conjugation. Similarly, in the vicinity of x_2 consider

$$\Psi=A_2\Psi_2(x)+B_2\bar{\Psi}_2(x)\ , \tag{A5}$$

where Ψ_2 yields Eqs. (A1)—(A3) with the subscript 1 replaced by the subscript 2, and x_1 replaced by x_2. Introduce a transfer matrix from A_2,B_2 to A_1,B_1:

$$\begin{bmatrix}A_1\\B_1\end{bmatrix}=\theta\begin{bmatrix}A_2\\B_2\end{bmatrix},\quad \theta=\begin{bmatrix}\theta_{11}&\theta_{12}\\\theta_{21}&\theta_{22}\end{bmatrix}. \tag{A6}$$

The complex conjugated $\bar{\Psi}$ yields the same Schrödinger equation as Ψ. Therefore, $\bar{B}_2\Psi_2+\bar{A}_2\bar{\Psi}_2$ generates $\bar{B}_1\Psi_1+\bar{A}_1\bar{\Psi}_1$. Thus, according to Eq. (A6),

$$\begin{bmatrix}\bar{B}_1\\\bar{A}_1\end{bmatrix}=\theta\begin{bmatrix}\bar{B}_2\\\bar{A}_2\end{bmatrix} \tag{A7}$$

for arbitrary A_2,B_2. Comparing Eq. (A6) and Eq. (A7), one obtains

$$\theta=\begin{bmatrix}\theta_{11}&\theta_{12}\\\bar{\theta}_{12}&\bar{\theta}_{11}\end{bmatrix}. \tag{A8}$$

Another general relation is the current conservation: $[\Psi(d\Psi/dx)-\Psi(d\bar{\Psi}/dx)]$ must be the same at $x=x_1$ and $x=x_2$. By Eqs. (A3)—(A5) this implies

$$k_1(|A_1|^2-|B_1|^2)=k_2(|A_2|^2-|B_2|^2)\ .$$

Now Eq. (A8) leads to

$$\theta=\left[\frac{k_2}{k_1}\right]^{1/2}\begin{bmatrix}\cosh(S)\exp(i\alpha)&\sinh(S)\exp(i\tilde{\beta})\\\sinh(S)\exp(-i\tilde{\beta})&\cosh(S)\exp(-i\alpha)\end{bmatrix}. \tag{A9}$$

I shall also use another presentation of Eq. (A9). Introduce $\cos(h)=\operatorname{sech}(S)$, where $0\le h\le\pi/2$, and $\sinh(S)=\tan(h)$. Then

$$\theta=\left[\frac{k_2}{k_1}\right]^{1/2}\begin{bmatrix}\exp(i\alpha)/\cosh(h)&\exp(i\tilde{\beta})\tan(h)\\\exp(-i\tilde{\beta})\tan(h)&\exp(-i\alpha)/\cos(h)\end{bmatrix}. \tag{A10}$$

This form of the transfer matrix in the case of $V_{1,2}=0$ in the infinitesimal vicinity of $x=x_1,x_2$ was presented in Ref. 11. The above derivation makes no assumptions about the potential.

The transfer matrix from x_1 to x_2 is θ_l:

$$\begin{bmatrix}B_2\\A_2\end{bmatrix}=\theta_l\begin{bmatrix}B_1\\A_1\end{bmatrix}. \tag{A11}$$

It is easily related to θ:

$$\theta_l=\frac{k_1}{k_2}\begin{bmatrix}\theta_{11}&\bar{\theta}_{12}\\-\theta_{12}&\bar{\theta}_{11}\end{bmatrix}. \tag{A12}$$

By Eq. (A12), the transmission coefficient of the current is the same from the left to the right and vice versa. The solution to Eqs. (A1) and (A3) can be presented as

$$\Psi_1(\xi)=1+ik_1\xi+\int_0^\xi(\xi-\xi')[V_1(\xi')-k_1^2]\Psi_1(\xi')d\xi'\ , \tag{A13}$$

$$\xi=x-x_1,\quad V_1(0)=0\ . \tag{A14}$$

If $V_1\equiv 0$ in some vicinity of x_1, then Ψ_1, by Eqs. (A1) and (A3), is a plain wave. There,

$$\Psi_1(x)=\exp[ik_1(x-x_1)]\ . \tag{A15a}$$

Elsewhere the difference between $\Psi_1(x)$ and this plain wave is, in virtue of Eq. (A13), of order of $\xi\int_0^\xi V_1(\xi')d\xi'$. Generally the best choice for x_1,x_2 are the points where $V(x)$ has local minima. In the vicinity of, e.g., x_1,

$$\Psi_1(x)-\exp[ik_1(x-x_1)]\sim(x-x_1)^4\ . \tag{A15b}$$

[At any point other than the minimum, this difference is $\sim(x-x_1)^3$.]

If at $x>x_2$ there is no barrier and thus there is only a transmitted wave $B_2=0$, then

$$\tau\equiv A_2/A_1=1/\theta_{11},\quad \rho\equiv B_1/A_1=\theta_{21}/\theta_{11}\ . \tag{A16a}$$

The current transmission t and reflection r coefficients equal

$$\begin{aligned} r&=|B_1|^2/|A_1|^2=|\rho|^2,\\ t&=k_2|A_2|^2/k_1|A_1|^2=k_2|\tau|^2/k_1\ . \end{aligned} \tag{A16b}$$

If a barrier is symmetric (and, in particular, $k_1=k_2\equiv k$), then the Schrödinger equation allows for odd and even (with respect to the middle of the barrier) solutions. This leads to

$$\tilde{\beta}=\frac{\pi}{2} \tag{A17}$$

in Eq. (A9). Thus $\pi/2-\tilde{\beta}\equiv\beta$ characterizes the asymmetry of the barrier. To elucidate the physical meaning of the quantities S,α,β in Eq. (A9), consider several special cases.

In a quasiclassical case, the comparison of Eq. (A16a) to the formulas 31(a1) for τ,ρ leads to

$$S\approx S_1=\left|\int_{x_1'}^{x_1''}k(x)dx\right|, \tag{A18}$$

$$\begin{aligned} -\alpha&\simeq\int_{x_1}^{x_1'}k(x)dx+\int_{x_1''}^{x_2}k(x)dx\ ,\\ -\tilde{\beta}&\simeq\frac{\pi}{2}+\int_{x_1}^{x_1'}k(x)dx-\int_{x_1''}^{x_2}k(x)dx\ , \end{aligned} \tag{A19}$$

$$k(x)=[(2m/\hbar^2)E-V(x)]^{1/2}\ . \tag{A20}$$

Thus, $\hbar S$ is the absolute value of the classically unavailable phase area (action).

The straightforward calculations in the case of a rectangular barrier, $V(x)=0$ at $x<x_1$, $V(x)\equiv(k_1^2+\kappa^2)\hbar^2/2m$ at $x_1<x<x_2$ and $V(x)\equiv(k_2^2+\kappa^2)$ at $x>x_2$ lead to θ with

$$\cosh(S)\exp(i\tilde{\alpha})=\tfrac{1}{2}\exp(ik_0d)\{(1+k_2/k_1)\cosh(\kappa d)-i[(\kappa/k_1)-(k_2/\kappa)]\sinh(\kappa d)\}\ , \tag{A21}$$

$$\sinh(S)\exp(i\tilde{\beta})=\tfrac{1}{2}\exp(-\tfrac{1}{2}ik'd)\{(1-k_2/k_1)\cosh(\kappa d)-i[(\kappa/k_1)+(k_2/\kappa)]\sinh(\kappa d)\}\ , \tag{A22}$$

$$k_0=(k_1+k_2)/2,\quad k'=k_2-k_1,\quad d=x_2-x_1\ , \tag{A23}$$

$$\tilde{\alpha}=\alpha+k_0d\ . \tag{A24}$$

A δ-function potential energy,

$$V(x)=v\delta(x-x_0)\ , \tag{A25}$$

allows us to choose $x_1=x_0-0$, $x_2=x_0+0$, and leads to

$$\cosh(S)\exp(i\alpha)=1-iv/2k,\quad \tilde{\beta}=\frac{\pi}{2}\ . \tag{A26}$$

APPENDIX B: RESONANCE TUNNELING

Consider a transfer matrix θ_{12} through two consequent barriers [see Fig. 11(a)] with the transfer matrices θ_1 and θ_2. Then

$$\theta_{12}=\theta_1\theta_2\ . \tag{B1}$$

Denote S,α,β by the subscripts 1 and 2 for the first and the second barrier correspondingly. Consider beyond the second barrier only a transmitted wave function. Then one obtains

$$\begin{bmatrix}A_2\\B_2\end{bmatrix}=\theta_2\begin{bmatrix}1\\0\end{bmatrix}=\left[\frac{k_1}{k_2}\right]^{1/2}\begin{bmatrix}\cosh(S_2)\exp(i\alpha_2)\\ \sinh(S_2)\exp(-i\beta_2)\end{bmatrix}. \tag{B2}$$

Suppose $S_2>S_1\gg1$. Then the probability density ρ at $x\simeq x_2$ is $\sim\exp(2S_2)$ times higher than in the transmitted wave function, whatever is the particle energy. Now consider $x\approx x_1$. By Eq. (B1),

$$\begin{bmatrix}A_1\\B_1\end{bmatrix}=\theta_1\begin{bmatrix}A_2\\B_2\end{bmatrix}=\begin{bmatrix}\exp(i\omega_1)[\cosh S_1\cosh S_2+\sinh S_1\sinh S_2\exp(i\omega)]\\ \exp(-i\omega_2)[\sinh S_1\cosh S_2+\cosh S_1\sinh S_2\exp(i\omega)]\end{bmatrix}, \tag{B3}$$

where

$$\omega=\beta_1-\beta_2-\alpha_1-\alpha_2,\quad \omega_1=\alpha_1+\alpha_2,\quad \omega_2=\beta_1-\alpha_2\ . \tag{B4}$$

In the case of interest ($S_2>S_1\gg 1$), by Eqs. (A19) and (A20),

$$\begin{aligned}\omega&\approx 2\int_{x_1''}^{x_2'}k(x)dx,\\ -\omega_1&\approx\tfrac{1}{2}\omega+\int_{x_1''}^{x_2'}k(x)dx+\int_{x_1}^{x_1'}k(x)dx,\\ \omega_2&\approx\tfrac{1}{2}(\omega-\pi)-\int_{x_1''}^{x_2'}k(x)dx+\int_{x_2''}^{x_3}k(x)dx\ .\end{aligned} \tag{B5a}$$

The quasiclassical eigenenergies in the potential well are determined by the Bohr quantization,[31(a5)]

$$\tfrac{1}{2}\omega^*=\int_{x_1''}^{x_2'}k(x)dx=(Q+\tfrac{1}{2})\pi\ . \tag{B5b}$$

An asterisk denotes the values at the eigenenergy and Q is an integer. When ω is not too close to $(2Q+1)\pi$, then the probability densities in the incident and reflected wave functions are $\sim\exp(2S_1)$ higher than in the vicinity of $x\simeq x_2$ and $\sim\exp(2S_1+2S_2)$ higher than in the transmitted wave [see Fig. 1(b), dotted curve]. The current transmission coefficient t_{12} through both barriers is $\sim t_1t_2$, where $t_{1,2}\sim\exp(-2S_{1,2})$. However, when

$$\omega=\omega^*+\omega',\quad \omega^*=(2Q+1)\pi,\quad \omega'\ll 1\ ,$$

then the particle density is

$$\sim\exp(2S_2^*-2S_1^*)[1+\omega'^2\exp(4S_1)]$$

in the incident and reflected waves. Thus, $\omega'\lesssim\exp(-2S_1^*)$ determines the width [dashed lines in Fig. 11(a)] of the resonances and of the eigenenergies $\omega^*=(2Q+1)\pi$. In the resonance

$$t_{1,2}^*\sim t_2^*/t_1^*\sim\exp(-2\,|\,S_1^*-S_2^*\,|\,)\ .$$

Particle density accumulates at the quasilevel which is localized inside the potential well, and the probability density of incident particles with such an energy is $\sim\exp(2S_1^*)$ times *less* than in the vicinity of $x=x_2$. When $S_1^*=S_2^*$ and $\omega'=0$, then $t_{12}\approx 1$. This is the origin of an allowed energy band (with no scattering) in a periodic system.

APPENDIX C: RECURRENCE RELATION

Equations (A12) and (A10) lead to the transfer matrix of Eq. (32) for an arbitrary wave function, described by Eqs. (A4) and (A5). But any wave function reduces to two independent currentless (i.e., real) wave functions, e.g.,

$$\begin{aligned}A_1\Psi_1+B_1\bar{\Psi}_1=&\left[\frac{A_1+\bar{B}_1}{2}\Psi_1+\frac{\bar{A}_1+B_1}{2}\bar{\Psi}_1\right]\\&+i\left[\frac{A_1-\bar{B}_1}{2i}\Psi_1+\frac{\bar{A}_1-B_1}{-2i}\bar{\Psi}_1\right].\end{aligned} \tag{C1}$$

Since the Schrödinger equation is linear, it is sufficient to consider only currentless wave functions. (The current is conserved; therefore, then the wave function is everywhere currentless.) Present such wave functions in the form

$$\Psi=C_k\Psi_k+\bar{C}_k\bar{\Psi}_k\quad\text{at } x\simeq x_k\ , \tag{C2}$$

$$C_k=\exp(\tfrac{1}{2}g_k-\tfrac{1}{2}i\phi_k)\ . \tag{C3}$$

Now the transformation from, e.g., x_2 to x_3 by the transfer matrix from Eq. (32) leads to

$$\exp(\tfrac{1}{2}g_3-\tfrac{1}{2}i\phi_3)=(k_2/k_3)^{1/2}[\exp(\tfrac{1}{2}g_2-\tfrac{1}{2}i\phi_2+\tfrac{1}{2}i\alpha_2)\operatorname{sech}h_2+\exp(\tfrac{1}{2}g_2+\tfrac{1}{2}i\phi_2+i\beta_2)\tanh_2]\ , \tag{C4}$$

where α_2,β_2,h_2 describe the barrier to the right of x_2. Introduce

$$w=\tan^2(\tfrac{1}{4}\pi-\tfrac{1}{2}h)\ . \tag{C5}$$

Then

$$\operatorname{sech}h=(1+w)/2\sqrt{w},\quad \tanh=(1-w)/(2\sqrt{w})\ , \tag{C6}$$

and Eq. (C4) can be rewritten as

$$\exp(\tfrac{1}{2}g_3-\tfrac{1}{2}i\phi_3)=(k_2/k_3)^{1/2}w_2^{-1/2}\exp[\tfrac{1}{2}g_2+\tfrac{1}{2}i(\alpha_2+\beta_2)]\{\cos[\tfrac{1}{2}(\phi_2-\alpha_2+\beta_2)]-iw_2\sin[\tfrac{1}{2}(\phi_2-\alpha_2+\beta_2)]\}\ , \tag{C7}$$

or as

$$\begin{aligned}\exp(\tfrac{1}{2}g_3-iF_3)=&(k_2/k_3)^{1/2}w_2^{-1/2}\\&\times\exp(\tfrac{1}{2}g_2)(\cos D_2-iw_2\sin D_2)\ ,\end{aligned} \tag{C8}$$

where

$$F_3=\tfrac{1}{2}(\phi_3+\alpha_2+\beta_2),\quad D_2=\tfrac{1}{2}(\phi_2-\alpha_2+\beta_2)=F_2-\xi_2\ , \tag{C9}$$

$$\xi_2=\tfrac{1}{2}(\alpha_2-\beta_2+\alpha_1+\beta_1)\ . \tag{C10}$$

By Eq. (C8),

$$\exp(\tfrac{1}{2}g_3)\cos F_3=(k_2/k_3)^{1/2}w_2^{-1/2}\exp(\tfrac{1}{2}g_2)\cos D_2\ , \tag{C11}$$

$$\exp(\tfrac{1}{2}g_3)\sin F_3=(k_2/k_3)^{1/2}w_2^{1/2}\sin D_2\ . \tag{C12}$$

Thus

$$\tan F_3=w_2\tan D_2=w_2\tan(F_2-\xi_2)\ , \tag{C13}$$

$$g_3=g_2+\ln[w_2^{-1}(\cos^2 D_2+w_2^2\sin^2 D_2)]+\ln(k_2/k_3)$$
$$=g_2+\ln[(w_2^{-1}+w_2\tan^2 D_2)/(1+\tan^2 D_2)]$$
$$+\ln(k_2/k_3)\ . \tag{C14}$$

By Eq. (C13),

$$g_3=g_2+\ln[(1+X_3^2)/(w_2+w_2^{-1}X_3^2)]+\ln(k_2/k_3) \tag{C15}$$

and

$$X_3=w_2(X_2-r_2)/(1+r_2X_2)\ , \tag{C16}$$

where

$$r_2=\tan\xi_2,\quad X_2=\tan F_2\ . \tag{C17}$$

Obviously, Eqs. (C15)—(C17) are valid for the recurrence from any x_κ to $x_{\kappa+1}$. In the case of δ-function potentials Eqs. (C15)—(C17) were derived in Ref. 15; in the case of nonoverlapping potentials in Ref. 11. The knowledge of X_κ determines, by Eq. (C17), $F_\kappa=\arctan X_\kappa+\tilde{Q}\pi$, where $\tilde{Q}$ is an integer. By Eqs. (C9), (C2), (C3), and (C15), the wave function is thus determined, except for its sign [which is unimportant, but can be derived from Eqs. (C11) and (C17)].

For an arbitrary number of barriers, Eqs. (C15) and (C16) provide the recurrence relations,

$$g_{\kappa+1}=\sum_{\mu=1}^{\kappa}\ln[(1+X_\mu^2)/(w_{\mu-1}+w_{\mu-1}^{-1}X_\mu^2)]+\ln(k_0/k_{\kappa+1})$$
$$\equiv\sum_{\mu=1}^{\kappa}H_\mu+\ln(k_0/k_{\kappa+1})\ , \tag{C18}$$

$$X_{\mu+1}=w_\mu(X_\mu-r_\mu)/(1+r_\mu X_\mu)\ . \tag{C19}$$

Suppose that to the left of the first barrier (it is $\kappa=1$, $k_1=k$) there is only a transmitted plain wave

$$\exp(-ikx)=\cos(kx)-i\sin(kx)\ .$$

By Eqs. (C2) and (C3), $\cos(kx)$ corresponds to $\phi_0=0$, $-\sin(kx)$ corresponds to $\phi_0=\pi$. It is convenient to introduce formally the barrier at $x=x_0+0$ with $\theta_0=1$. By Eq. (A10), its $h_0=\alpha_0=0$, $k_0=k$. Choose its $\beta_0=+\frac{1}{2}\pi$, in agreement with Eq. (A17). Then the initial conditions for $g_\kappa^{(\pm)}$ read

$$X_0=\pm 1\ . \tag{C20}$$

To the right of the last barriers (it is $\kappa=L$), by Eqs. (C2) and (C3),

$$\psi_{L+1}=\exp(\tfrac{1}{2}g_{L+1}^{(+)})\cos(kx-\tfrac{1}{2}\phi_{L+1}^{(+)})$$
$$+i\exp(\tfrac{1}{2}g_{L+1}^{(-)})\sin(kx-\tfrac{1}{2}\phi_{L+1}^{(-)})\ .$$

Now the current transmission coefficient t is easily related to $g_L^{(\pm)}$ and equals[15]

$$t=4[\exp(g_{L+1}^{(+)})+\exp(g_{L+1}^{(-)})+2]^{-1}\ . \tag{C21}$$

By Landauer,[12] the dimensionless resistance

$$R=t^{-1}-1=[\exp(g_{L+1}^{(+)})+\exp(g_{L+1}^{(-)})-2]/4\ . \tag{C22}$$

Some general remarks on the recurrence. The recurrence equations (36) and (37) are invariant with respect to the change $w_\mu\to w_\mu^{-1}$, $r_\mu\to(-r_\mu)$, $X_\mu\to X_\mu^{-1}$. The first change is equivalent to $h\to -h$ in Eq. (38). Another invariant change is $X_\mu\to -X_\mu$, $r_\mu\to -r_\mu$, which is equivalent (in virtue of the previous change) to $X_\mu\to -X_\mu^{-1}$, $w_\mu\to w_\mu^{-1}$. Consider the last change in Eq. (38),

$$X_{\mu+1}=w_\mu(X_\mu+r_\mu)/(1-r_\mu X_\mu)\ , \tag{C23}$$

and rewrite it as

$$\Delta X_\mu\equiv X_{\mu+1}-X_\mu=[(w_\mu-1)/(1-r_\mu X_\mu)]X_\mu$$
$$+r_\mu(w_\mu+X_\mu^2)/(1-r_\mu X_\mu)\ . \tag{C24}$$

Suppose $r_\mu>0, w_\mu>1$ and $0<X_\mu<r_\mu^{-1}$. Then

$$X_{\mu+1}-X_\mu>\Omega_\mu X_\mu,\ \Omega_\mu=(w_\mu-1)/(1-r_\mu X_\mu)>0\ ,$$

and $X_\mu>(1+\Omega_\mu)X_\mu$ exponentially increases with μ, until X_{μ_1} becomes larger than $r_{\mu_1}^{-1}$. Then $X_{\mu_1}+1$ becomes less than 0, and the first term in (C24) becomes negative. However, this negative value cannot become too large, since a large enough negative X_μ leads, by Eq. (C23), to $X_{\mu+1}\simeq -w_\mu/r_\mu$. Thus X_μ may change only between certain largest and lowest values, except for the special points, where $X_\mu=1/r_\mu$ implies $|X_{\mu+1}|=\infty$. The next value, however, by Eq. (C23) is a regular $X_{\mu+2}=-w_\mu/r_{\mu+1}$. When, e.g., $w_\mu=1+w'_\mu$ and w'_μ, $r_\mu\ll 1$ (weak narrow potentials), then the analytical solution to X_μ is possible. In the region where $|X_\mu|\ll 1/r_\mu$, in the leading over w'_μ, r_μ approximation,

$$X_{\mu+1}\simeq w'_\mu(X_\mu+r_\mu)(1+r_\mu X_\mu)\simeq(1+w'_\mu+r_\mu)X_\mu+r_\mu\ . \tag{C25}$$

When $|X_\mu|\gg r_\mu$, then $X_\mu\to -1/\tilde{X}_\mu$ leads to

$$\tilde{X}_{\mu+1}=(1-w'_\mu)\tilde{X}_\mu+r_\mu=w_\mu^{-1}\tilde{X}_\mu+r_\mu\ . \tag{C26}$$

The regions $|X_\mu|\ll 1/r_\mu$ and $|X_\mu|\gg r_\mu$ overlap when $r_\mu\ll 1$, while Eqs. (C25) and (C26) are easily solved. For instance, in Eq. (C26) the change $\tilde{X}_\mu=\Omega_\mu^{-1}y_\mu$, $\Omega_\mu=w_{\mu-1}w_{\mu-2}\cdots$ leads to

$$y_{\mu+1}=y_\mu+r_\mu\tilde{\Omega}_{\mu+1}=y_1+\sum_{\nu=1}^{\mu}r_\nu\tilde{\Omega}_{\nu+1}\ . \tag{C27}$$

APPENDIX D: PHASE RANDOMIZATION

To derive Eq. (42), present Eqs. (35) and (38) in the form

$$X_\kappa=\tan F_\kappa,\quad F_\kappa=\tfrac{1}{2}(\phi_\kappa+\alpha_\kappa+\beta_{\kappa-1})\ , \tag{D1}$$

$$r_\kappa=\tan\rho_\kappa,\quad \rho_\kappa=\tfrac{1}{2}(\alpha_\kappa-\beta_\kappa+\alpha_{\kappa-1}+\beta_{\kappa-1})\ . \tag{D2}$$

Now Eq. (37) can be rewritten as

$$\tan F_{\mu+1}=w_\mu\tan\tilde{F}_\mu,\quad \tilde{F}_\mu=F_\mu-\rho_\mu\ . \tag{D3}$$

Therefore, by Eq. (D1), $1+X_\mu^2=\sec^2 F_\mu$, and by Eq. (D3),

$w_{\mu-1}+w_{\mu-1}^{-1}X_{\mu}^{2}=w_{\mu-1}(1+\tan^2\tilde{F}_{\mu-1})=w_{\mu-1}\sec^2\tilde{F}_{\mu-1}$.

Therefore, Eq. (36) leads to

$$g_{\kappa+1}-\ln\left[\frac{k}{k_{\kappa+1}}\right]=\sum_{\mu=1}^{\kappa}\frac{\cos^2\tilde{F}_{\mu-1}}{w_{\mu-1}\cos^2F_{\mu}}=\sum_{\mu=0}^{\kappa-1}\frac{\cos^2\tilde{F}_{\mu}}{w_{\mu}\cos^2F_{\mu+1}}\ . \tag{D4}$$

On the other hand, by Eqs. (D1) and (D3),

$$\Delta F_{\kappa}=F_{\kappa}^{(+)}-F_{\kappa}^{(-)}=\tfrac{1}{2}(\phi_{\kappa}^{(+)}-\phi_{\kappa}^{(-)})\equiv\tfrac{1}{2}\Delta\phi_{\kappa}\ , \tag{D5}$$

$$\Delta\tilde{F}_{\kappa}=\tfrac{1}{2}\Delta\phi_{\kappa}\ . \tag{D6}$$

When $\Delta\phi_{\kappa}\ll 1$, then Eq. (D3) yields $\Delta(\tan F_{\mu+1})=w_{\mu}\Delta(\tan\tilde{F}_{\mu})$, i.e., accounting for Eqs. (D5) and (D6),

$$\Delta\phi_{\mu+1}/\cos^2F_{\mu+1}=w_{\mu}\Delta\phi_{\mu}/\cos^2\tilde{F}_{\mu}\ . \tag{D7}$$

Therefore,

$$\Delta\phi_{\mu+1}\propto\prod_{1}^{\mu}w_{\mu_1}\cos^2F_{\mu_1+1}/\cos^2\tilde{F}_{\mu}=\exp\left[-\sum_{1}^{\mu}\ln\frac{\cos^2\tilde{F}_{\mu_1}}{w_{\mu_1}\cos^2F_{\mu+1}}\right]\ . \tag{D8}$$

The comparison of Eqs. (D4) and (D8) yield

$$\Delta\phi_{\mu+1}\alpha\exp(-g_{\mu+2})\ . \tag{D9}$$

In virtue of Eq. (41), where[11] $g_{L+1}^{+}-g_{L+1}^{-}\to$ const (because the memory of the initial conditions vanish),

$$\Delta\phi_{\mu+1}\alpha 1/R_{\mu+1}\ . \tag{D10}$$

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